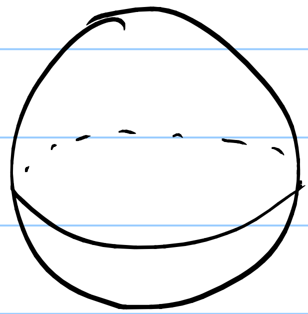
Enumeration of planar maps.

Titre de la note 12/03/2015

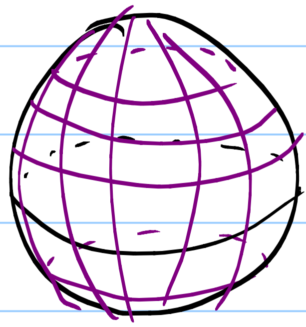
Introduction

Imagine a network of particles that spans a surface, like the sphere.

What is a good model for this?

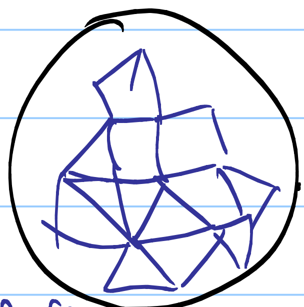


Regular mesh? not realistic



(But regular lattices are widely studied in stat. physics)

- We would like a structure with more irregularities...



→ maps form a good discrete model for surfaces

① Definitions

1 - Embeddings

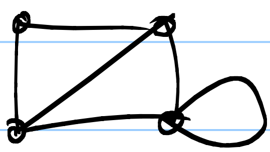
Drawing a graph onto a surface in such a way that two edges never cross each other (except potentially at their endpoints) defines an <u>embedding</u> of the graph.

Only some graphs can be embedded in the sphere: these graphs are said to be planar

(<u>RK</u>: A graph can be embedded in the sphere iff it ____________ plane.
<u>Reason</u>: Stereographic projection)

<u>Def</u> A <u>planar map</u> is an embedding of a connected and planar graph onto the sphere, considered up to continuous deformation.

<u>Ex</u>



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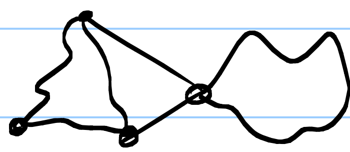

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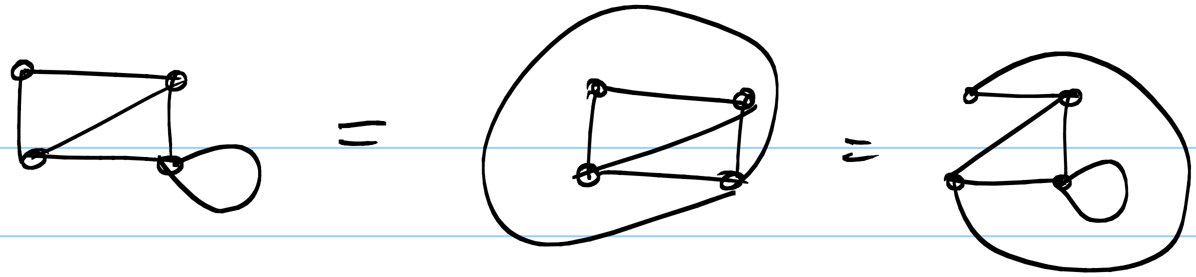
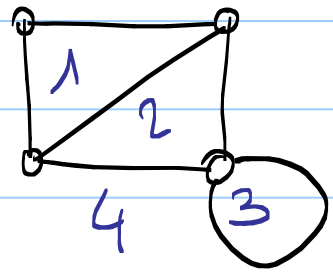


Maps contain more information than graphs,
e.g. the notion of faces.

A face is a connected component of the sphere minus the embedding.

ASK

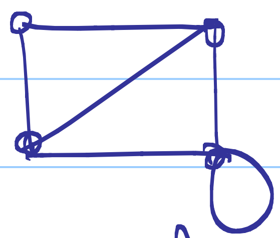
Ex



→ 4 faces

4 is the exterior face

A face has a degree: it is the number of incident ~~edges~~ corners (cf later definition)



the degree of exterior face is 5

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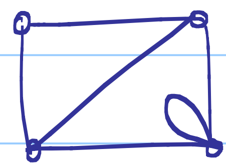

no face of degree 5.

2 - Rootings

These objects have a lot of underlying symmetries

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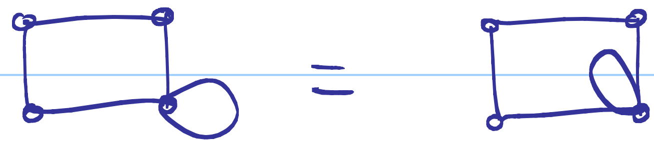
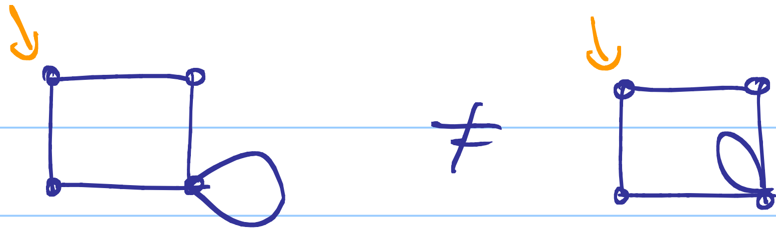


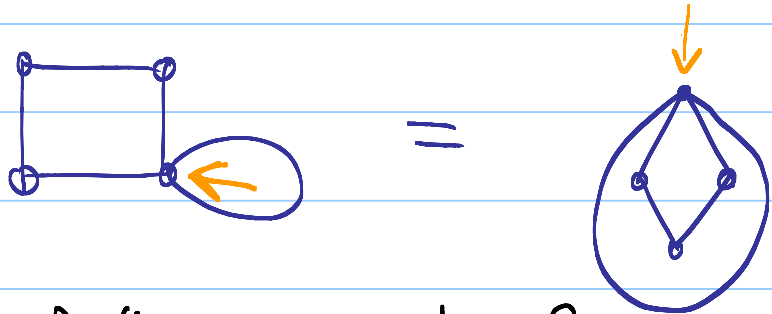
In order to kill these symmetries, we **root** the maps.

Def When we cut an edge in its middle, we get two half-edges.
A corner is a pair of two consecutive half-edges

A map is **rooted** if a corner of the map is marked.
This corner is called the **root** of the map.

Ex:



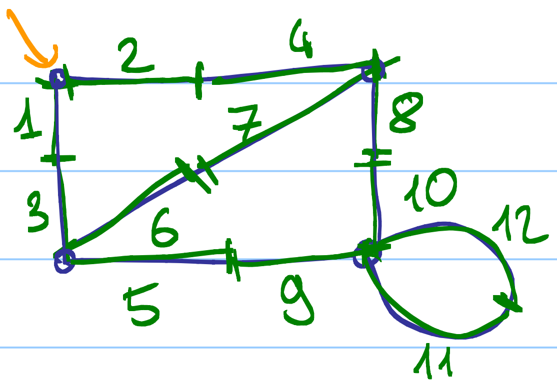


Why rooting kills symmetry?

i.e under a certain convention

When we root a map, we can canonically order the half-edges.

For instance, we can order them via a Breadth First Search, starting with the half-edge that follows the root in a counterclockwise order.



Prop Let M and M' be two rooted planar maps with m edges.

For $i \in \{1,\dots,2m\}$, we denote by $h_M(i)$ the half-edge labeled by i via the previous BFS.

M and M' are equal iff for $(i,j) \in \{1,\dots,2m\}^2$ the equivalences

1. $(h_M(i), h_M(j))$ is an edge in M

$$\Updownarrow$$

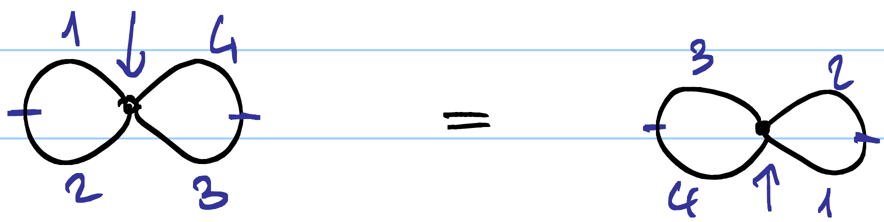
$(h_{M'}(i), h_{M'}(j))$ is an edge in M'

2. $h_M(i)$ and $h_M(j)$ are incident to a same vertex in M

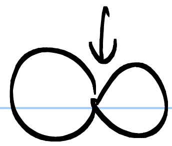
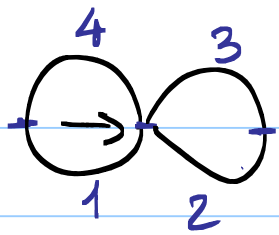
$$\Updownarrow$$

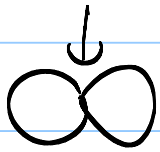
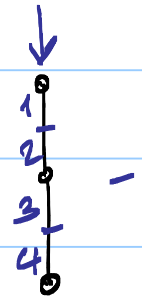
$h_{M'}(i)$ and $h_{M'}(j)$ are incident to a same vertex in M'.

Ex



But

 $\neq$ 

and  $\neq$ 

From now on, every map is rooted.

Is rooting too abusive / too artificial?
No and No.

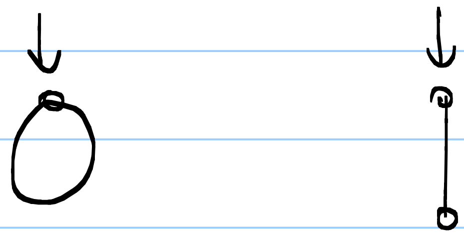
→ Natural problems with maps have an anchor point (as an origin, a position in space.

→ The enumeration of unrooted maps lies on the enumeration of rooted maps (cf Polya enumeration)

3. Counting planar maps

a. Small values

There are 2 planar maps with one edge:



Exercice ① How many planar maps with two edges?

② How many planar maps with one vertex? two vertices?

③ How many 4-valent planar maps with one vertex? two vertices?

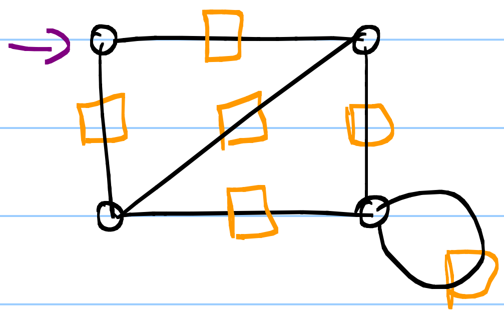
A graph is 4-valent if the degree of every vertex is 4.

It looks like the numbers between 1. and 3. are the same.

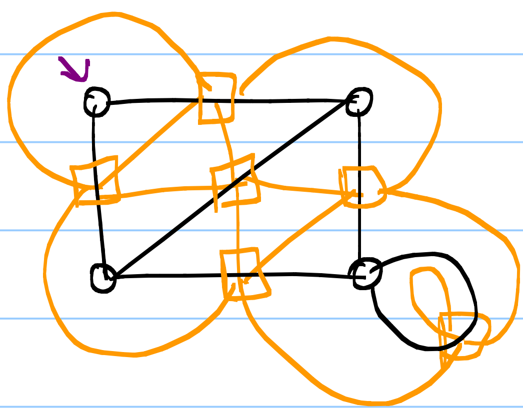
6. A bijection between maps with n edges and 4-valent maps with n vertices.

Recipe:

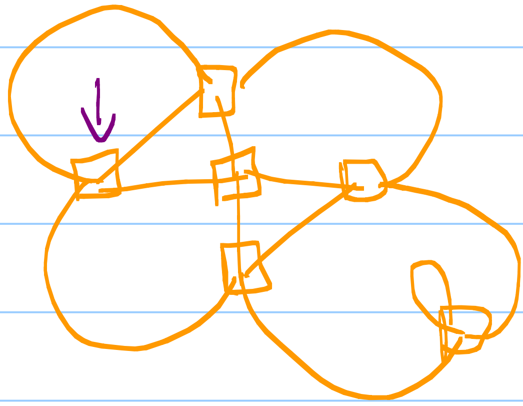
0. Take a map with n edges.
1. Put a square vertex on each edge.



2. Each edge is incident to 2 faces. Link every edge to the 4 adjacent edges in the two faces.

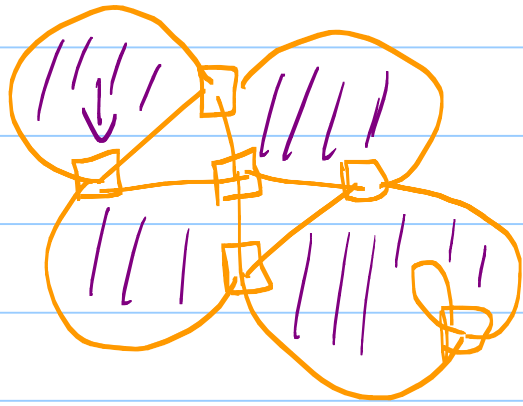


3 - Forget the original map -

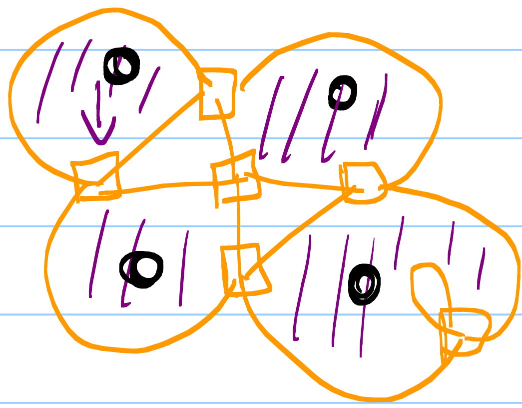


Is it bijective? Here is the converse transformation -

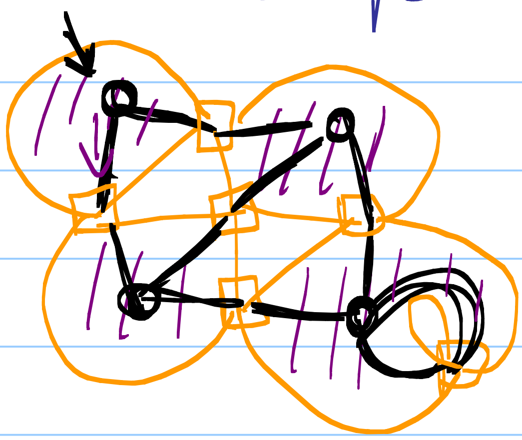
1 - The faces of a 4 - valent map form a bipartite graph, i.e., we can colour the faces with 2 colours in such a way that 2 adjacent faces never have the same colour -



2 - Put a vertex in each face with the same colour as the face containing the root -



3. Link each new vertex to the corners incident to the corresponding face.



c. Motivations

→ Tutte was the forerunner of maps enumeration. It wanted to prove the four colours theorem.

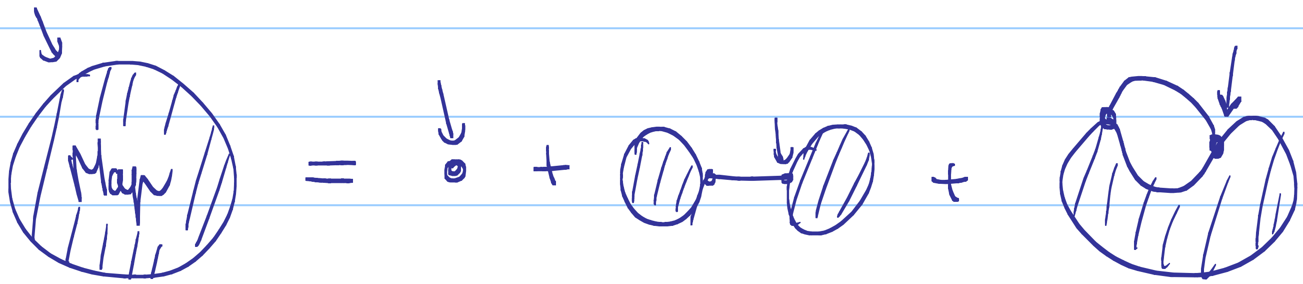
→ Nowadays this domain contributes to numerous models of statistical physics. ("quantum gravitation", ...)

→ The asymptotic enumeration of planar graphs lies on enumeration of planar maps (Gimenez, Noy, 2005).

② Tutte's recursive approach

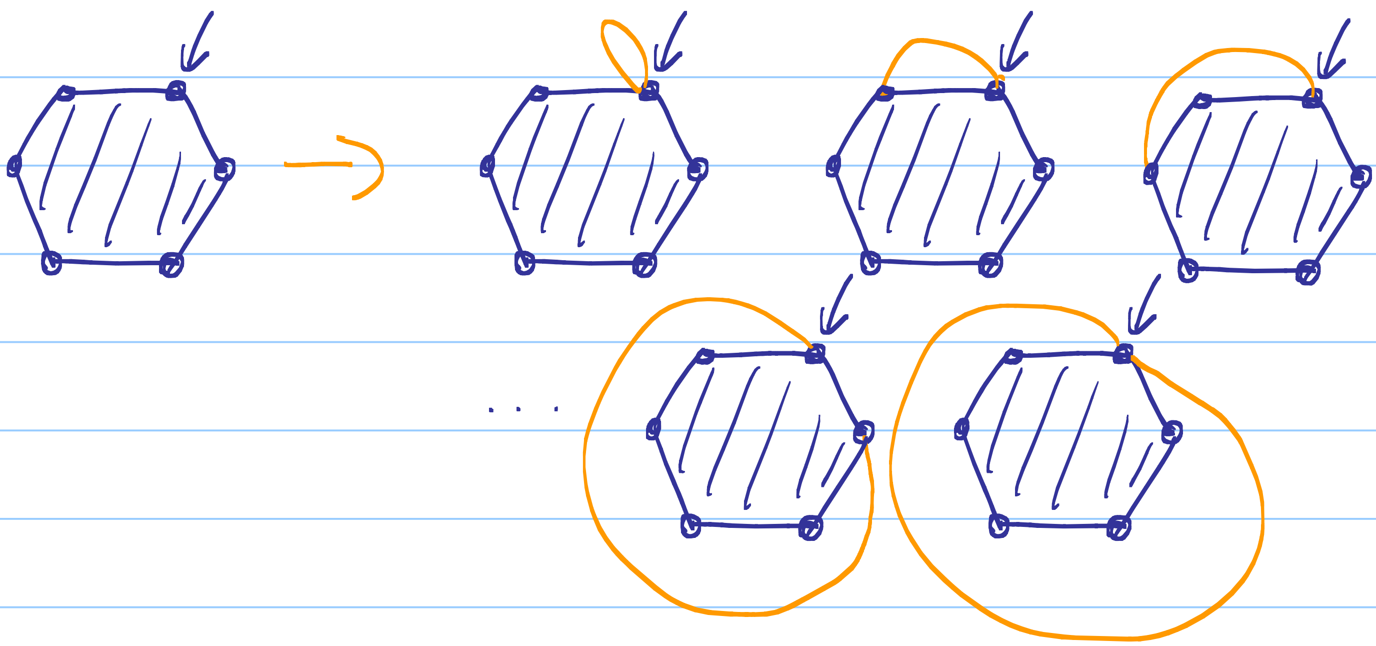
1 - Equations for the GF of planar maps

Natural decomposition of a map:



However this decomposition is not injective:

from a planar map, there are several ways to add an edge at the root



How many exactly? $d+1$, if d denotes the degree of the root face, i.e the face that contains the root.

To derive a functional equation for their generating function, we need to take into account the degree of the root face, by an additional variable.

Let M be the GF of planar maps

$$M(t,y) = \sum_{M \text{ planar map}} t^{\#\text{edges}(M)} \, y^{\text{root degree}(M)}$$

$$= \sum_{d \geq 0} M_d(t)\, y^d$$

Then

$$M(t,y) = 1 + y^2 t M(t,y)^2 + t \sum_{d \geq 0} M_d(t) \times (y + \dots + y^{d+1})$$

$$\boxed{M(t,y) = 1 + y^2 t M(t,y)^2 + t y \frac{y M(t,y) - M(t,1)}{y-1}} .$$

We want to get $M(t,1)$ but we cannot set $y=1$ because of $\frac{yM(t,y)-M(t,1)}{y-1}$.

The variable y is said to be catalytic.

Equations with catalytic variables also arise in the enumeration of

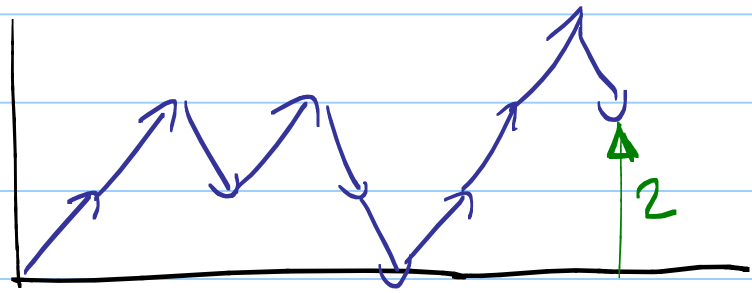
- polyominoes
- lattice walks
- permutations ...

2. How to study functional equations with a catalytic variable?

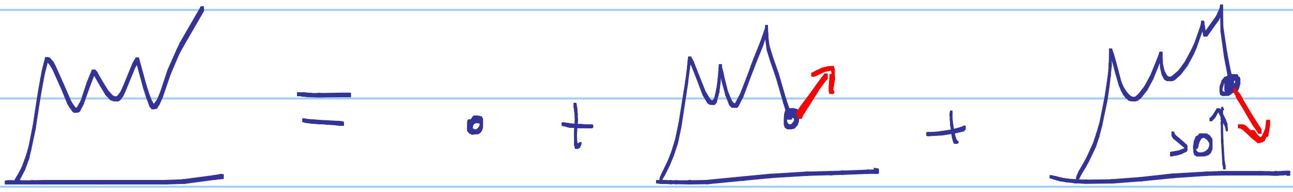
Consider the example of Dyck paths.

Let $D(t,y)$ the GF of Dyck prefixes counted by the size and the height of the endpoint.

Contribution of



$$= t^{10} y^{2}$$



$$D(t,y) = 1 + ty\,D(t,y) + ty^{-1}\big(D(t,y) - D(t,0)\big)$$

We want $D(t,0)$ but y is a catalytic variable.

<u>Kernel method</u> :

$$\underbrace{\left(ty^2 - y + t\right)}_{\text{kernel}} D(t,y) = tD(t,0) - y$$

Let us rewrite this as

$$K(t,y) \times D(t,y) = tD(t,0) - y$$

with $\quad K(t,y) = ty^2 - y + t.$

There exists a unique series $\alpha(t)$ such that $\quad K(t,\alpha(t)) = 0$

then we have $\quad \begin{cases} \alpha(t) = t\alpha(t)^2 + t \\ tD(t,0) = \alpha(t) \end{cases}$

$$\left(\alpha(t) = \frac{1-\sqrt{1-4t^2}}{2t}\right)$$

Hence $\quad D(t,0) = t^2 D(t,0)^2 + 1$

→ We recover the usual functional equation for Dyck paths.

(Thereafter we only need Lagrange's inversion)

→ This method cannot be used for planar maps since there is a square (M^2)

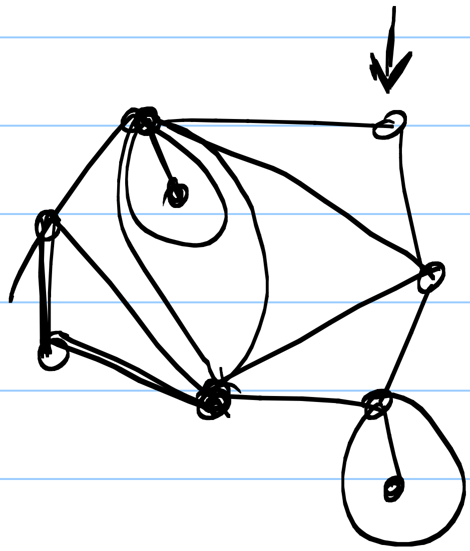
We use in this case the quadratic method (invented by Brown), which induces more equations.

→ The general case has been solved by Bousquet-Mélou and Jehanne.

Exercices

① A <u>quasi-triangulation</u> is a planar map where each face not containing the root has degree 3.

Ex:



Write an equation with a catalytic variable for the GF of quasi-triangulations.

Answer:
$$T = 1 + t y^2 T^2 + t \, \frac{T - T_0 - y T_1}{y}$$

② Write an equation for walks in a half-line with steps $+1, -1, -2$ with a catalytic variable.

Give an exact formula for the number of such paths thanks to Lagrange inversion.

③ Guess and try.

The method used in the 60's is actually based on a "guess and try" method.

1. You compute the first terms of the GF thanks to the equation with a catalytic variable.

$$1, 2, 9, 54, 378, 2916, 24057, 208494 \ldots$$

2. We try to guess:
   - OEIS
   - look for factorization

3. And we inject this in the equation.

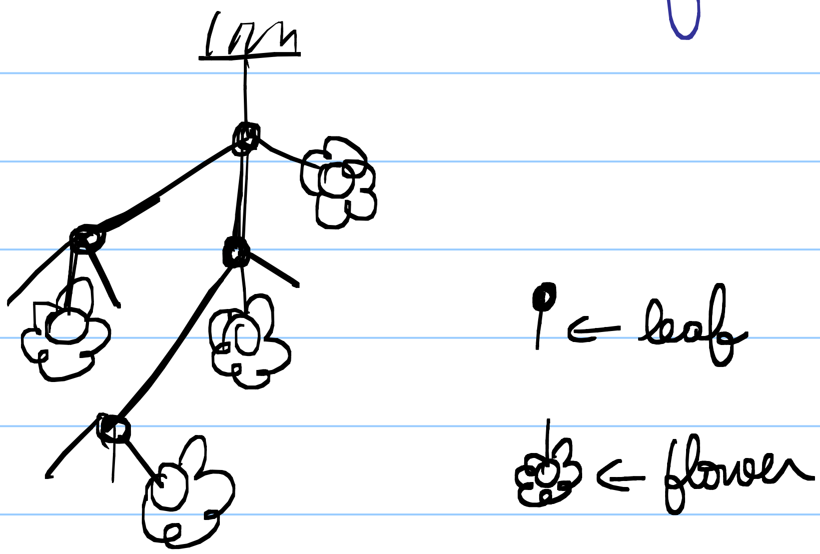
In the case of planar maps, we find

$$2\, 3^n \frac{(2n)!}{n!\,(n+2)!}$$

Can we find a simple explanation for this formula?

③ Schaeffer's bijection between 4-valent maps and blossoming trees

A blossoming tree is a plane binary tree, rooted at a leaf, such that every inner node carries a flower (+ its 2 children of course)



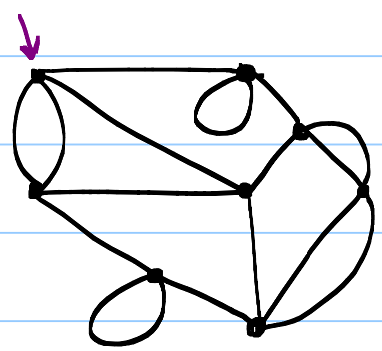
Prop: There are $\frac{(2n)!}{n!(n+1)!} \times 3^n$ blossoming trees with n inner nodes

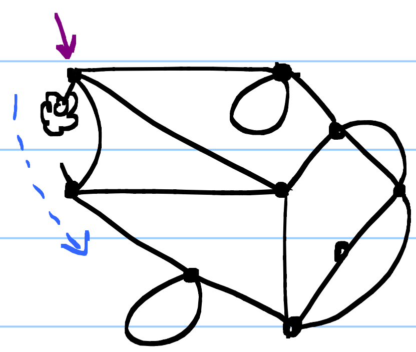
(there are 3 possible positions for a flower)

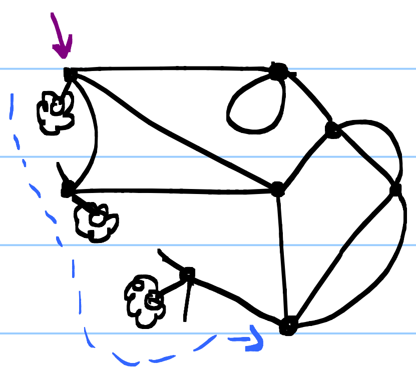
Given a 4-valent map, we want to cut some edges into leaves and flowers in order to obtain a blossoming tree

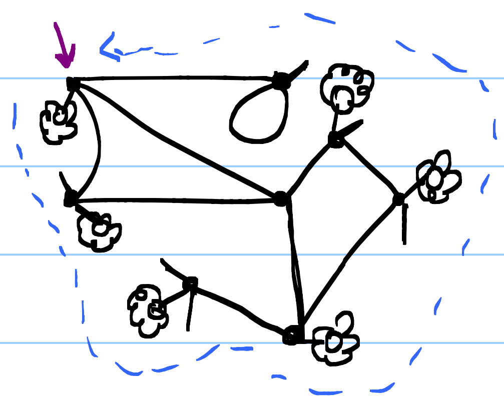
Draw the map on the plane so that the root is in the exterior face.
Walk around the map in counterclockwise order starting from the root.
Every time you walk along a non-isthmus* edge, we cut this edge in two half-edges: the first one is a flower, the second one is a leaf.

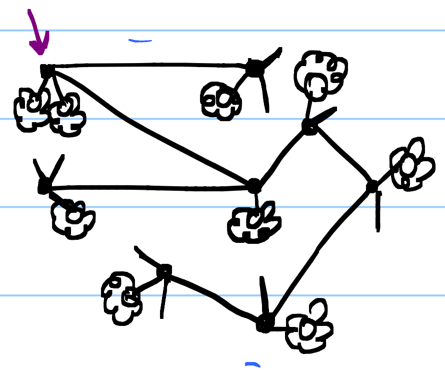
* An edge is an isthmus if its deletion disconnects the map.











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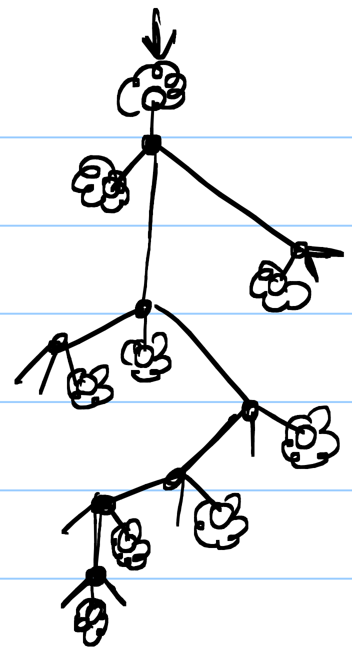
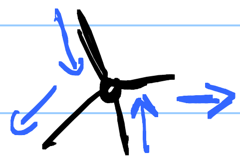


Cut the flower after the root and root the tree on the obtained leaf: we obtain a <u>blossoming tree</u>.

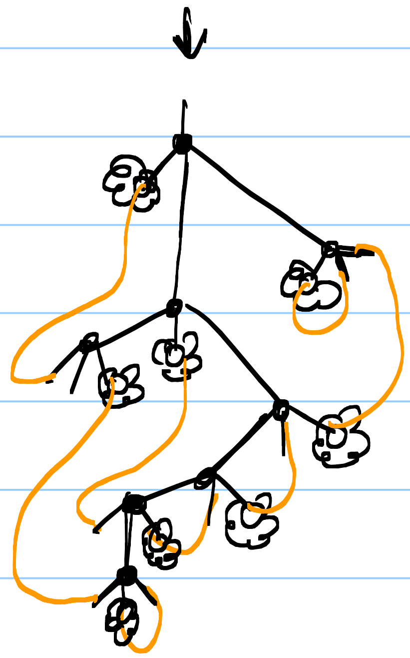


Let $\psi(M)$ be the image of a 4-valent map under this transform.

We can recover the original map M by matching leaves and flowers as follows:

Consider the root as a leaf.

Whenever a leaf immediately follows a flower in the cyclic leaves-flowers sequence, merge them into an edge.

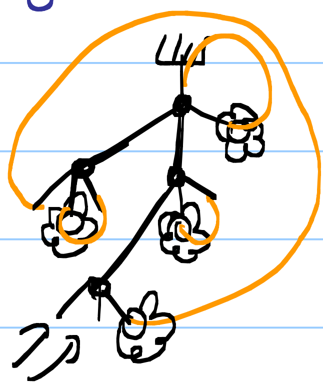


Stop when all flowers have been matched.
Then match the two remaining leaves.
We can prove that we always recover the original map M, the order of the mergings does not matter.

In other terms, φ is injective?

Is φ surjective? No, of course, look at the numbers.

Moreover, observe that when we match leaves and flowers, there always remain two unmatched leaves.
the root is unmatched if the blossoming tree is some $\varphi(M)$, where M is a 4-valent map.
But it is not always the case.



the 2 unmatched leaves

A blossoming tree is said to be balanced if the root is an unmatched leaves.

Prop : Ψ is a bijection between 4-valent maps and balanced blossoming trees.

How many balanced blossoming trees are there?

We can get this number via a "double rooting".

What is the number of blossoming trees with two marked leaves, the first one being unmatched, the other one having no restriction.

On one hand, it is $\underbrace{(n+2)}_{\#\text{ leaves}} \times b_n$

where b_n be the number of balanced blossoming trees.

On the other hand, it is

$$\underbrace{\frac{(2n)!}{n!(n+1)!}\times 3^n}_{\#\text{ blossoming trees}} \times \underbrace{2}_{\#\text{ unmatched leaves}}$$

Cor : The number of maps with n edges is $b_n = 2 \cdot 3^n \dfrac{(2n)!}{(n+2)!\, n!}$.