Spanning Forests in Regular Planar Maps

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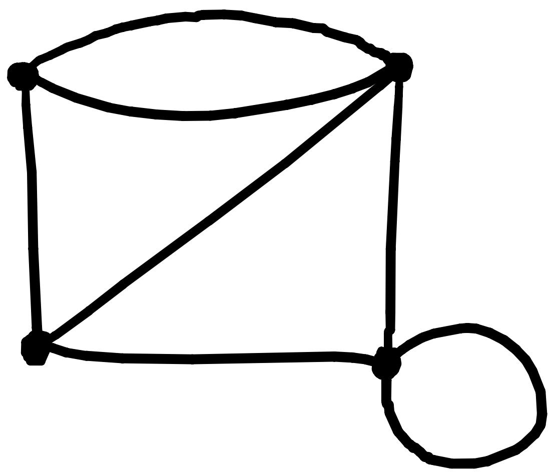
with Mireille BOUSQUET-MÉLOU, LaBRI (Bordeaux)

arXiv : 1306.4536

PLANAR MAPS : DEFINITION

Planar map = connected graph

+ embedding of this graph in the plane, considered up to continuous deformation.



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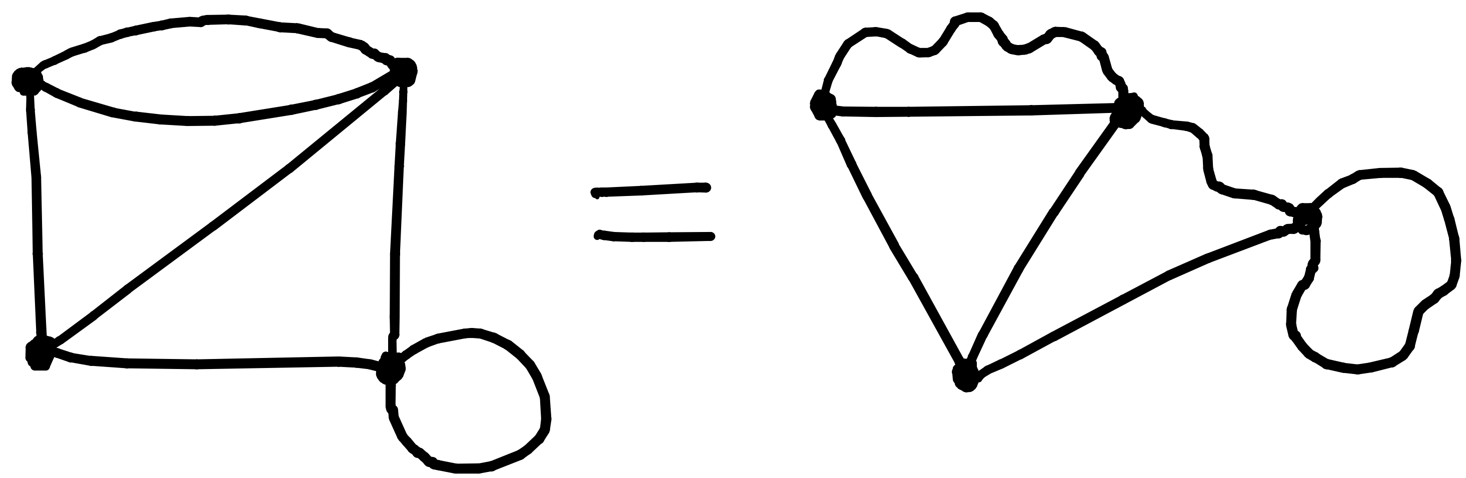

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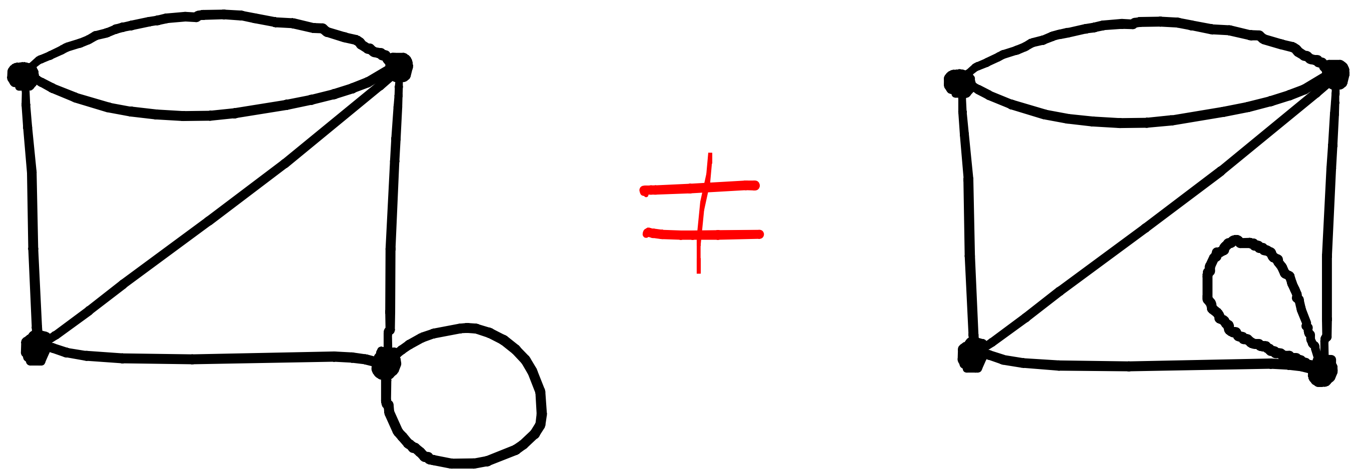

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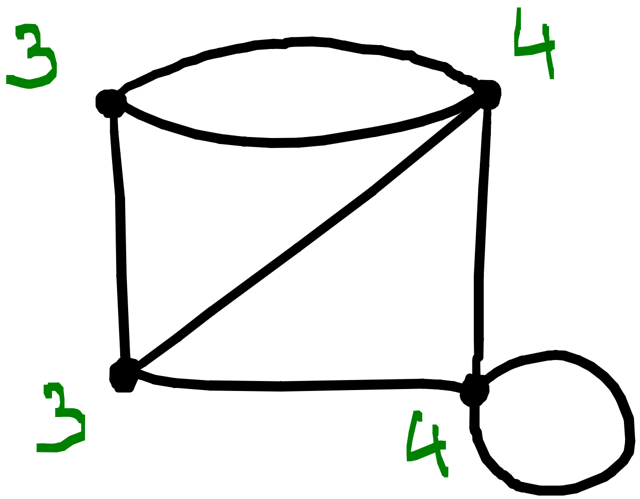

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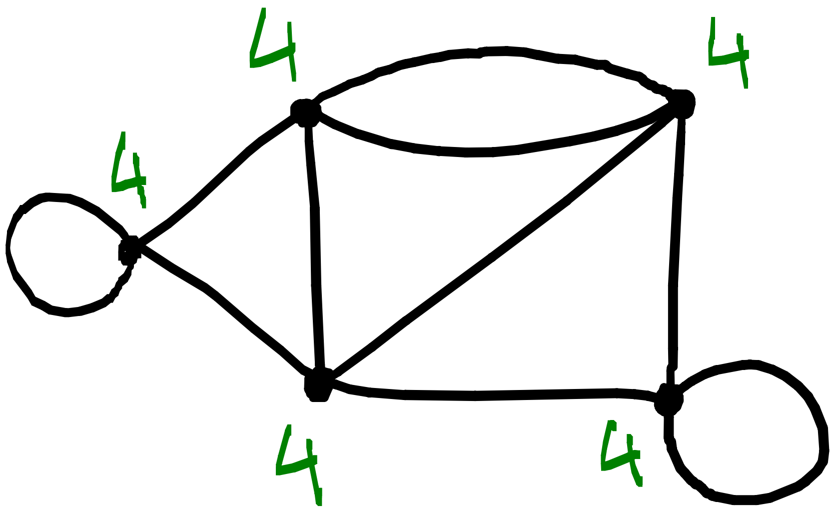

Regular map = All vertices have the same degree.
(In this talk : 4)

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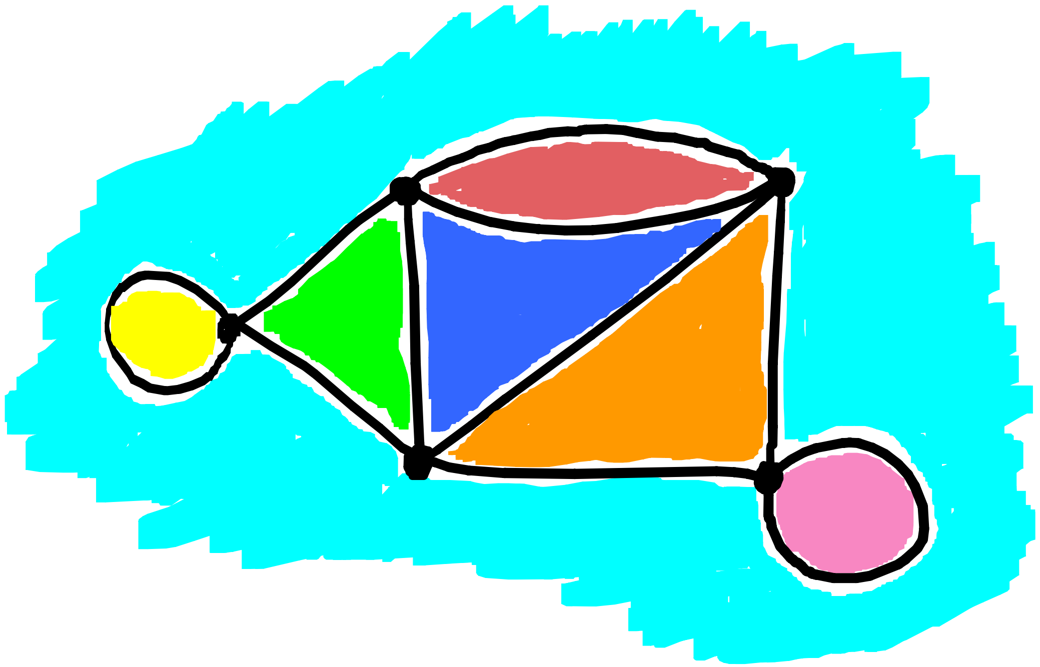

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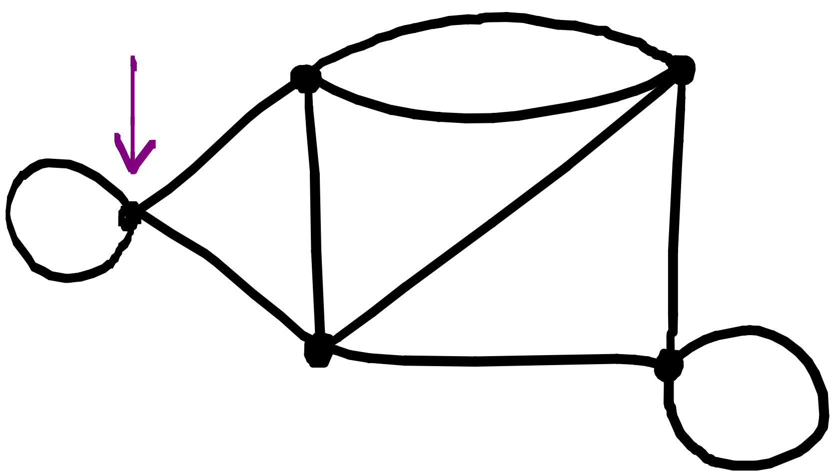

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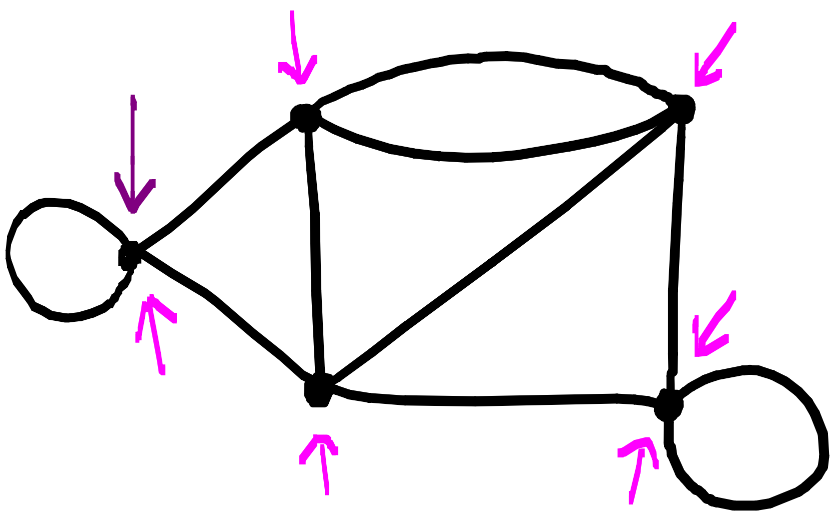

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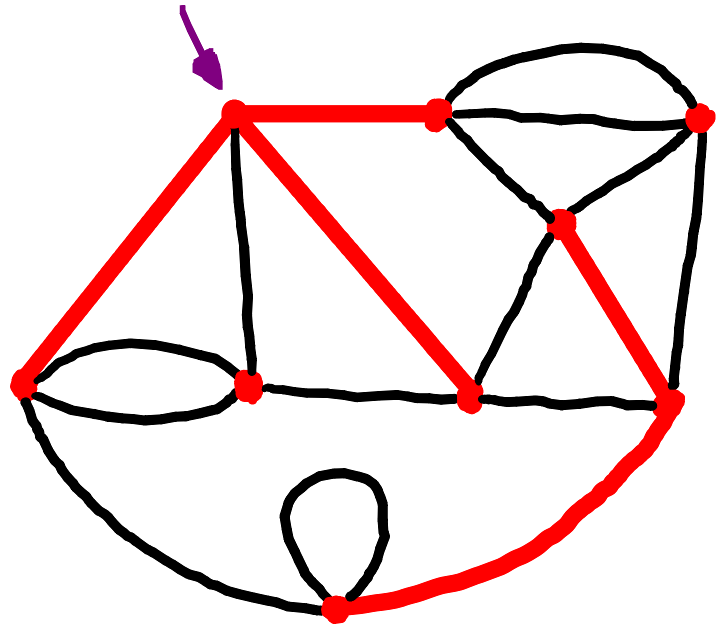
FORESTED MAPS : DEFINITION

Spanning forest of M =
graph F such that:
- $V(F) = V(M)$
- $E(F) \subseteq E(M)$ has no cycle.



Forested map (M,F) = Rooted map M with a spanning forest F.

Some other structures: Spanning trees, colourings, percolation, Ising/Potts model, self-avoiding walks... [Tutte, Mullin, Kazakov, Borot, Bouttier, Guitter, Sportiello, Eynard, Duplantier, Bousquet-Mélou, Schaeffer, Bernardi, Angel ...]

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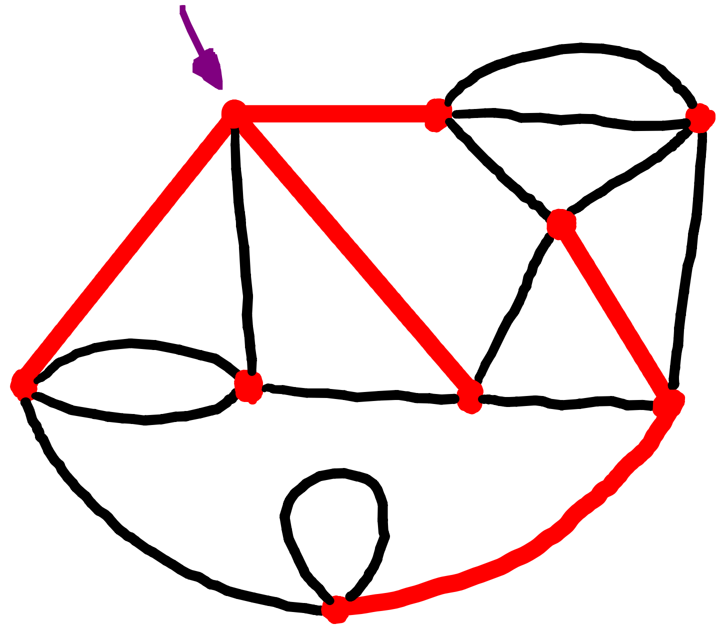

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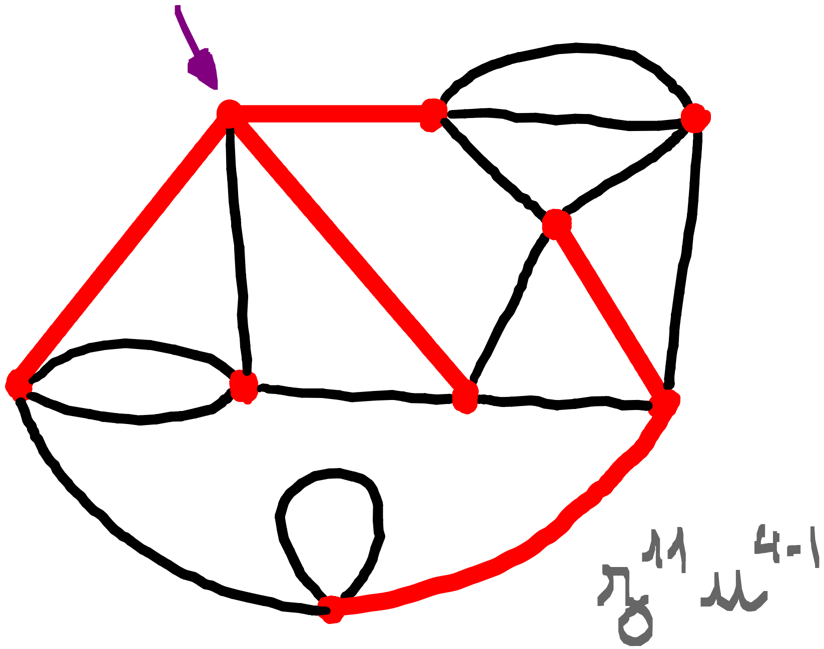

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SPECIAL VALUES OF u

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* $u = 1$: spanning forests

* $u = 0$: spanning trees [Mullin, 1967]

* $u = -1$: root-connected acyclic orientations on (dual) quadrangulations. [Las Vergnas, 1984]

Generic Values of μ

1) Connected subgraphs on quadrangulations (counted by cycles)

2) Tutte polynomial $T_M(\mu+1, 1)$

3) Sandpile model [Merino Lopez, Cori, Le Borgne]

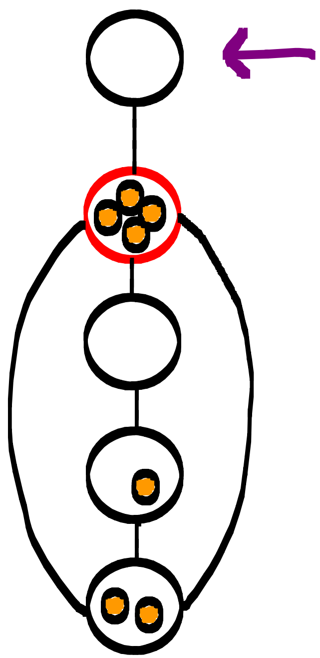
4) Limit $q \to 0$ of the Potts model.

Natural domain $\mu \in [-1, +\infty)$

---

3)

$$F(z,\mu) = \sum_{\substack{\text{quadrangulation} \\ \text{with recurrent} \\ \text{configuration } C}} z^{\#\text{vertices}} (\mu+1)^{\text{level}(C)}$$



GENERIC VALUES OF u

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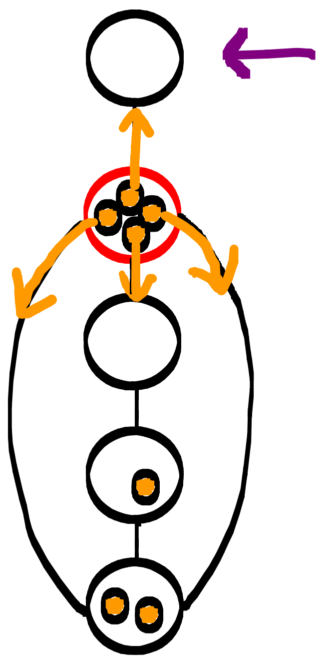

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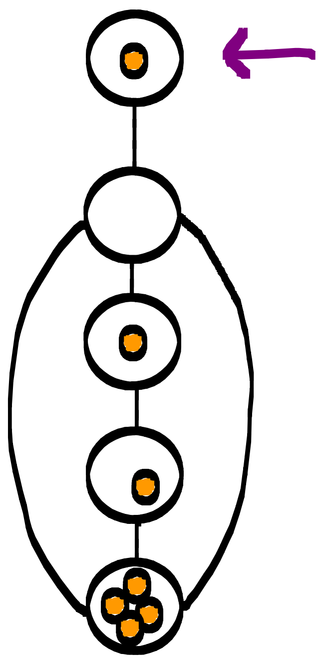

OBJECTIVES

EXACT ENUMERATION

→ Generating function F of forested maps.

→ Nature of F:

D-finite ? D-algebraic ?

ASYMPTOTIC ENUMERATION

→ For $u \geq -1$, asymptotic behaviour of $f_n(u) = [z^n] F(z,u)$

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EXACT ENUMERATION

→ Generating function F of forested maps.

→ Nature of F:

D-finite? i.e satisfies a linear differential equation.

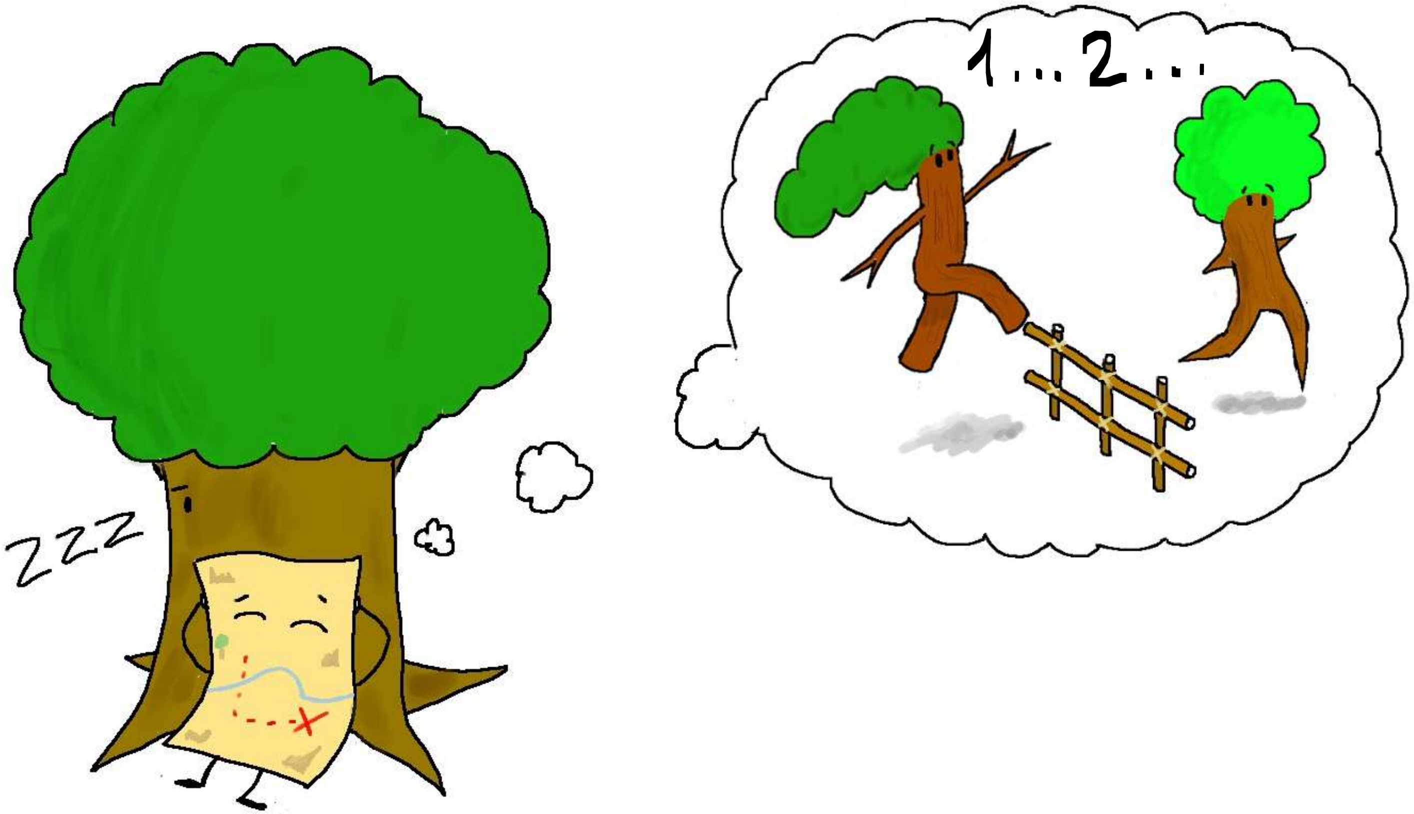
D-algebraic? i.e satisfies a polynomial differential equation.

(non-trivial equations with coefficients in $\mathbb{Q}(z,u)$)

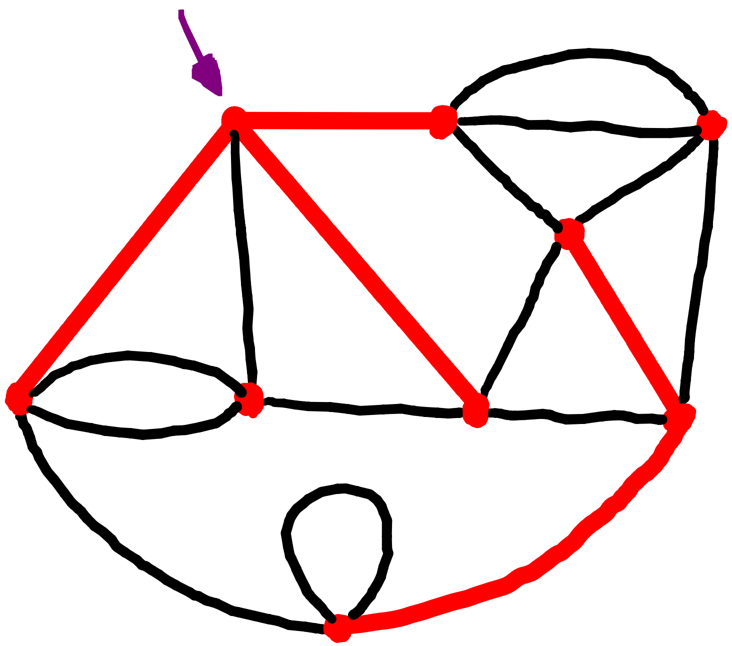
ASYMPTOTIC ENUMERATION

→ For $u \geq -1$, asymptotic behaviour of $f_n(u) = [z^n]F(z,u)$

EXACT ENUMERATION



From Forested To General Maps



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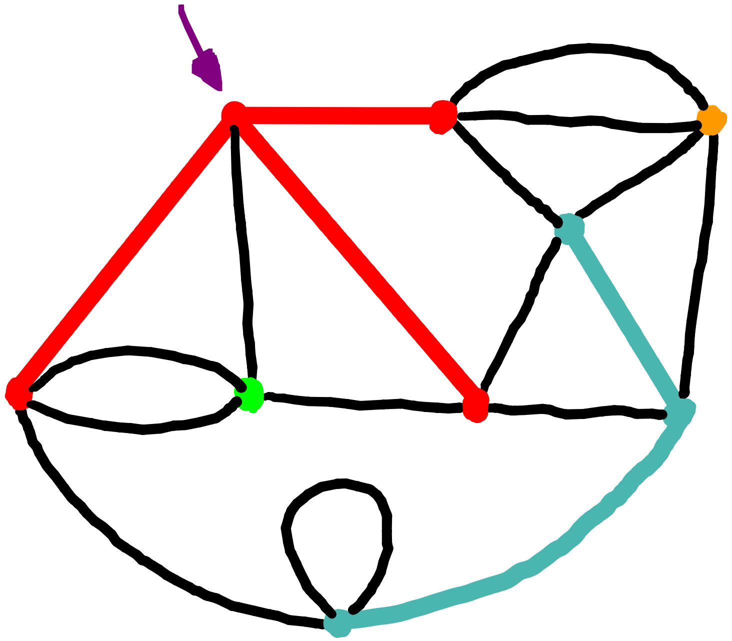

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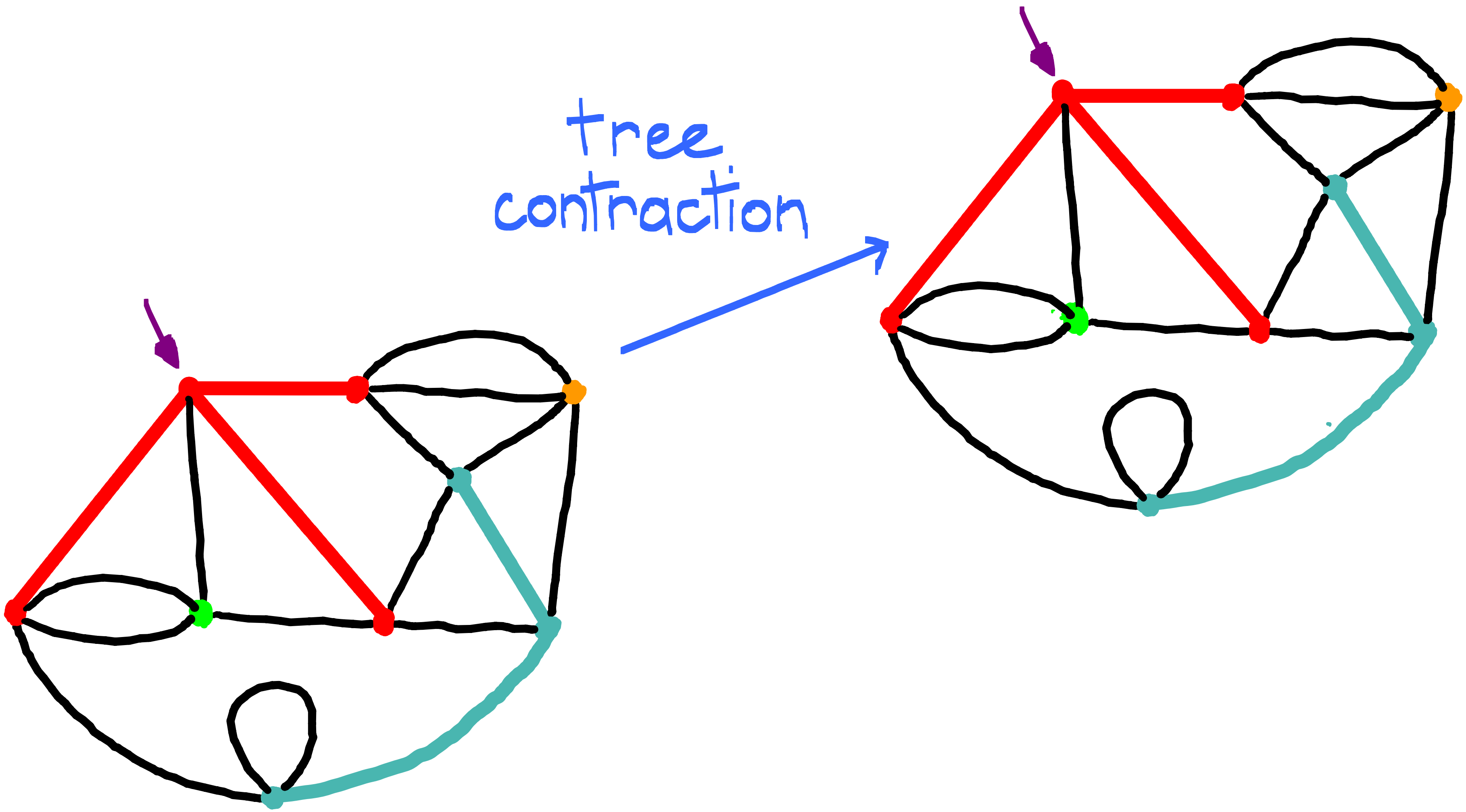

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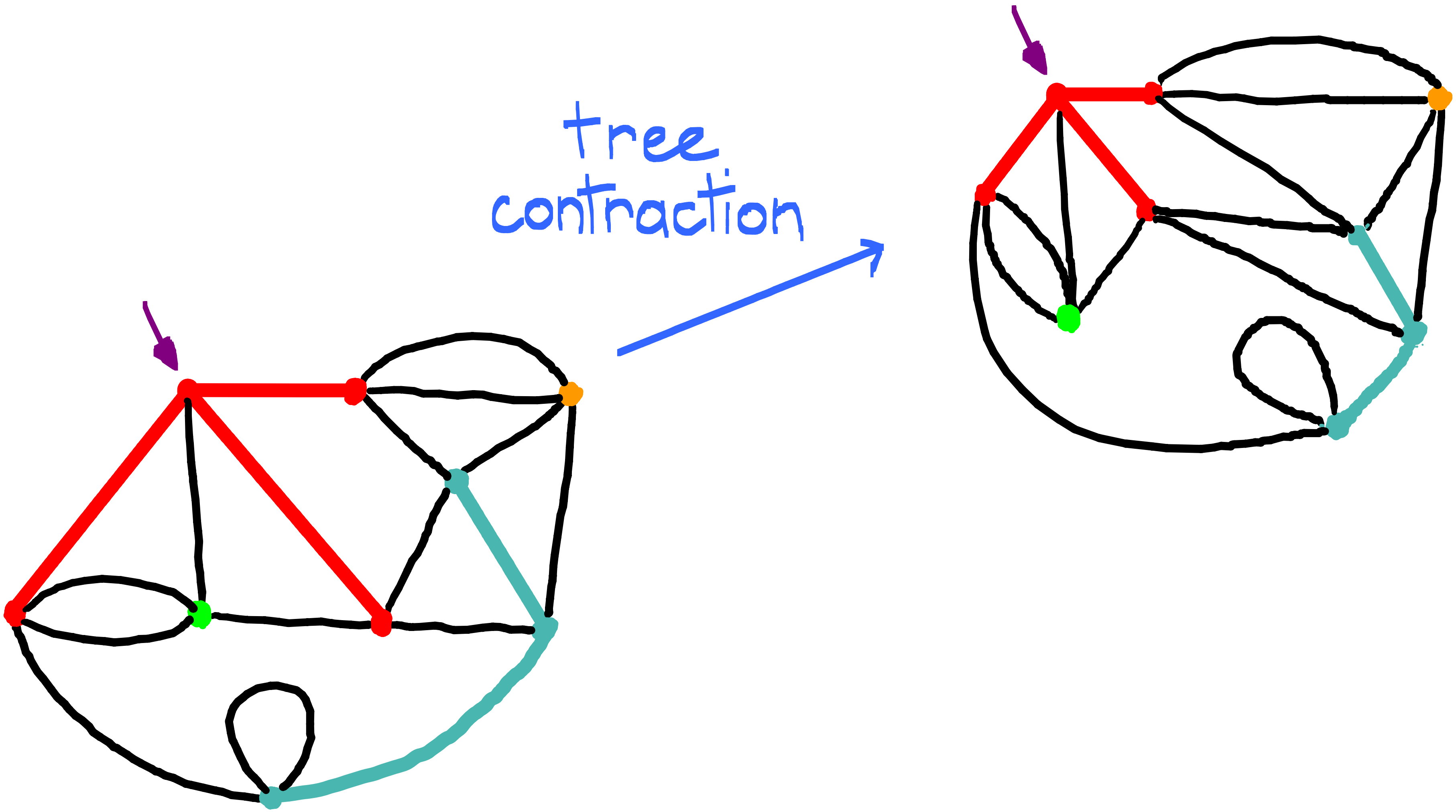

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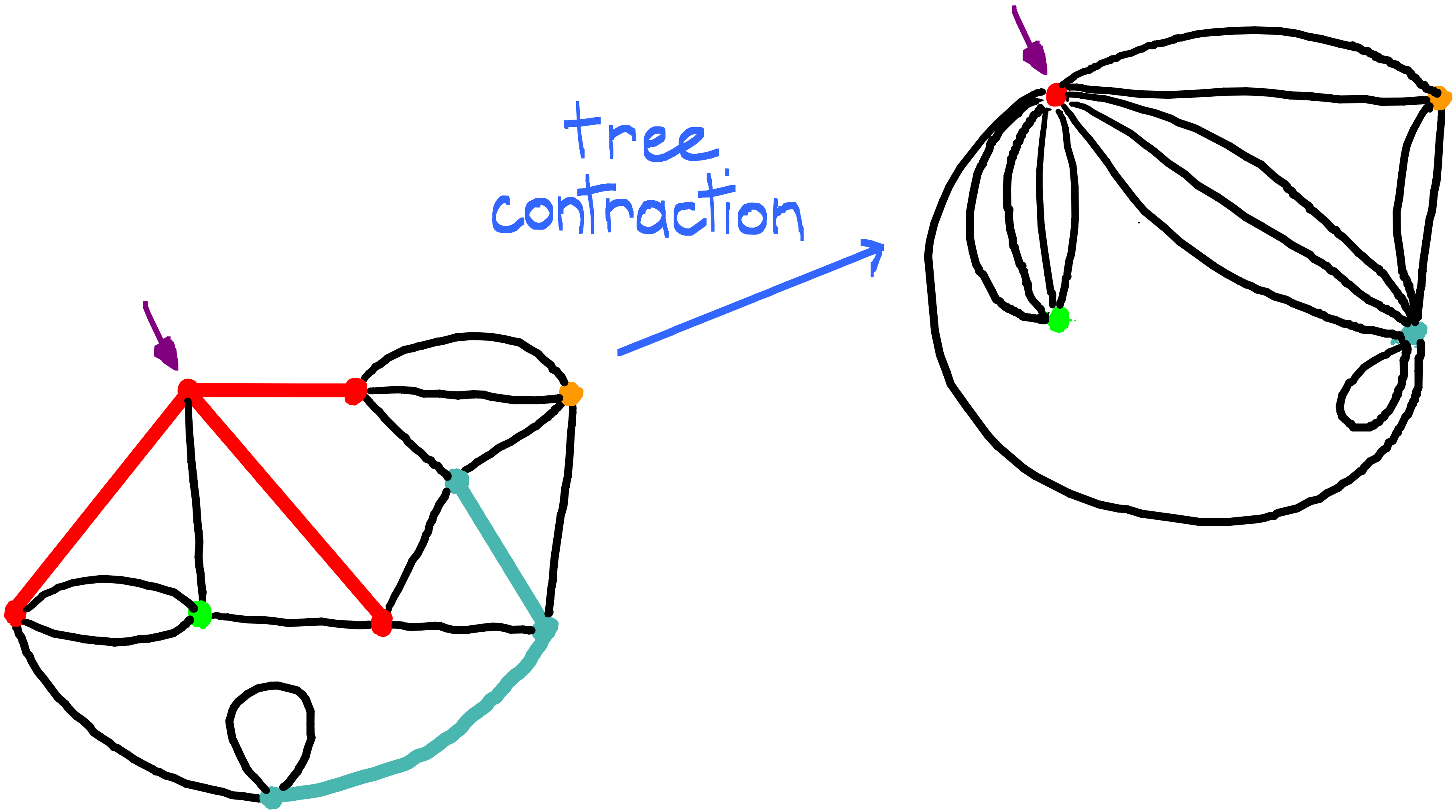

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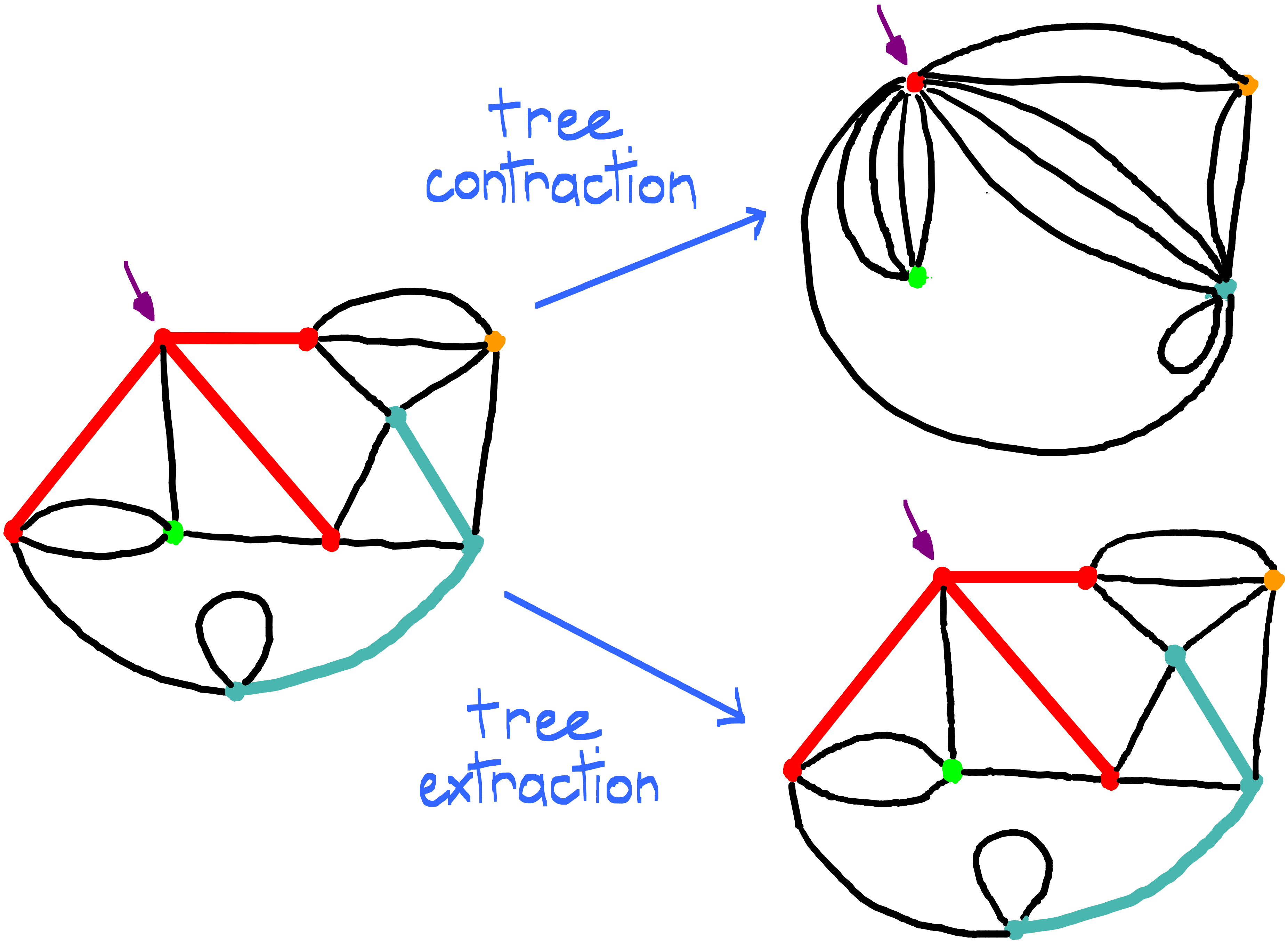

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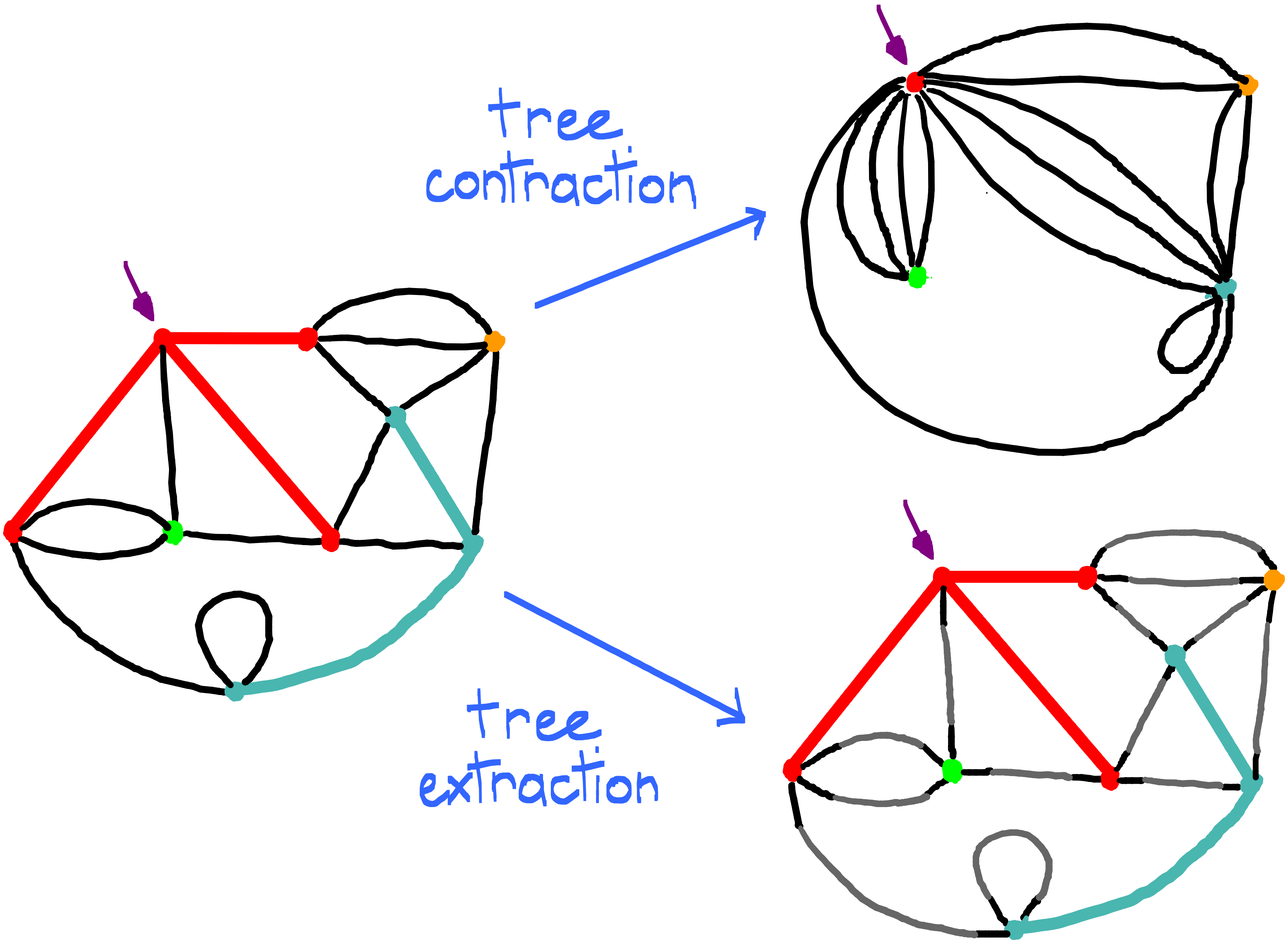

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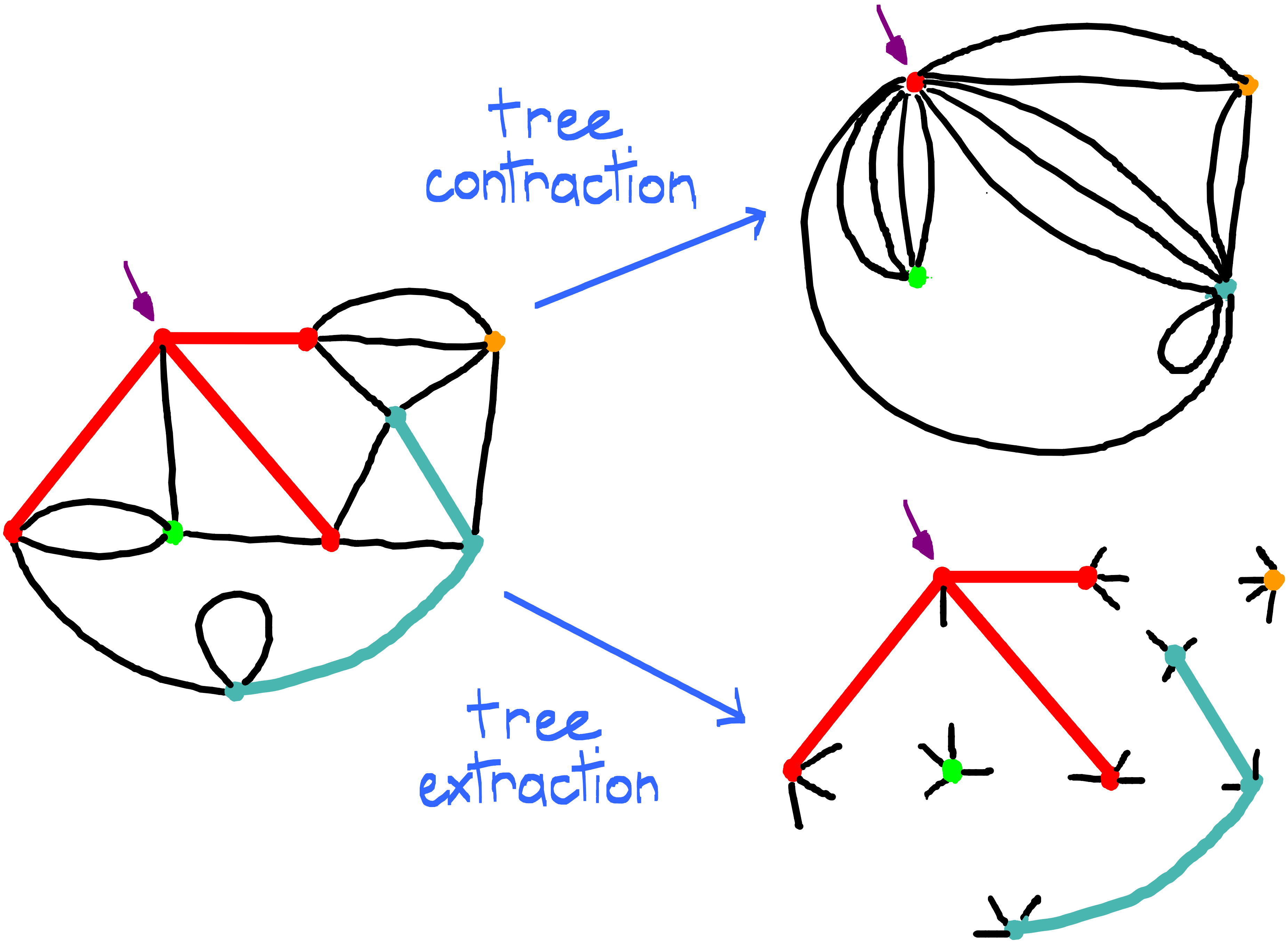

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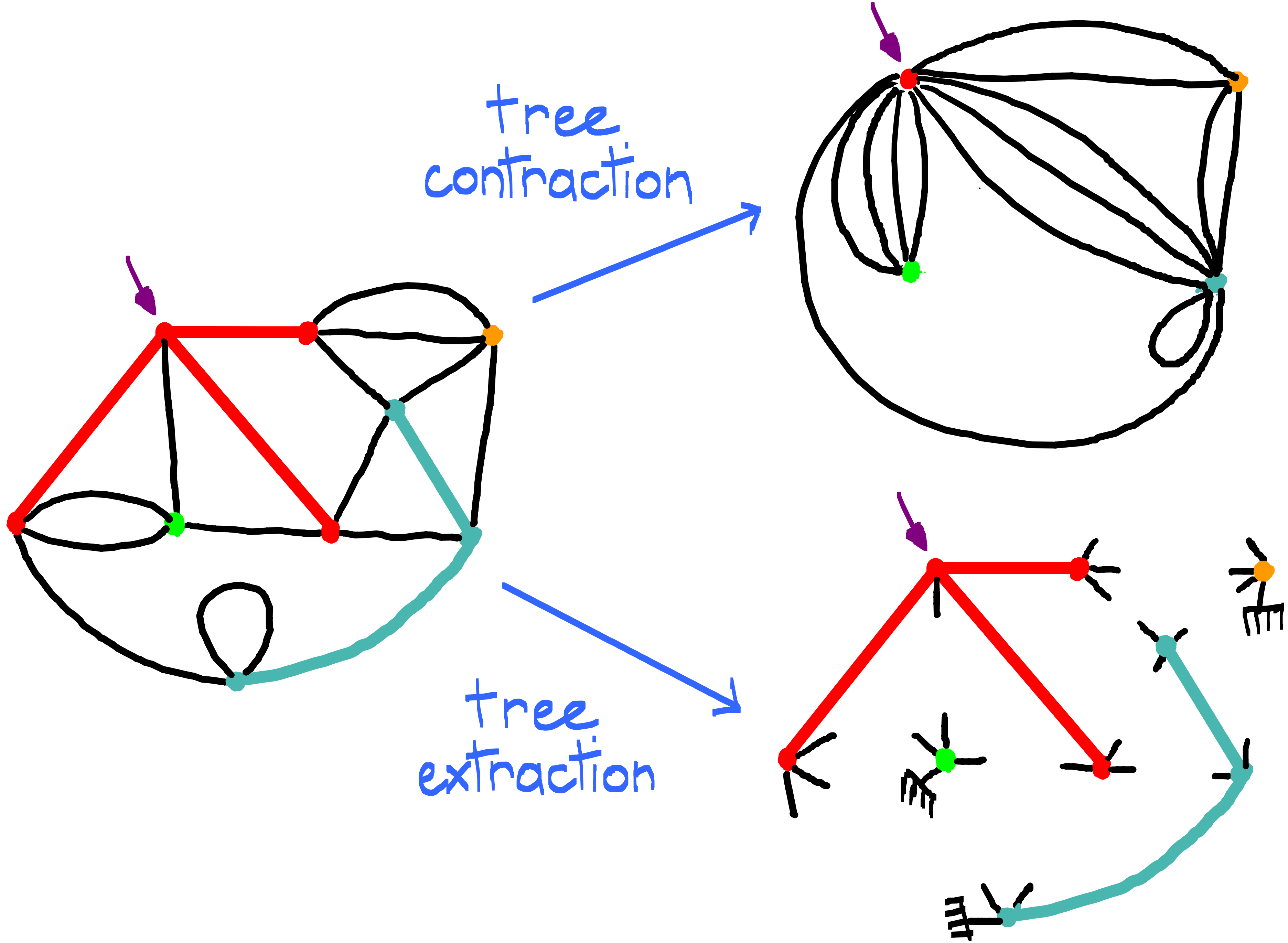
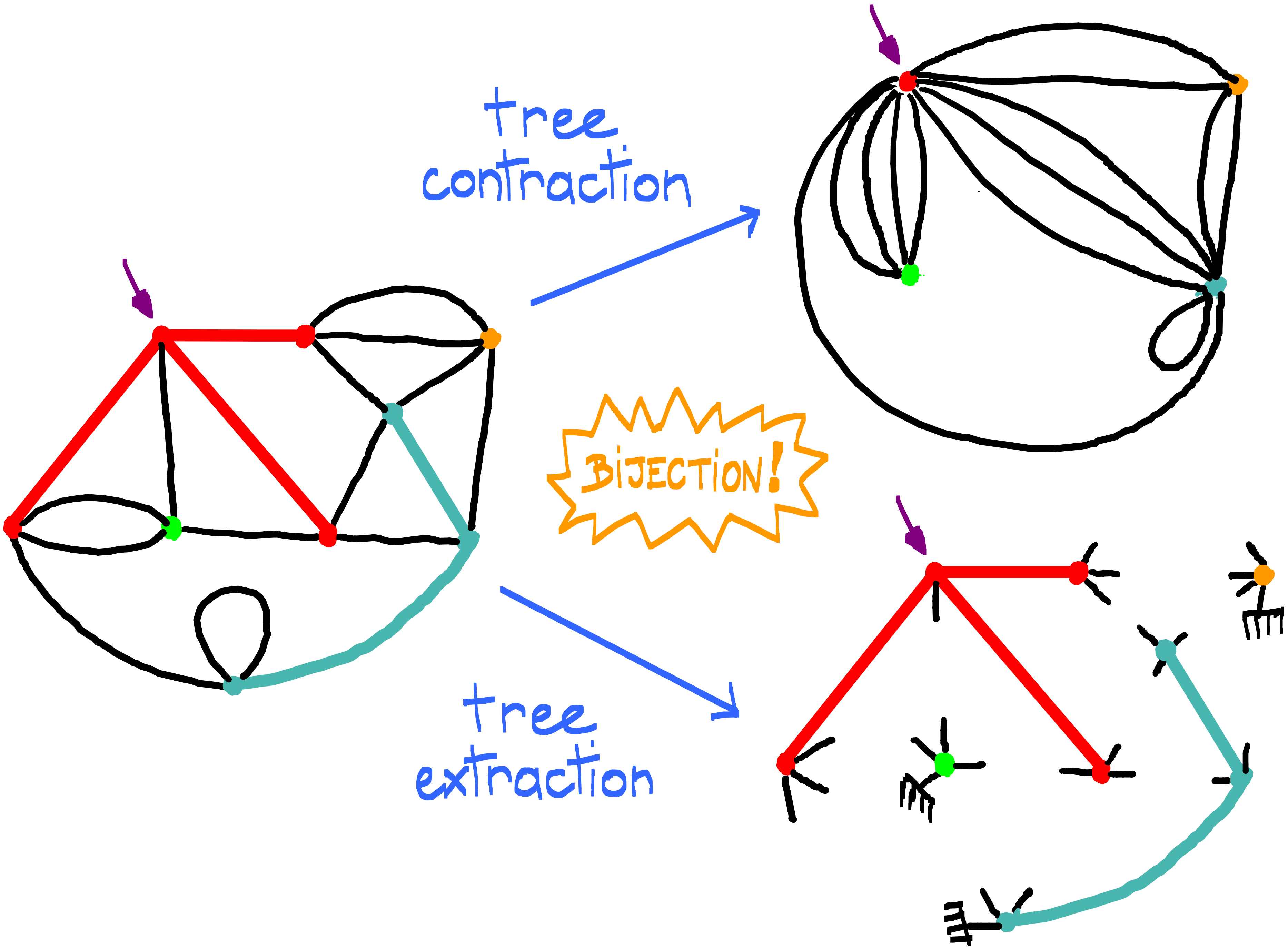


FROM FORESTED TO GENERAL MAPS



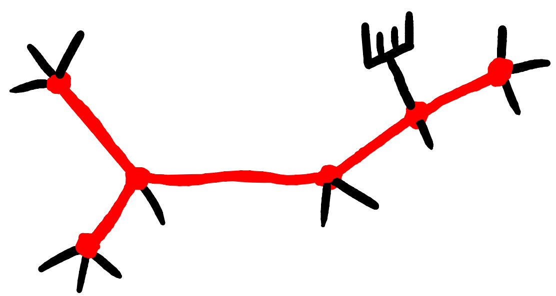
TRANSLATION INTO GENERATING FUNCTIONS

$M(z, u; g_1, g_2, g_3, \ldots; h_1, h_2, h_3, \ldots) =$
Generating function of rooted maps with a weight:

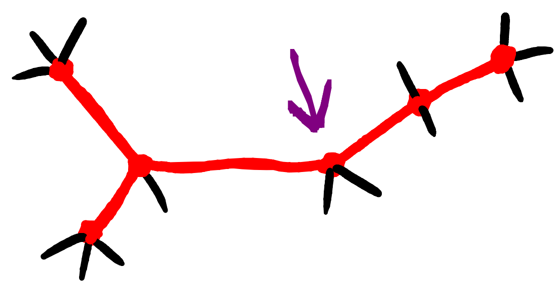
- z per face,
- $u\, g_k$ per non-root vertex of degree k,
- h_k if the root vertex has degree k.

$$\boxed{F(z, u) = M(z, u; t_1, t_2, t_3, \ldots; t_1^c, t_2^c, t_3^c, \ldots)}$$

$t_k = \#$ 4-valent leaf-rooted trees with k leaves



$t_k^c = \#$ 4-valent corner-rooted trees with k leaves



GENERATING FUNCTION FOR GENERAL MAPS

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- z per face,
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This generating function is known.
cf [Bouttier-Guitter, 2012]

(M' is even nicer.)

Notation: $X' = \frac{\partial X}{\partial z}$

THE GENERATING FUNCTION OF FORESTED MAPS

Theorem

There exists a unique series R in z with constant term 0 and coefficients in $\mathbb{Q}[u]$ such that

$$R = z + u \sum_{i \geq 2} \frac{(3i-3)!}{(i-1)!^2 \, i!} R^i$$

Then :

$$F' = 4 \sum_{i \geq 2} \frac{(3i-2)!}{(i-2)! \; i!^2} R^i$$

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For $u=0$, [Mullin]

$$R = z \quad \text{and} \quad F' = 4 \sum_{i \geq 2} \frac{(3i-2)!}{(i-2)!\, i!^2} z^i \quad \text{D-finite.}$$

THE GENERATING FUNCTION OF FORESTED MAPS

Theorem

There exists a unique series R in z with constant term 0 and coefficients in $\mathbb{Q}[u]$ such that

$$R = z + u \quad \phi(R)$$

Then:

$$F' = \quad \Theta(R)$$

where

$$\phi(x) = \sum_{i \geq 2} \frac{(3i-3)!}{(i-1)!^2\, i!} x^i, \qquad \Theta(x) = 4 \sum_{i \geq 2} \frac{(3i-2)!}{(i-2)!\; i!^2} x^i.$$

A DIFFERENTIAL EQUATION FOR F

$$R = z + u\,\phi(R) \qquad\qquad F' = \Theta(R)$$

Prop F is D-algebraic.

(Fundamental reason: ϕ and Θ are D-finite.)

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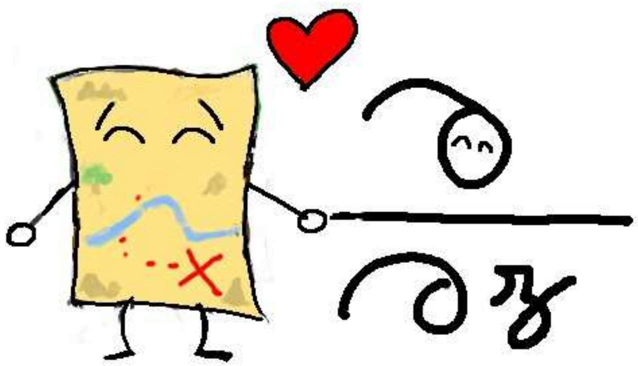
YES!



A DIFFERENTIAL EQUATION FOR F

$$
\begin{aligned}
& 9F'^2F''^5u^6 + 36F'^2F''^3F'''u^5z + 144F'^2F''^4u^5 - 12(21z-1)F'F''^5u^5 + 432F'^2F''^2F'''u^4z \\
& - 48(24z-1)F'F''^3F'''u^4z + 864F'^2F''^3u^4 - 96(27z-2)F'F''^4u^4 + 4(27z-1)(15z-1)F''^5u^4 \\
& + 1728F'^2F''F'''u^3z - 288(21z-2)F'F''^2F'''u^3z + 10368F'F'''^2u^2z^3 + 16(27z-1)(21z-1)F''^3F'''u^3z \\
& + 2304F'^2F''^2u^3 - 288(31z-4)F'F''^3u^3 - 64(6uz-162z^2+33z-1)F''^4u^3 + 2304F'^2F'''u^2z \\
& - 2304(6z-1)F'F''F'''u^2z - 192(8uz-54z^2+29z-1)F''^2F'''u^2z - 768(2u+189z-7)F'''^2uz^3 \\
& + 2304F'^2F''u^2 - 3072(3z-1)F'F''^2u^2 - 192(24uz-27z^2+55z-2)F''^3u^2 - 1536(21z-2)F'F'''uz \\
& - 768(12uz+81z^2+24z-1)F''F'''uz + 1536(9z+2)F'F''u - 512(39uz+81z^2+51z-2)F''^2u \\
& + 36864F'z - 1024(12uz-162z^2+33z-1)F'''z - 1024(36uz+27z-1)F'' - 24576z = 0.
\end{aligned}
$$

Differential equation of order 2 in F' and degree 7.

(but not in F)

A DIFFERENTIAL EQUATION FOR F

$$R = z + u\,\phi(R) \qquad F' = \Theta(R)$$

Prop F is D-algebraic.

(Fundamental reason: ϕ and Θ are D-finite.)

- A differential equation for F can be explicitly computed.
- cf Bernardi - Bousquet-Mélou's result:

The Potts generating function of planar maps is D-algebraic.

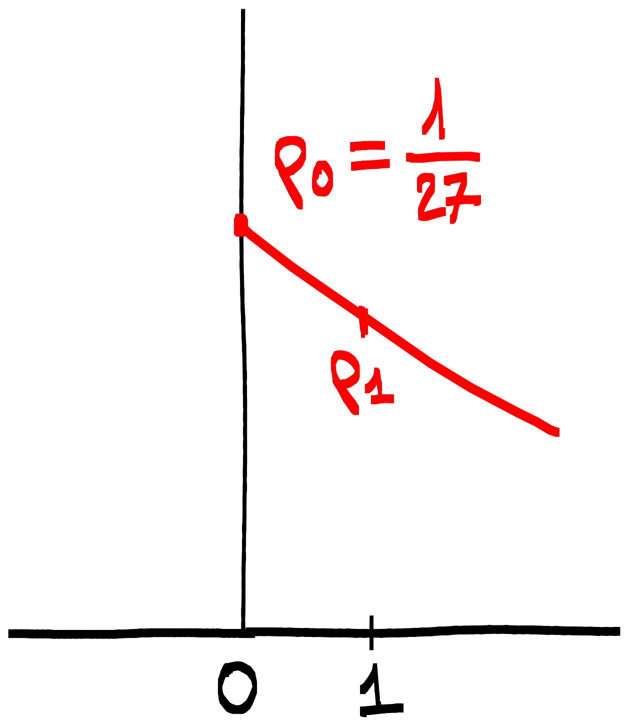
(established in a more painful way.)

Asymptotic Results

Radius of Convergence

Fix u,

ρ_u = radius of convergence of $F(z,u) = \sum_n f_n(u)\, z^n$.



$$\begin{cases} \rho_u = \tau_u - u\,\phi(\tau_u) \\ \phi'(\tau_u) = \frac{1}{u} \end{cases}$$

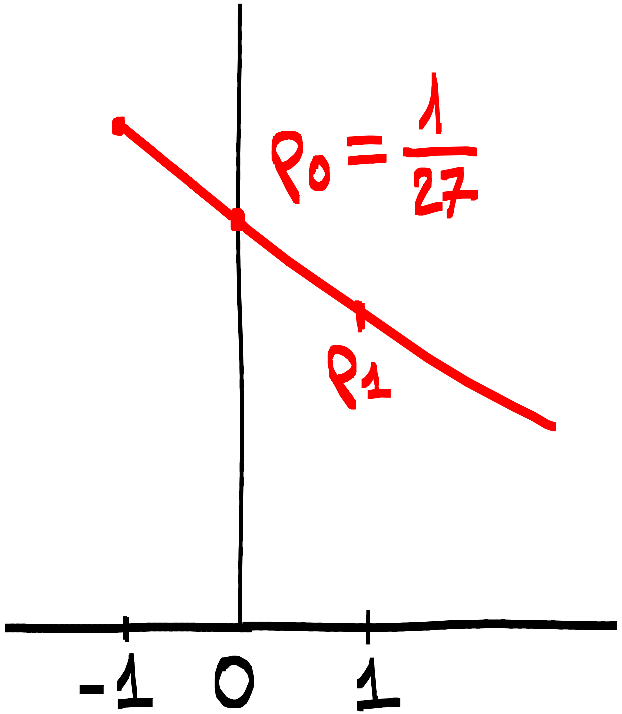
$(u > 0)$

Radius of Convergence

Fix μ in $[-1,+\infty)$,

ρ_μ = radius of convergence of $F(z,\mu)=\sum_n f_n(\mu)\, z^n$.

ρ_μ is affine on $[-1,0]$!



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RADIUS OF CONVERGENCE

Fix u in $[-1,+\infty)$,

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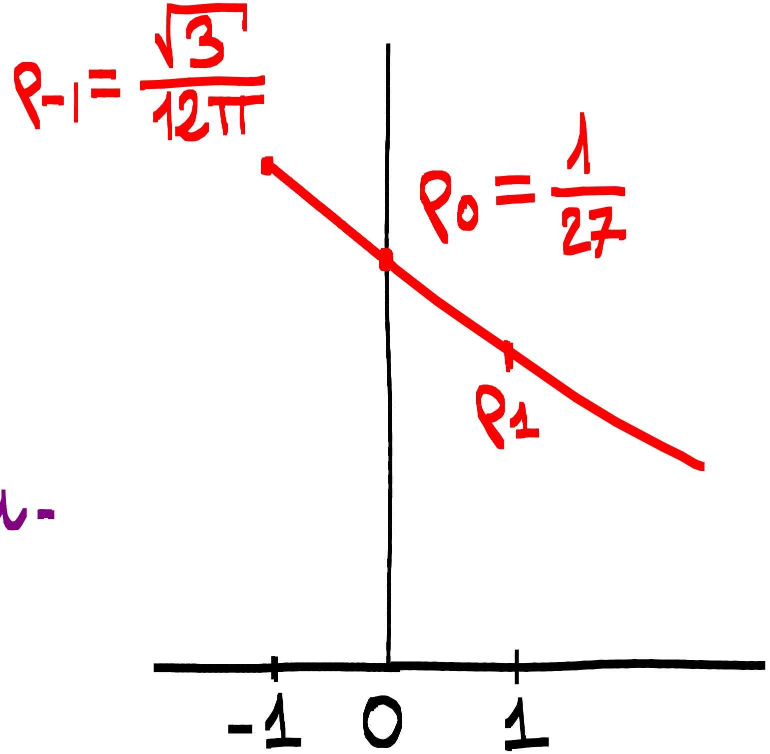
$$\rho_u = \frac{1}{27}(1+u) - \frac{\sqrt{3}}{12\pi}u.$$



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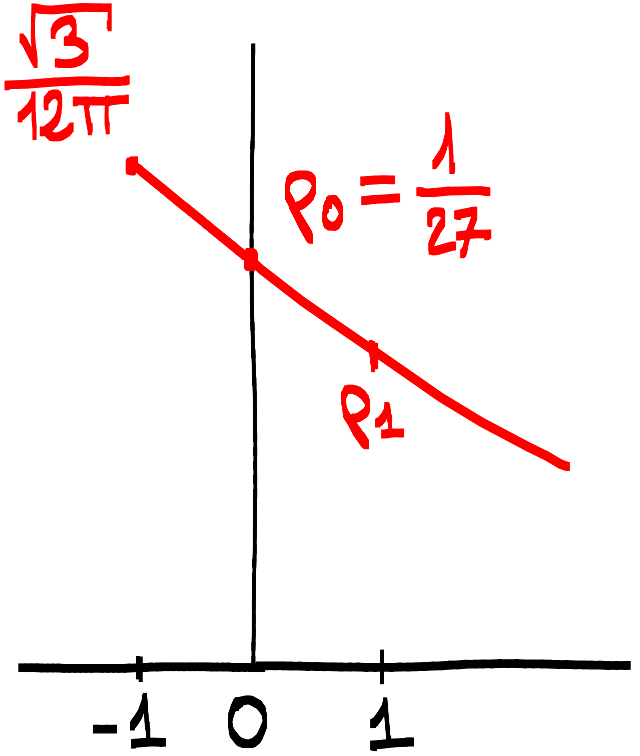

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$(u>0)$

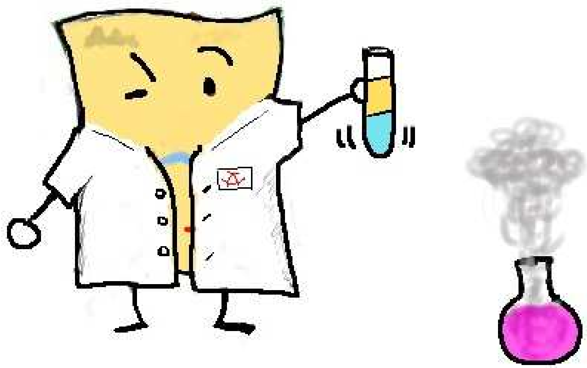
Cor ρ_{-1} is transcendental:

$F(z,-1)$ is not D-finite.

Phase Transition At 0

$f_n(u) = [z^n]\, F(z,u)$

$-1 \leqslant u < 0$	$u = 0$	$0 < u$
$f_n(u) \sim \dfrac{c_u\, \rho_u^{-n}}{n^3 \ln^2 n}$	$f_n(u) \sim \dfrac{c_u\, \rho_u^{-n}}{n^3}$	$f_n(u) \sim \dfrac{c_u\, \rho_u^{-n}}{n^{5/2}}$
New "Universality class" for maps	maps with a spanning tree	standard



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D-finite

Cor

For $u \in [-1,0)$, $F(z,u)$ is not D-finite.

IDEA OF THE PROOF

<u>Singularity analysis</u> [Flajolet - Odlyzko]

A link between the singular behaviour of $F(z, u)$ near ρ_u and the asymptotic behaviour of $f_n(u)$.

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$$\begin{cases} R = z + u\,\phi(R) \\ R(0) = 0 \end{cases}$$

radius of convergence of $\phi = \frac{1}{27}$

Idea of the Proof

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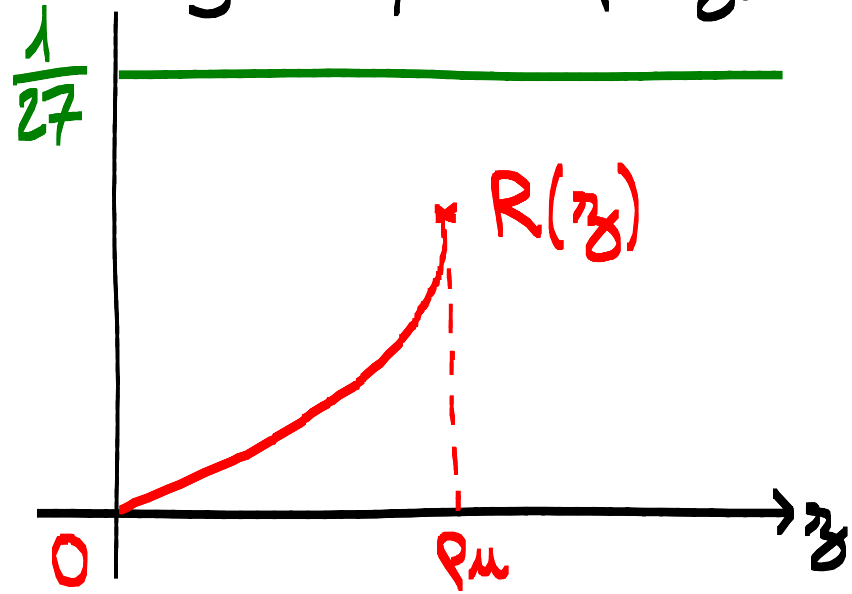
radius of convergence of $\phi = \frac{1}{27}$

$\boxed{u > 0}$

singularity in $(\rho_u - z)^{\frac{1}{2}}$



IDEA OF THE PROOF

Singularity analysis [Flajolet - Odlyzko]

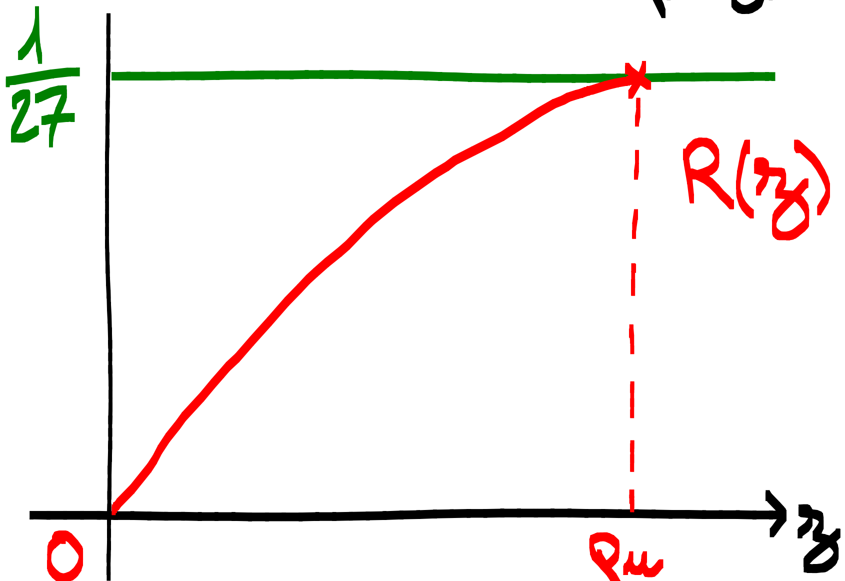
A link between the singular behaviour of $F(z,u)$ near ρ_u and the asymptotic behaviour of $f_n(u)$.

$$\begin{cases} R = z + u\,\phi(R) \\ R(0) = 0 \end{cases}$$

radius of convergence of $\phi = \frac{1}{27}$

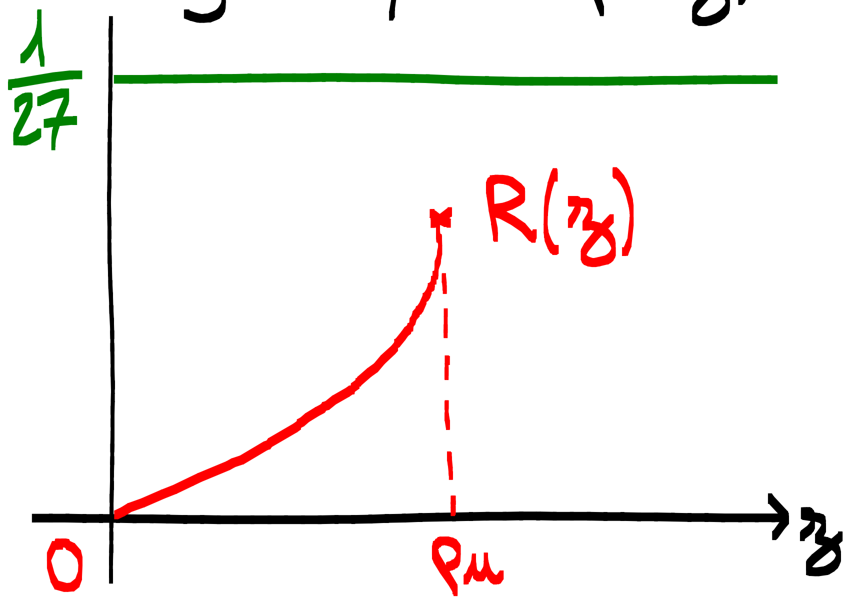
$\boxed{u < 0}$

singularity in $\frac{\rho_u - z}{\ln(\rho_u - z)}$



$\boxed{u > 0}$

singularity in $(\rho_u - z)^{\frac{1}{2}}$



IDEA OF THE PROOF

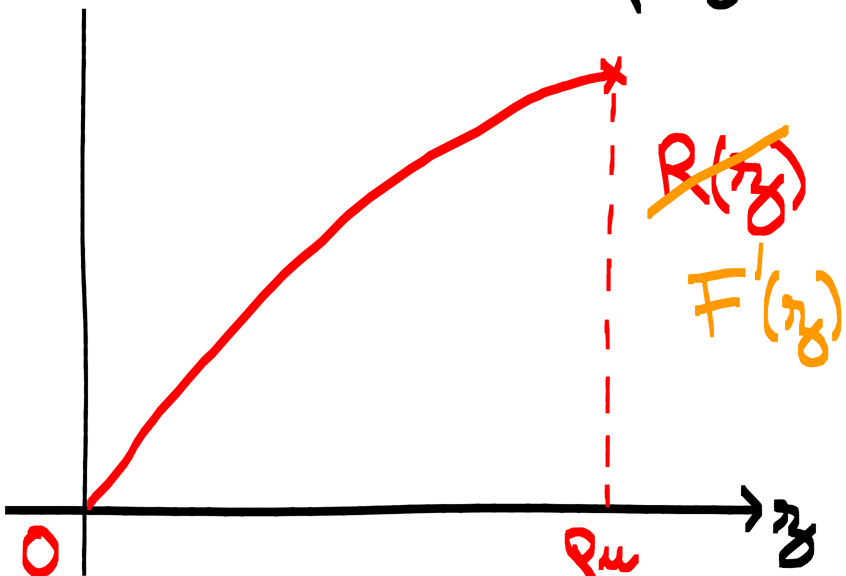
Singularity analysis [Flajolet - Odlyzko]

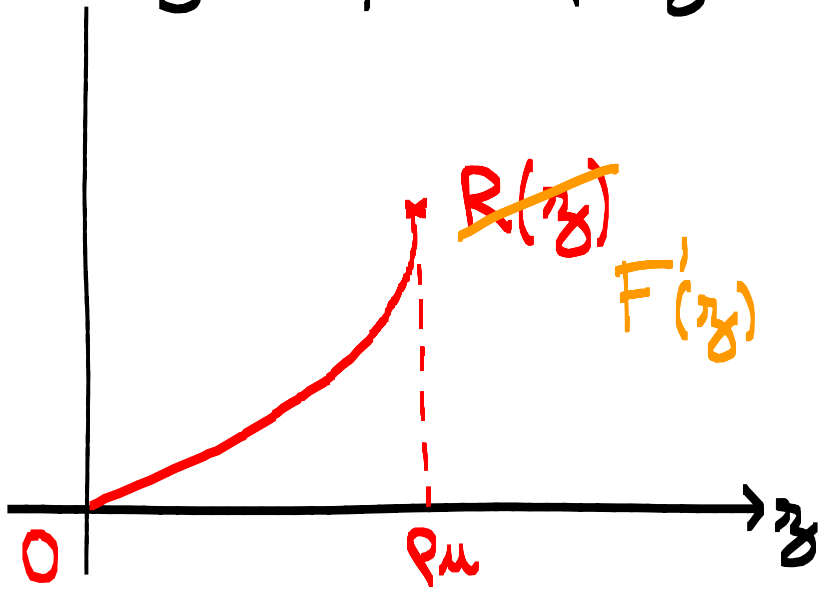
A link between the singular behaviour of $F(z, \mu)$ near ρ_μ and the asymptotic behaviour of $f_n(\mu)$.

$$\begin{cases} R = z + \mu\,\phi(R) \\ R(0) = 0 \end{cases}$$

radius of convergence of $\phi = \frac{1}{27}$

$\boxed{\mu < 0}$	$\boxed{\mu > 0}$
singularity in $\dfrac{\rho_\mu - z}{\ln(\rho_\mu - z)}$	singularity in $(\rho_\mu - z)^{\frac{1}{2}}$





More Results

→ Exact enumeration of p-valent forested maps, $p > 3$.

→ Asymptotic behaviour of p-valent forested maps, $p = 3$, p even.
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→ Random forested maps:
- size of the root component
- number of components

FUTURE WORK

- Enumeration of all (non-regular) forested maps.
- Random forested maps:
  - size of the largest component.
    (Conjecture: mean $\sim C \ln n$)
  - Random sampling.


THANK YOU!