Planar Maps and Spanning Forests

Julien COURTIEL (SFU/PIMS)
DM seminar, Nov. 4th
BASIC NOTIONS
Planar map = connected graph + embedding of this graph in the plane, considered up to continuous deformation.
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+ embedding of this graph in the plane, considered up to continuous deformation.

faces
Planar map = connected graph + embedding of this graph in the plane, considered up to continuous deformation.

We root every planar map at an outer corner.
Planar map = connected graph + embedding of this graph in the plane, considered up to continuous deformation.

We root every planar map at an outer corner.
LARGE MAPS
LARGE MAPS
APPLICATION AREAS

- statistical physics
- probability theory (matrix integrals, random continuous objects...)
- algorithmic geometry
- permutation factorizations
- every area that involves some surface...
ENUMERATION OF 4-VALENT MAPS

3 ♦ + 2 ♦ = ?
Enumeration of 4-valent maps

4-valent map = map where every vertex has degree 4.
**Enumeration of 4-valent Maps**

$4$-valent map = map where every vertex has degree $4$. 

![Diagram of a 4-valent map with vertex degrees labeled]
**Enumeration of 4-valent maps**

4-valent map = map where every vertex has degree 4.

\[ q_n = \text{number of 4-valent maps with } (n+2) \text{ faces} \]

\[ = 2 \times \frac{3^n}{(n+1)(n+2)} \binom{2n}{n} \]
**Enumeration of 4-valent Maps**

4-valent map = map where every vertex has degree 4.

\[ q_1 = 2 \]

\[ q_n = \text{number of 4-valent maps with } (n+2) \text{ faces} \]

\[ = 2 \frac{3^n}{(n+1)(n+2)} \binom{2n}{n} \]
ASYMPTOTIC BEHAVIOUR

\[ q_n = 2 \frac{3^n}{(n+1)(n+2)} \binom{2n}{n} \]

Stirling formula:

\[ q_n \sim \frac{2}{\sqrt{\pi}} 12^n n^{-5/2} \]
ASYMPTOTIC BEHAVIOUR

\[ q_n = 2 \frac{3^n}{(n+1)(n+2)} \binom{2n}{n} \]

Stirling formula:

\[ q_n \sim \frac{2}{\sqrt{\pi n}} 12^n n^{-\frac{5}{2}} \]

Typical for planar maps
GENERATING FUNCTION

Generating function of the 4-valent maps:

\[ Q(x) = \sum_{n \geq 1} q_n x^n \]

Nature of the generating function?

\[ Q = \frac{P_1}{P_2}, \quad \exists \text{ polynomial that annihilates } Q \text{ satisfies a linear } \text{DE} \phantom{and polynomial DE} \]
Nature of the Generating Function

Generating function of the 4-valent maps:

\[ Q(z) = \sum_{n \geq 1} q_n z^n \]

Nature of the generating function?

\[ Q = \frac{P_1}{P_2} \]

\( \exists \) polynomial that annihilates \( Q \) that satisfies a linear DE satisfies a polynomial DE

\[ Q = T - 8T^3 \quad T = 1 + 3g_8 T^2 \]
NATURE OF THE GENERATING FUNCTION

Generating function of the 4-valent maps:

\[ Q(z) = \sum_{n \geq 1} q_n z^n \]

Nature of the generating function?

rational $\rightarrow$ algebraic $\rightarrow$ D-finite $\rightarrow$ D-algebraic

\[ Q = \frac{P_1}{P_2} \]

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NATURE OF THE GENERATING FUNCTION

Generating function of the 4-valent maps:

\[ Q(z) = \sum_{n \geq 1} q_n z^n \]

Nature of the generating function?

- rational → algebraic → D-finite → D-algebraic
- \[ Q = \frac{P_1}{P_2} \]
  - \exists \text{ polynomial that annihilates } Q
  - satisfies a linear DE
  - satisfies a polynomial DE

Form of the asymptotics

\[ Q = T - g_3 T^3 \]
\[ T = 1 + 3 g_3 T^2 \]
THE POTTS MODEL
FORESTED MAPS

with Mireille BOUSQUET-MÉLOU (Bordeaux)
Forest Maps & Definition

Spanning forest of $M$ = graph $F$ such that:
- $V(F) = V(M)$
- $E(F) \leq E(M)$ has no cycle.

Forested map $(M,F) = \text{Rooted map } M \text{ with a spanning forest } F$.

Some other structures: Spanning trees, colourings, percolation, Ising/Potts model, self-avoiding walks... [Tutte, Mullin, Kazakov, Borot, Bouttier, Guitter, Sportiello, Eynard, Duplantier, Bousquet-Mélou, Schaeffer, Bernardi, Angel...]

Spanning forest of $M = \text{graph } F$ such that:
- $V(F) = V(M)$
- $E(F) \leq E(M)$ has no cycle.

Forested map $(M,F) = \text{Rooted map } M \text{ with a spanning forest } F.$

$$F(z, w) = \sum_{(M,F) \text{ 4-valent forested map}} z^\# \text{ faces} w^\# \text{ components} - 1$$
**Forest Maps & Definition**

Spanning forest of $M$ = graph $F$ such that:
- $V(F) = V(M)$
- $E(F) \subseteq E(M)$ has no cycle.

Forest map $(M,F) = \text{Rooted map } M \text{ with a spanning forest } F$.

$$F_4(M) = \sum_{(M,F) \text{ 4-valent forested map}} \binom{\# \text{ faces}}{3} \binom{\# \text{ components}}{4} - 1$$
SPECIAL VALUES OF $\mu$

\[ F(z, \mu) = \sum_{(M,F) \text{ 4-valent forested map}} z^g \mu^{\#\text{faces} - \#\text{components} - 1} \]

* $\mu = 1$: spanning forests

* $\mu = 0$: spanning trees \[\text{[Mullin, 1967]}\]

* $\mu = -1$: root-connected acyclic orientations on (dual) quadrangulations. \[\text{[Las Vergnas, 1984]}\]
Generic Values of $\mu$

1) Connected subgraphs on quadrangulations (counted by cycles)

2) Tutte polynomial $T_M(\mu + 1, 1)$

3) Sandpile model $\left[\text{Merino Lopez, Cori, Le Borgne}\right]$ Natural domain $\mu \in [-1, +\infty)$

4) Limit $q \to 0$ of the Potts model.

3)

$$F(g, \mu) = \sum_{\text{quadrangulation with recurrent configuration } C} g^\text{# vertices} (\mu + 1)^{\text{level}(C)}$$
**Generic Values of $\mu$**

1) Connected subgraphs on quadrangulations (counted by cycles)

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$$F(\gamma,\mu) = \sum_{\text{quadrangulation with recurrent configuration } C} \gamma^\# \text{ vertices} \ (\mu+1)^\text{level}(C)$$
**Generic Values of $\mu$**

1) Connected subgraphs on quadrangulations (counted by cycles)

2) Tutte polynomial $T_m(\mu + 1, 1)$

3) Sandpile model \[\text{Merino Lopez, } \text{Cori, Le Borgne}\] \(\mu \in [-1, +\infty)\)

4) Limit $\sigma \to 0$ of the Potts model

3)

$$F(\sigma, \mu) = \sum_{\text{quadrangulation with recurrent configuration } C} \sigma \# \text{vertices} (\mu + 1)^{\text{level}(C)}$$
Questions

→ Characterization of $F$?

→ Asymptotic behaviour?

→ Nature of $F$?

→ Statistical properties on large maps?
From Forested To General Maps

Idea from [Boutrier, Di Francesco, Guitter, 2008]
FROM FORESTED TO GENERAL MAPS
From Forested To General Maps

tree contraction
FROM FORESTED TO GENERAL MAPS

tree contraction
FROM FORESTED TO GENERAL MAPS

Tree contraction

Tree extraction
From Forested To General Maps

tree contraction

tree extraction
FROM FORESTED TO GENERAL MAPS

Tree contraction

Tree extraction
FROM FORESTED TO GENERAL MAPS

- Tree contraction
- Tree extraction
FROM FORESTED TO GENERAL MAPS

- Tree contraction
- Tree extraction

BIJECTION!
Translation Into Generating Functions

\[ M(\gamma, \mu; g_1, g_2, g_3, \ldots; h_1, h_2, h_3, \ldots) = \]

Generating function of rooted maps with a weight:

* \( \gamma \) per face,
* \( \mu \) per non-root vertex of degree \( k \),
* \( \gamma \) if the root vertex has degree \( k \).

\[ F(\gamma, \mu) = M(\gamma, \mu; k_1, k_2, k_3, \ldots; c_1, c_2, c_3, \ldots) \]

\( k_k = \# \text{4-valent leaf-rooted trees with} \ k \text{ leaves} \)

\( c_k = \# \text{4-valent corner-rooted trees with} \ k \text{ leaves} \)
Generating Function For General Maps

\[ M(\bar{z}, \mu; g_1, g_2, g_3, \ldots; h_1, h_2, h_3, \ldots) = \]

Generating function of rooted maps with a weight:

- \( \bar{z} \) per face,
- \( \mu \) per non-root vertex of degree \( \bar{k} \),
- \( h_k \) if the root vertex has degree \( k \).

This generating function is known of [Bouttier - Guitter, 2012]

\( (M' \text{ is even nicer}) \)

Notation: \( X' = \frac{e^X}{\partial X} \)
The Generating Function of Forested Maps

**Theorem**

There exists a unique series $R$ in $z^8$ with coefficients in $\mathbb{Q}[u]$ such that

$$R = z^8 + \mu \sum_{i \geq 2} \frac{(3i-3)!}{(i-1)!^2 i!} R^i$$

Then:

$$F' = 4 \sum_{i \geq 2} \frac{(3i-2)!}{(i-2)! i!^2} R^i$$
The Generating Function of Forested Maps

**Theorem**

There exists a unique series \( R \) in \( z_8 \) with coefficients in \( \mathbb{Q}[u] \) such that

\[
R = z_8 + u \sum_{i \geq 2} \frac{(3i-3)!}{(i-1)!^2 i!} R^i
\]

Then:

\[
F' = 4 \sum_{i \geq 2} \frac{(3i-2)!}{(i-2)! i!^2} R^i
\]

For \( u = 0 \), [Mullin]

\[
R = z_8 \quad \text{and} \quad F' = 4 \sum_{i \geq 2} \frac{(3i-2)!}{(i-2)! i!^2} z_8^i \quad \text{D-finite.}
\]
The Generating Function of Forested Maps

**Theorem**

There exists a unique series $R$ in $x^2$ with coefficients in $\mathbb{Q}[x]$ such that

$$R = x^2 + \mu \phi(R)$$

Then:

$$F' = \Theta(R)$$

where

$$\phi(x) = \sum_{i \geq 2} \frac{(3i-3)!}{(i-2)! i!} x^i, \quad \Theta(x) = 4 \sum_{i \geq 2} \frac{(3i-2)!}{(i-2)! i!^2} x^i.$$
A Differential Equation For F

\[ R = \gamma + \mu \phi(R) \quad F' = \Theta(R) \]

Prop F is D-algebraic.

(Fundamental reason: \( \phi \) and \( \Theta \) are D-finite.)
A Differential Equation For $F$

\[ R = z + \mu \phi(R) \quad F' = \Theta(R) \]

**Prop** $F$ is D-algebraic.

(Fundamental reason: $\phi$ and $\Theta$ are D-finite.)

Cf Bernardi - Bousquet-Mélou's result:
The Potts generating function of planar maps is D-algebraic.
(established in a more painful way.)
A **DIFFERENTIAL EQUATION** for \( F \)

\[
R = z + \nu \phi(R) \quad F' = \Theta(R)
\]

**Prop** \( F \) is \( D \)-algebraic.

(Fundamental reason: \( \phi \) and \( \Theta \) are \( D \)-finite.)

Can a differential equation for \( F \) be explicitly computed?
A Differential Equation for $F$

$$R = rz + \mu \phi(R) \quad F' = \Theta(R)$$

**Prop** $F$ is $D$-algebraic.

(Fundamental reason: $\phi$ and $\Theta$ are $D$-finite.)

Can a differential equation for $F$ be explicitly computed?

**YES!**
A DIFFERENTIAL EQUATION FOR $F$

\[ 9F'^2F''\mu^6 + 36 F'^2F''^3F''F''\mu^3 + 144 F'^2F''^4\mu^5 - 12(21\mu - 1)F'^2F''^3\mu^4 + 432 F'^2F''^2\mu^6 \]
\[- 48(24\mu - 4)F'^2F''^3F''F''\mu^3 + 864 F'^2F''^3\mu^4 - 36(27\mu - 2)F'^2F''^2\mu^5 + 4(27\mu - 2)(15\mu - 1)F'^2F''\mu^6 \]
\[ + 1728 F'^2F''F''\mu^5 \]

Differential equation of order 2 in $F'$ and degree 7. (but not in $F$)
A Differential Equation For $F$

\[
9 F^{12} F''^5 \mu^6 + 36 F^{12} F'''^3 F''^5 \mu^2 + 144 F^{12} F''^4 F''^6 \mu - 12(21\beta - 1) F F''^5 F''^6 \mu^2 + 432 F''^6 F''^6 \mu^6 - 48(24\beta - 1) F F''^3 F''^6 \mu^2 + 864 F^{12} F''^4 F''^4 \mu^4 - 36(27\beta - 2) F F''^4 F''^4 \mu^4 + 4(27\beta - 4)(15\beta - 1) F F''^4 F''^4 \mu^4 + 1728 F^{12} F''^4 F''^4 \mu^4 - 726(21\beta - 2) F F''^4 F''^4 \mu^4 + 10368 F F''^4 F''^4 \mu^4 + 16(27\beta - 1)(24\beta - 1) F F''^4 F''^4 \mu^4 + 2304 F^{12} F''^4 F''^4 \mu^4 - 288(31\beta - 4) F F''^4 F''^4 \mu^4 - 64(6\mu \gamma - 162 \mu \gamma^2 + 33 \mu \gamma - 1) F''^5 F''^5 \mu^4 + 2304 F^{12} F''^4 F''^4 \mu^4 - 2304(6\beta - 1) F F''^4 F''^4 \mu^4 - 768(2\mu + 18\beta - 7) F''^5 F''^5 \mu^4 + 2304 F^{12} F''^4 F''^4 \mu^4 - 3072(3\beta - 1) F''^5 F''^5 \mu^2 - 192(24 \mu \gamma - 27 \gamma^2 + 55 \gamma - 2) F''^5 F''^5 \mu^2 - 1536(24 \beta - 7) F''^5 F''^5 \mu^2 - 768(12 \mu \gamma + 81 \gamma^2 + 24 \gamma - 1) F''^5 F''^5 \mu^2 + 1536(9 \gamma + 2) F''^5 F''^5 \mu^2 - 512(33 \mu \gamma + 81 \gamma^2 + 55 \gamma - 2) F''^5 F''^5 \mu^2 + 36864 F''^5 F''^5 \mu - 1024(12 \mu \gamma - 162 \gamma^2 + 33 \gamma - 1) F''^5 F''^5 \mu - 1024(36 \mu \gamma + 27 \gamma - 1) F''^5 F''^5 \mu - 24576 \gamma \beta = 0
\]

Differential equation of order 2 in $F'$ and degree 7.

(but not in $F$)

(Moreover, this is the equation with minimal order.)

Thanks Alin Bostan, Bruno Salvy & Michael Singer!
Fix $\mu$, $\rho_\mu = \text{radius of convergence of } F(z, \mu) = \sum_{n=0}^{\infty} f_n(\mu) z^n$.

$\rho_0 = \frac{1}{27}$

\[
\begin{cases} 
\rho_\mu = z\mu - \mu \phi(z\mu) \\
\phi'(z\mu) = \frac{1}{\mu} \\
(\mu > 0)
\end{cases}
\]
Fix $\mu$ in $[-1, +\infty)$,

$\rho_\mu = \text{radius of convergence of } F(z, \mu) = \sum_n f_n(\mu) z^n$.

$\rho_\mu$ is affine on $[-1, 0]$!

\[ \rho_0 = \frac{1}{27} \]

\[ \rho_1 \]

\[ \begin{cases} \rho_\mu = 2\mu - \mu \phi(\mu) \\ \phi'(\mu) = \frac{1}{\mu} \end{cases} \quad (\mu > 0) \]
Fix $\mu$ in $[-1, +\infty)$, 

$\rho_\mu = \text{radius of convergence of } F(z, \mu) = \sum_{n} f_n(\mu) z^n$.

$\rho_\mu$ is affine on $[-1,0]$!

$\rho_\mu = \frac{1}{2\pi} (1 + \mu) - \frac{\sqrt{3}}{12\pi} \mu$. 

$\rho_{-1} = \frac{\sqrt{3}}{12\pi}$ 

$\rho_0 = \frac{1}{2\pi}$

$\rho_1$

\[ \begin{cases} 
\rho_\mu = z\mu - \mu \phi(z\mu) \\
\phi'(z\mu) = \frac{1}{\mu} \\
(\mu > 0) 
\end{cases} \]
**Radius of Convergence**

Fix \( \mu \) in \([-1, +\infty)\),

\[ p_\mu = \text{radius of convergence of } F(z_\mu, \mu) = \sum_{\infty} f_n(\mu) z_\mu^n. \]

\[ p_{-1} = \frac{\sqrt{3}}{12\pi} \]

\[ p_0 = \frac{1}{2\pi} \]

\[ p_1 \]

\[ p_{-1} \]

\[ p_0 = \frac{1}{2\pi} \]

\[ \{ p_\mu = z_\mu - \mu \phi(z_\mu) \}

\[ \phi'(z_\mu) = \frac{1}{\mu} \]

\((\mu > 0)\)

**Cor**

\( p_{-1} \) is transcendental:

\( F(z_\mu, -1) \) is not D-finite.
Phase Transition at $0$

\[ f_n(w) = \left[ \frac{1}{z^n} \right] F(z, w) \]

$-1 \leq w < 0$

\[ f_n(w) \sim \frac{c w^\mu \bar{w}^{-n}}{n^3 \ln^2 n} \]

"Universality class" for maps

$w = 0$

\[ f_n(w) \sim \frac{c w^\mu \bar{w}^{-n}}{n^3} \]

maps with a spanning tree

$0 < w$

\[ f_n(w) \sim \frac{c w^\mu \bar{w}^{-n}}{n^{5/2}} \]

\[ \uparrow \text{standard} \]
**Phase Transition At 0**

\[ b_n(w) = \lfloor z^n \rfloor F(z, w) \]

\[-1 \leq w < 0\]

\[ b_n(w) \sim \frac{cw R_n^{-n}}{n^3 \ln^2 n} \]

"Universality class" for maps

\[ 0 < w \]

\[ b_n(w) \sim \frac{cw R_n^{-n}}{n^{5/2}} \]

maps with a spanning tree

\[ \text{standard} \]


\[ \text{Cor} \]

For \( w \in [-1,0) \), \( F(z, w) \) is not \( D\)-finite.
Idea of the Proof

Singularity analysis [Flajolet - Odlyzko]

A link between the singular behaviour of $F(x, \mu)$ near $\rho_\mu$ and the asymptotic behaviour of $f_m(\mu)$. 
IDEA OF THE PROOF

Singularity analysis [Flajolet - Odlyzko]

A link between the singular behaviour of $F(z, \mu)$ near $p_\mu$ and the asymptotic behaviour of $f_m(\mu)$.

$$R = z_0 + \mu \phi(R) \quad \text{radius of convergence of } \phi = \frac{1}{2\pi}$$
**Idea of the Proof**

Singularity analysis [Flajolet - Odlyzko]

A link between the singular behaviour of $F(g, \mu)$ near $\rho_\mu$ and the asymptotic behaviour of $f_m(\mu)$.

\[ R = g_\delta + \mu \phi(R) \quad \text{radius of convergence of } \phi = \frac{1}{2\pi} \]

$\mu > 0$

Singularity in $(\rho_\mu - g_\delta)^{\frac{1}{2}}$
IDEA OF THE PROOF

Singularity analysis [Flajolet - Odlyzko]
A link between the singular behaviour of $F(g, \mu)$ near $p_\mu$ and the asymptotic behaviour of $f_\mu(\mu)$.

$$R = g + \mu \phi(R)$$

radius of convergence of $\phi = \frac{1}{2\pi}$

$\mu < 0$  $\mu > 0$

singularity in $\frac{p_\mu - g}{\ln(p_\mu - g)}$

singularity in $(p_\mu - g)^{1/2}$
**IDEA OF THE PROOF**

Singularity analysis [Flajolet - Odlyzko]

A link between the singular behaviour of \( F(y, \mu) \) near \( \rho_\mu \) and the asymptotic behaviour of \( \beta_\mu(\mu) \).

\[ R = y + \mu \phi(R) \]

radius of convergence of \( \phi = \frac{1}{2\pi} \)

\[ \begin{align*}
\mu < 0 & \quad \text{singularity in } \frac{\rho_\mu - y}{\ln(\rho_\mu - y)} \\
\mu > 0 & \quad \text{singularity in } (\rho_\mu - y)^{\frac{1}{2}}
\end{align*} \]
Fix $n \in \mathbb{N}$, consider a random forested map with $n$ faces - (under uniform distribution)
Some probability results

Fix $n \in \mathbb{N}$, consider a random forested map with $n$ faces.
(under uniform distribution)

$C_m = \text{r.v. that counts the number of components}$

\[\text{Th}\]

$C_m \xrightarrow{\text{distribution}} \text{gaussian law with linear mean }$

& linear variance.
SOME PROBABILITY RESULTS

Fix $n \in \mathbb{N}$, consider a random forested map with $n$ faces - (under uniform distribution)

$C_m = \text{r.v. that counts the number of components}$

$\implies C_m \xrightarrow{\text{distribution}} \text{gaussian law with linear mean} \& \text{linear variance}$
Fix \( n \in \mathbb{N} \), consider a random forested map with \( n \) faces - (under uniform distribution)

\[ S_n = \text{size of the root component (number of vertices)} \]

\[ \lim_{n \to +\infty} \Pr_n (S_n = k) = \frac{4}{(k-1)! k! (k+1)!} \frac{2^k}{\phi(2)} \]
**Extension of the Results**

Fix a set of permitted vertex degrees.

<table>
<thead>
<tr>
<th>Functional system</th>
<th>Eulerian</th>
<th>not Eulerian</th>
</tr>
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<tbody>
<tr>
<td>( R = 2g + \mu \phi(R) )</td>
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**Nature of \( F(\gamma,\mu) \)**

- \( D \)-algebraic if the set of permitted degrees is a finite union of arithmetic progressions.

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<th>periodic</th>
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<tbody>
<tr>
<td>4-valent ( \checkmark )</td>
<td>( (2\ell) )-valent ( \ell \geq 3 ) ( \checkmark )</td>
<td>cubic ( \checkmark )</td>
<td>( (2\ell+1) )-valent ( \ell \geq 2 ) ?</td>
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Fix a set of permitted vertex degrees -

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Nature of $F(g,\mu)$ - $D$-algebraic if the set of permitted degrees is a finite union of arithmetic progressions.

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<tr>
<td>(2l)-valent</td>
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<td>$l \geq 3$</td>
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**EXTENSION OF THE RESULTS**

Fix a set of permitted vertex degrees -

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<td>$S = u \phi_2(R, S)$</td>
<td>$F' = \Theta(R, S)$</td>
<td></td>
</tr>
</tbody>
</table>

| Nature of $F(g, u)$ | $D$-algebraic if the set of permitted degrees is a finite union of arithmetic progressions. |

<table>
<thead>
<tr>
<th>Asymptotic behaviour</th>
<th>aperiodic</th>
<th>periodic</th>
<th>aperiodic</th>
<th>periodic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eulerian maps</td>
<td>$4$-valent ✓</td>
<td>(22$)^r$-valent ✓</td>
<td>cubic ✓</td>
<td>($2l+1$)-valent ✓</td>
</tr>
<tr>
<td>$l \geq 3$ ✓</td>
<td>general maps ?</td>
<td>general maps ?</td>
<td>$l \geq 2$ ?</td>
<td></td>
</tr>
</tbody>
</table>

**Prospects**
OTHER PROSPECTS

→ Go further into probability results.

→ Maps equipped with a bond animal.

Objective: Bond percolation on maps.
OTHER PROSPECTS

- Go further into probability results.
- Maps equipped with a bond animal.

Objective: Bond percolation on maps.
THANK YOU! AND
HAPPY HALLOWEEN!