PLANAR MAPS AND SPANNING FORESTS

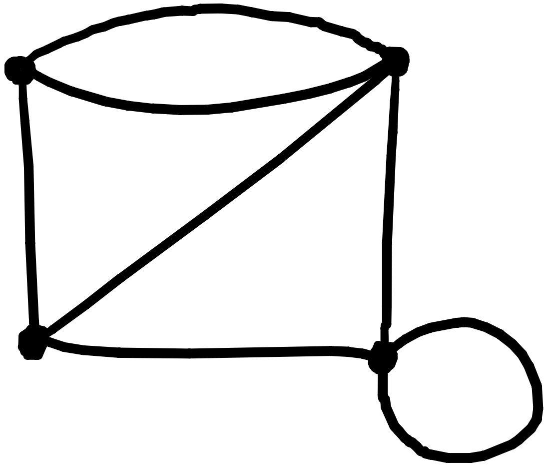
Julien COURTIEL (SFU/PIMS)
DM seminar, Nov. 4th

BASIC NOTIONS

PLANAR MAPS : DEFINITION

Planar map = connected graph

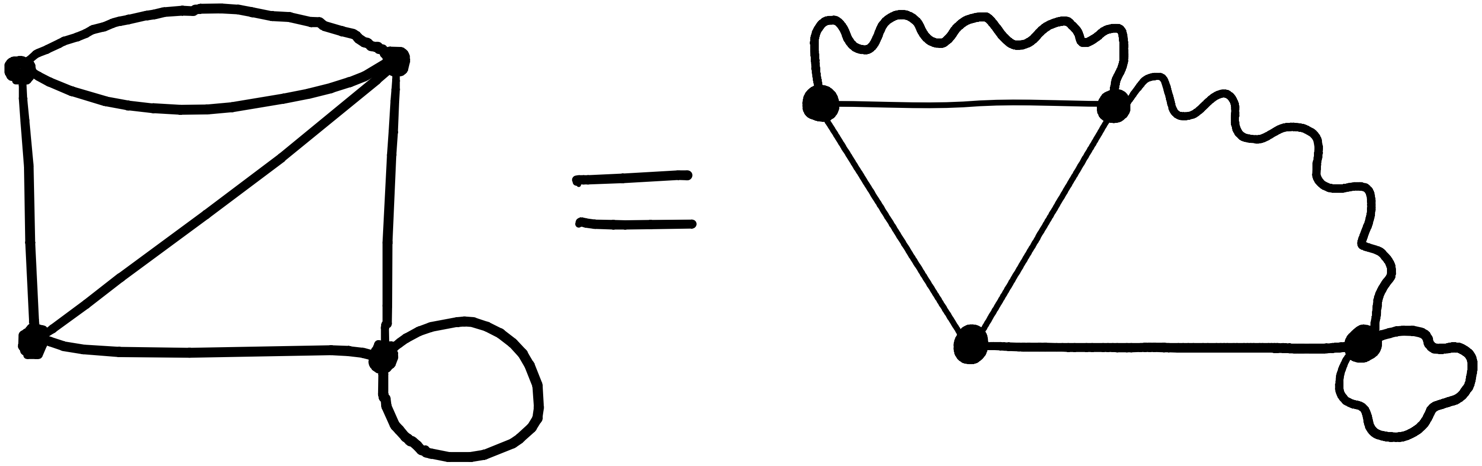
\+ embedding of this graph in the plane, considered up to continuous deformation.



PLANAR MAPS : DEFINITION

Planar map = connected graph

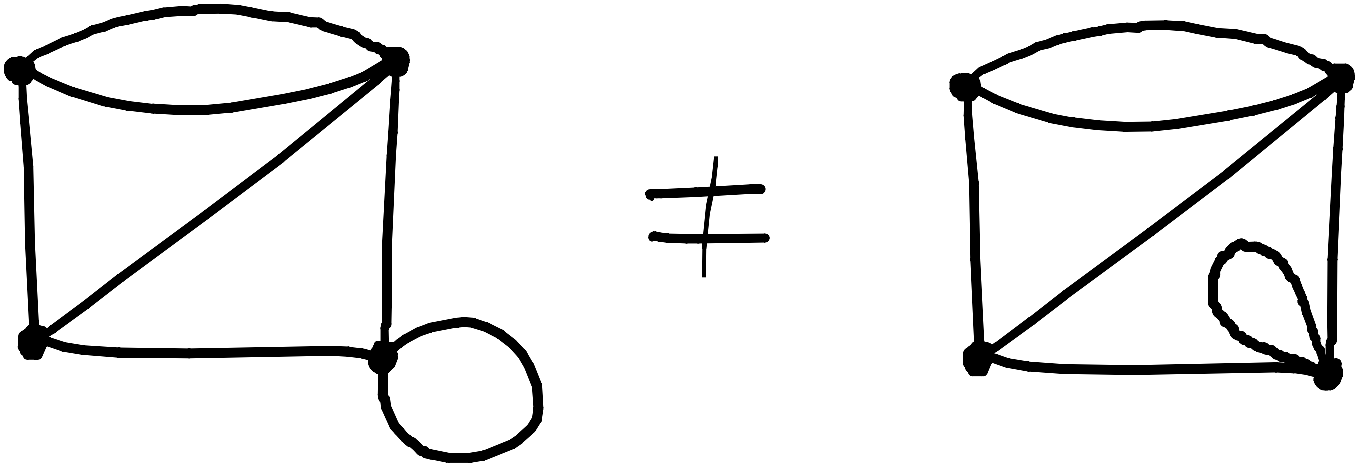
+ embedding of this graph in the plane, considered up to continuous deformation.



PLANAR MAPS : DEFINITION

Planar map = connected graph

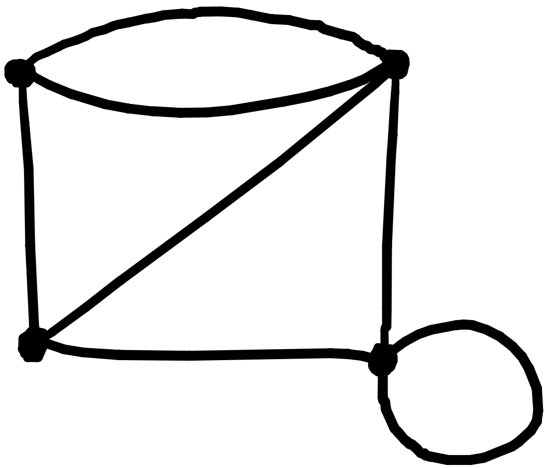
+ embedding of this graph in the plane, considered up to continuous deformation.



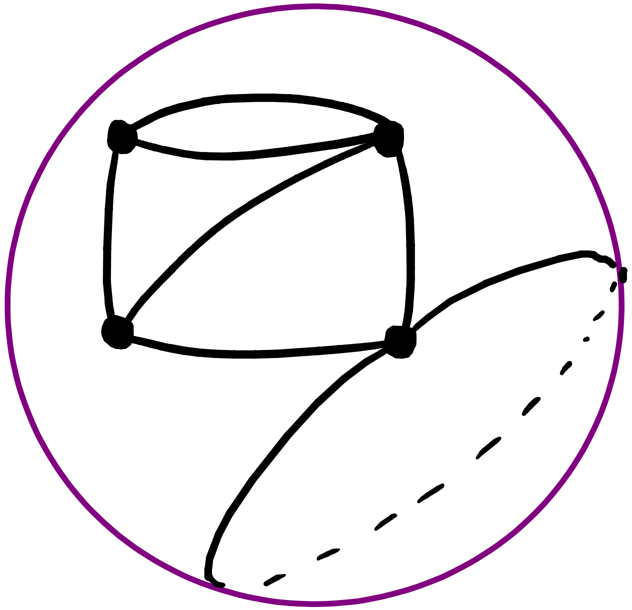
PLANAR MAPS : DEFINITION

Planar map = connected graph

\+ embedding of this graph in the ~~plane~~ sphere, considered up to continuous deformation.



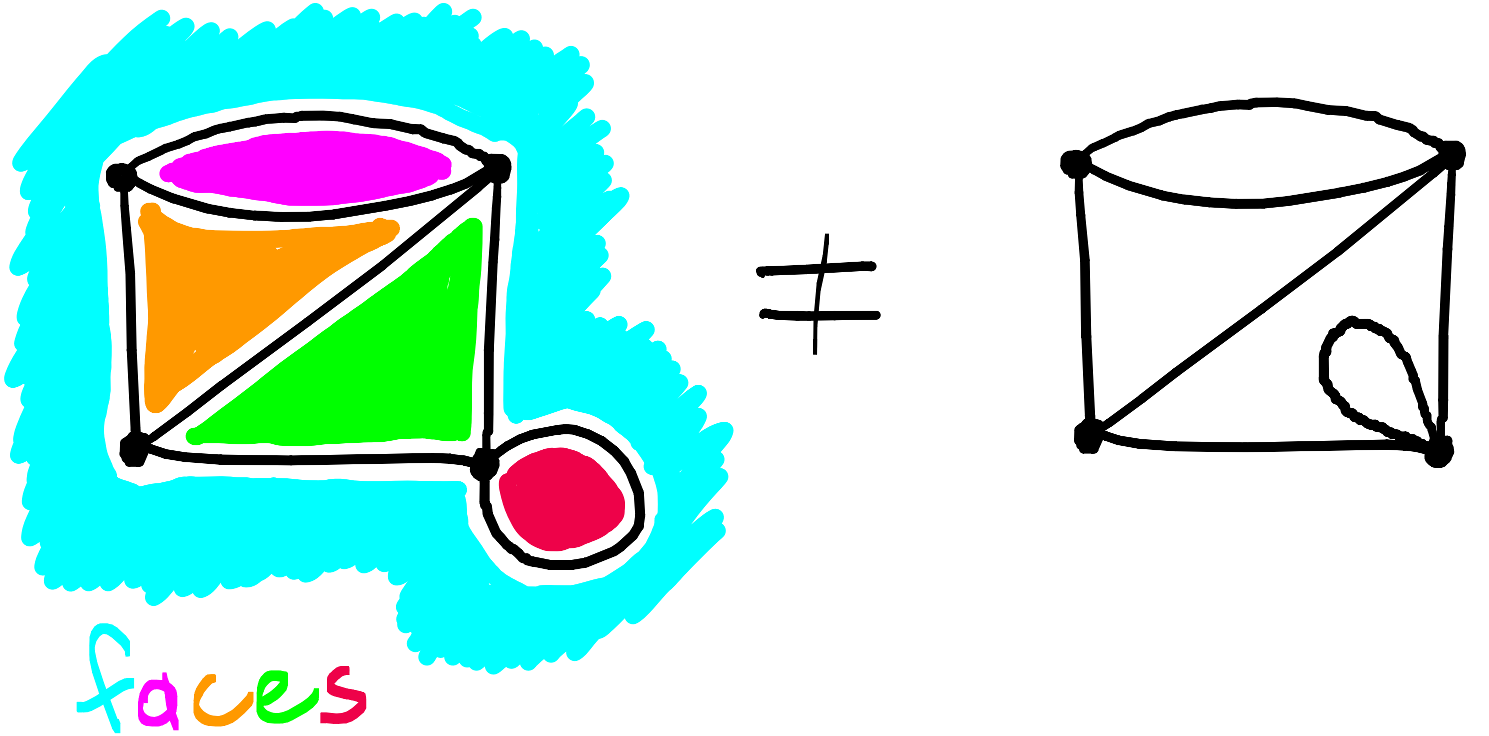
=



PLANAR MAPS : DEFINITION

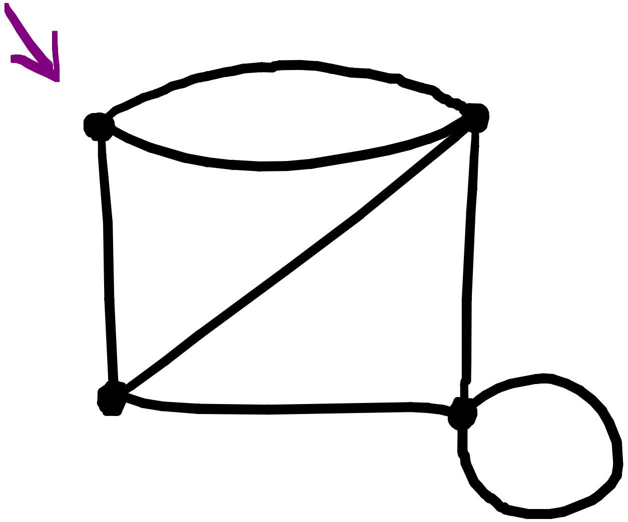
Planar map = connected graph

+ embedding of this graph in the plane, considered up to continuous deformation.



PLANAR MAPS : DEFINITION

Planar map = connected graph
+ embedding of this graph in the plane, considered up to continuous deformation.

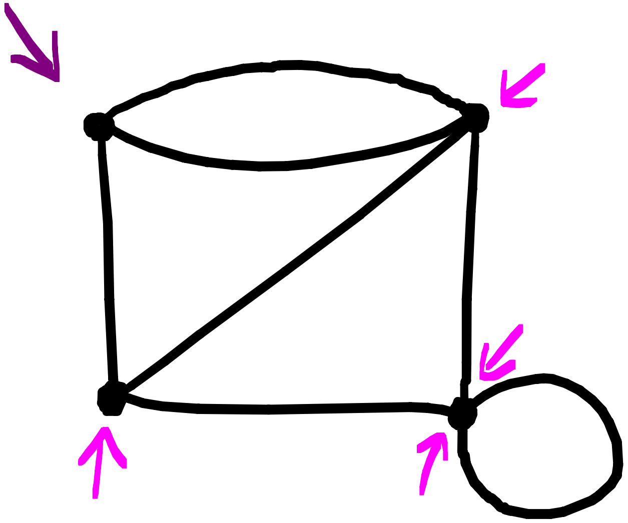


We root every planar map at an outer corner.

PLANAR MAPS : DEFINITION

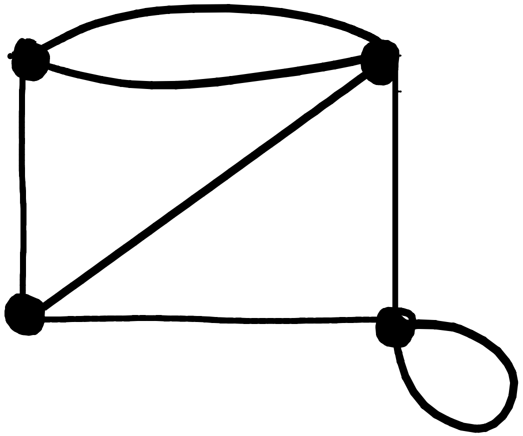
Planar map = connected graph

\+ embedding of this graph in the plane, considered up to continuous deformation.

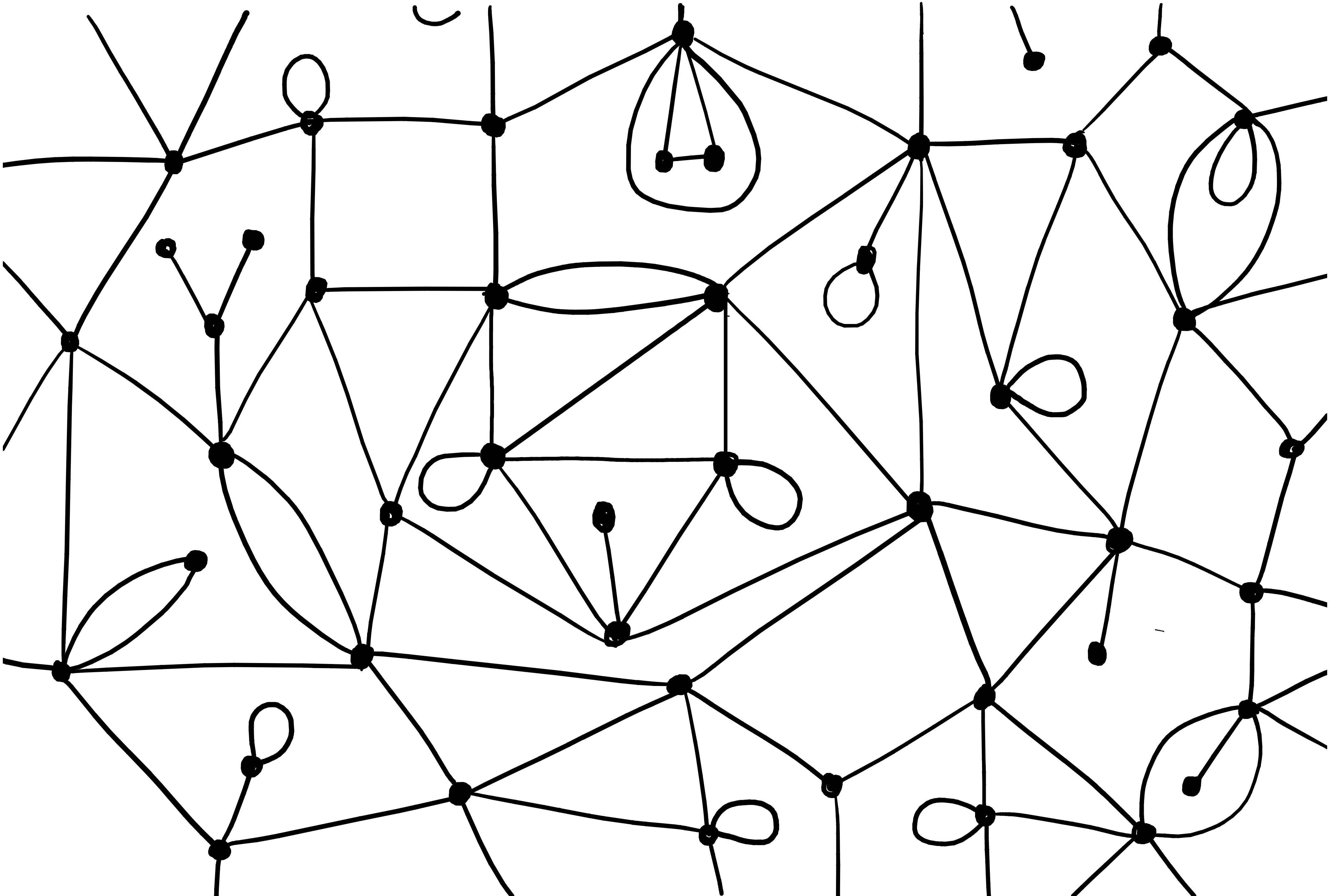


We root every planar map at an outer corner.

LARGE MAPS



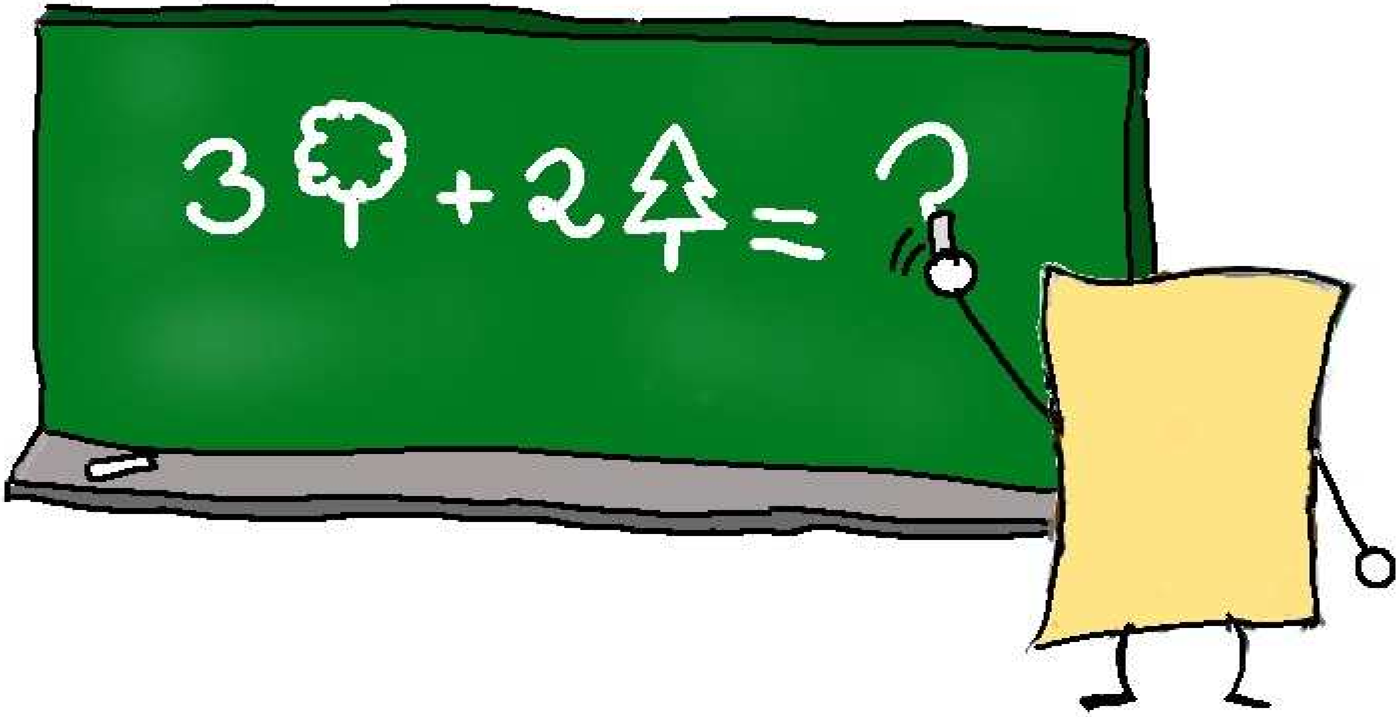
LARGE MAPS



APPLICATION AREAS

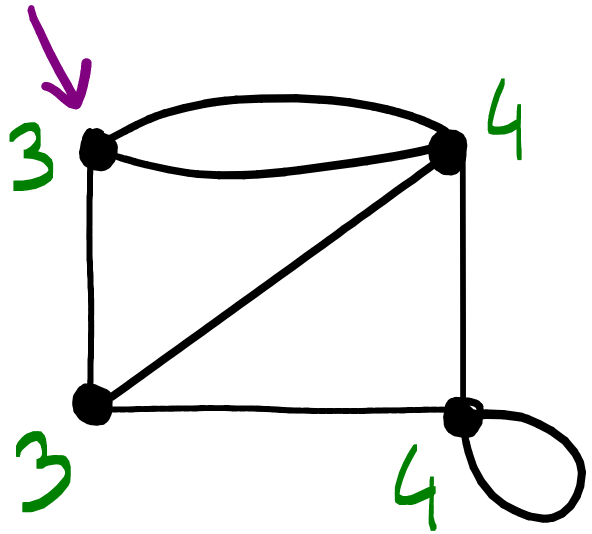
- statistical physics
- probability theory (matrix integrals, random continuous objects...)
- algorithmic geometry
- permutation factorizations
- every area that involves some surface ...

Enumeration of 4-Valent Maps



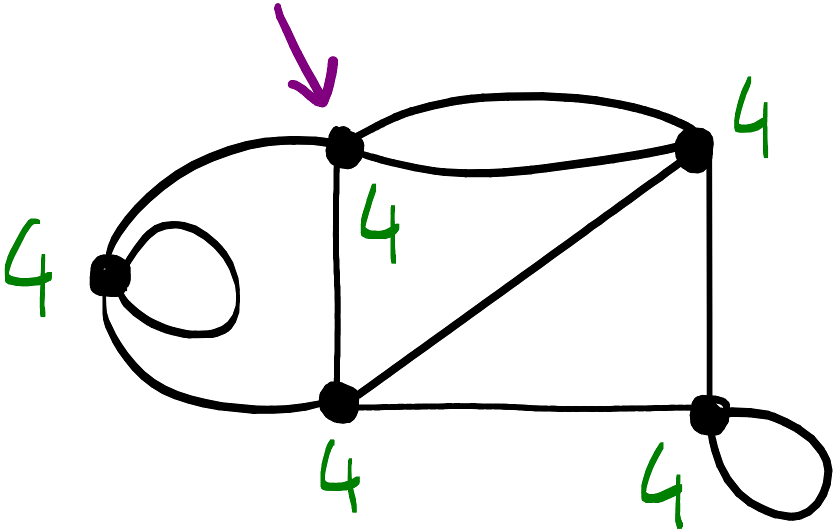
ENUMERATION OF 4-VALENT MAPS

4-valent map = map where every vertex has degree 4.



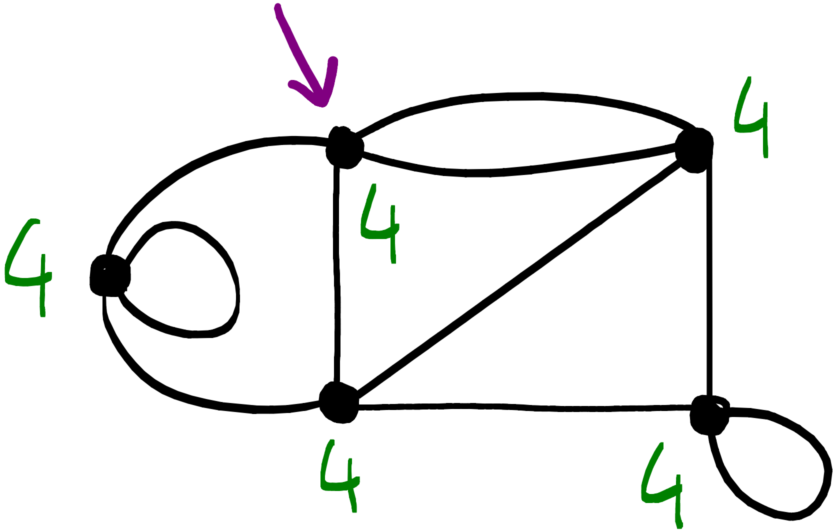
ENUMERATION OF 4-VALENT MAPS

4-valent map = map where every vertex has degree 4.



ENUMERATION OF 4-VALENT MAPS

4-valent map = map where every vertex has degree 4.



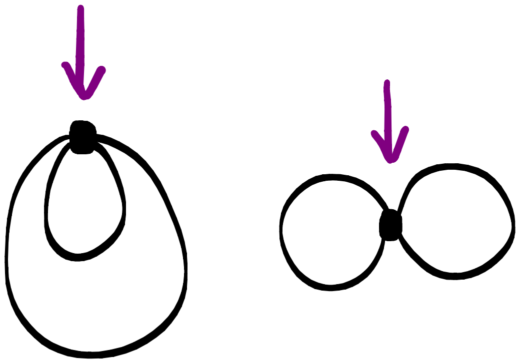
q_n = number of 4-valent maps with $(n+2)$ faces

$$= 2 \frac{3^n}{(n+1)(n+2)} \binom{2n}{n}$$

ENUMERATION OF 4-VALENT MAPS

4-valent map = map where every vertex has degree 4.

$q_1 = 2$



q_n = number of 4-valent maps with $(n+2)$ faces

$$= 2 \frac{3^n}{(n+1)(n+2)} \binom{2n}{n}$$

ASYMPTOTIC BEHAVIOUR

$$q_n = 2 \frac{3^n}{(n+1)(n+2)} \binom{2n}{n}$$

Stirling formula:

$$\boxed{q_n \sim \frac{2}{\sqrt{\pi}} 12^n n^{-5/2}}$$

ASYMPTOTIC BEHAVIOUR

$$q_n = 2 \frac{3^n}{(n+1)(n+2)} \binom{2n}{n}$$

Stirling formula:

$$\boxed{q_n \sim \frac{2}{\sqrt{\pi}} 12^n n^{-5/2}}$$

typical for planar maps

NATURE OF THE GENERATING FUNCTION

Generating function of the 4-valent maps:

$$Q(z) = \sum_{n\geq 1} q_n z^n$$

Nature of the generating function?

rational → algebraic → D-finite → D-algebraic

$Q = \frac{P_1}{P_2}$

$\exists$ polynomial that annihilates Q

satisfies a linear DE

satisfies a polynomial DE

NATURE OF THE GENERATING FUNCTION

Generating function of the 4-valent maps:

$$Q(z) = \sum_{n \geq 1} q_n z^n$$

Nature of the generating function?

rational → algebraic → D-finite → D-algebraic

$Q = \frac{P_1}{P_2}$

∃ polynomial that annihilates Q

satisfies a linear DE

satisfies a polynomial DE

$$Q = T - zT^3 \qquad T = 1 + 3zT^2$$

NATURE OF THE GENERATING FUNCTION

Generating function of the 4-valent maps:

$$Q(z) = \sum_{n \geq 1} q_n z^n$$

Nature of the generating function?

rational → algebraic → D-finite → D-algebraic

$Q = \frac{P_1}{P_2}$

∃ polynomial that annihilates Q

satisfies a linear DE

satisfies a polynomial DE

$$Q = T - zT^3 \qquad T = 1 + 3zT^2$$

NATURE OF THE GENERATING FUNCTION

Generating function of the 4-valent maps:

$$Q(z) = \sum_{n \geq 1} q_n z^n$$

Nature of the generating function?

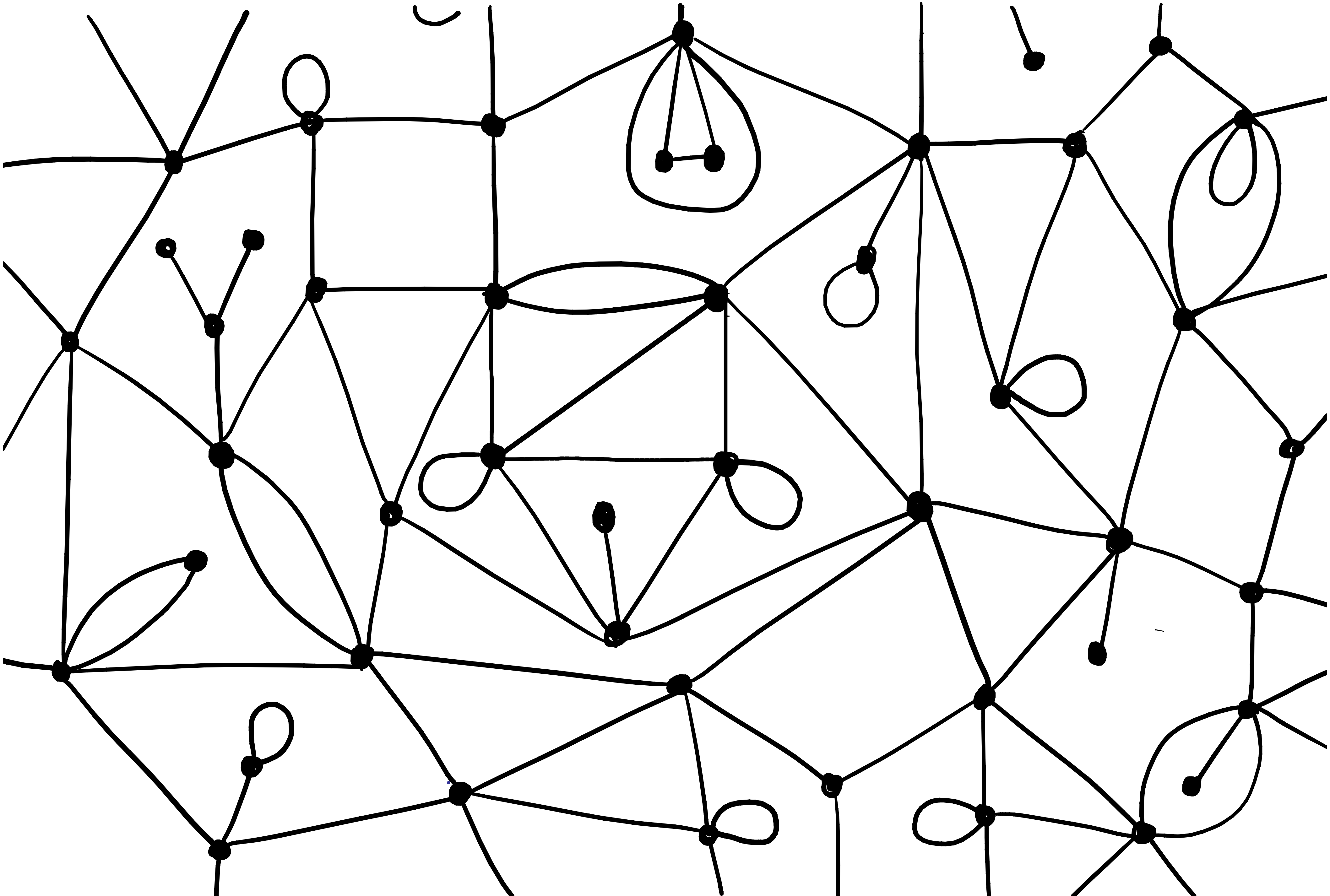
rational → algebraic → D-finite → D-algebraic

rational	algebraic	D-finite	D-algebraic
$Q = \frac{P_1}{P_2}$	$\exists$ polynomial that annihilates Q	satisfies a linear DE	satisfies a polynomial DE

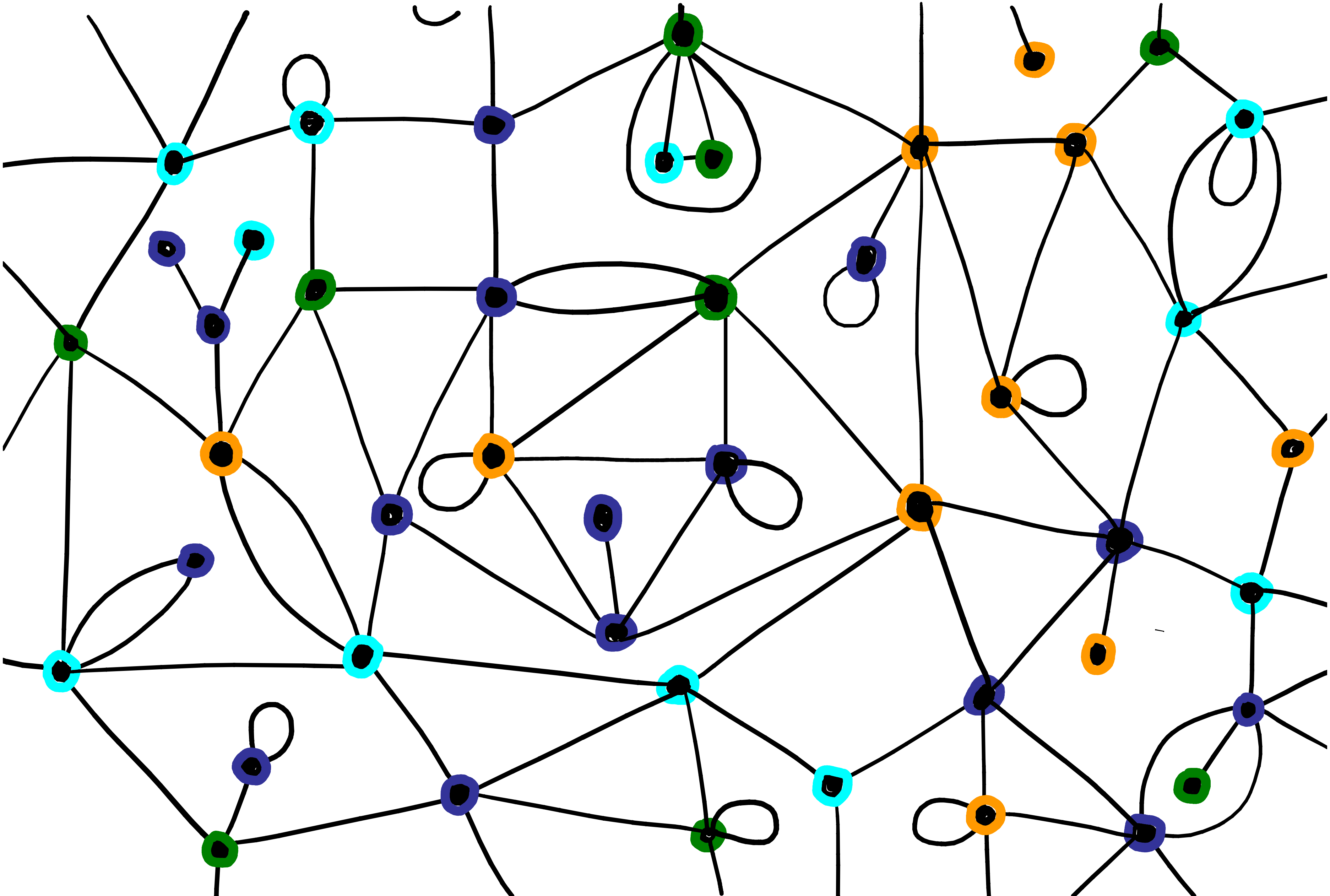
↯ form of the asymptotics

$$Q = T - zT^3 \qquad T = 1 + 3zT^2$$

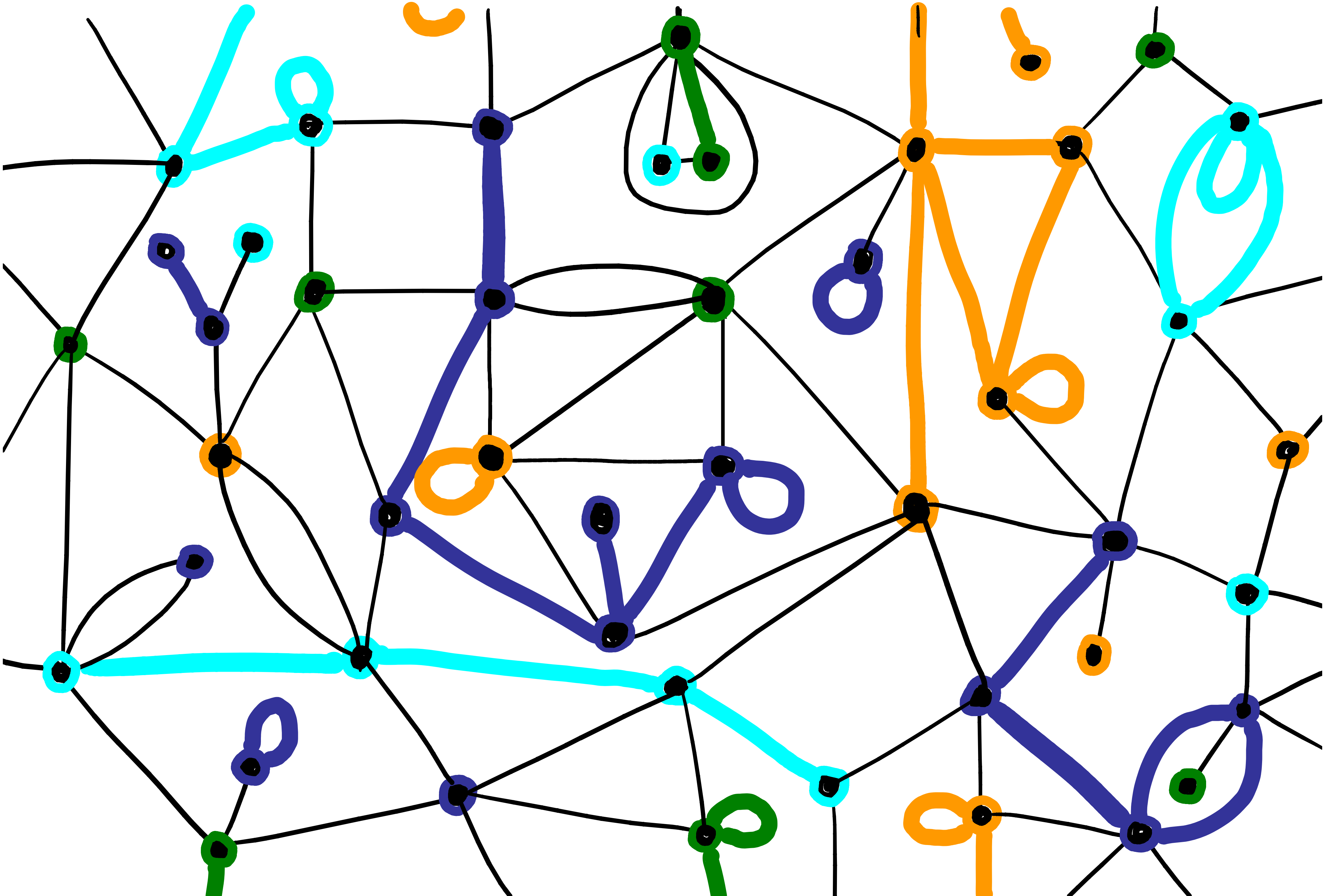
THE POTTS MODEL



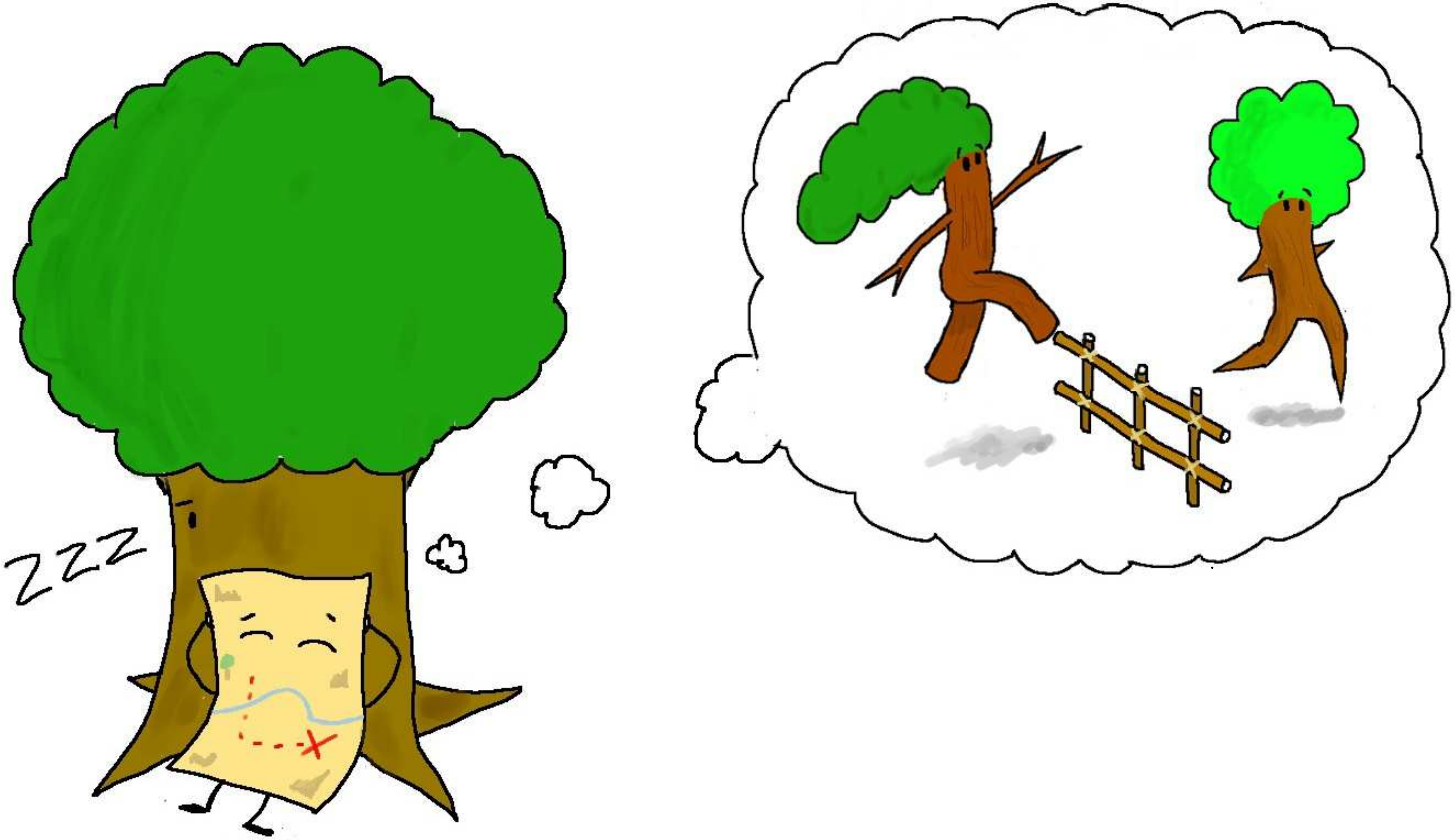
THE POTTS MODEL



THE POTTS MODEL

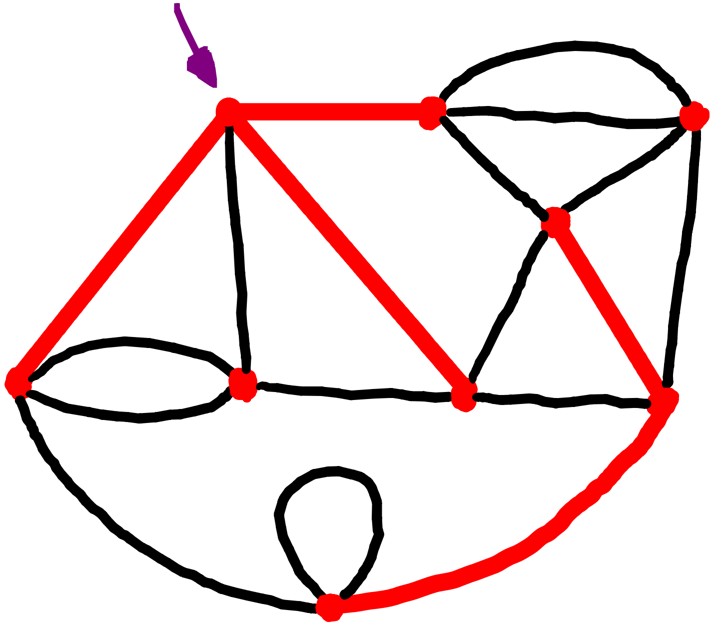


FORESTED MAPS



with Mireille BOUSQUET-MÉLOU (Bordeaux)

FORESTED MAPS : DEFINITION



Spanning forest of M = graph F such that:

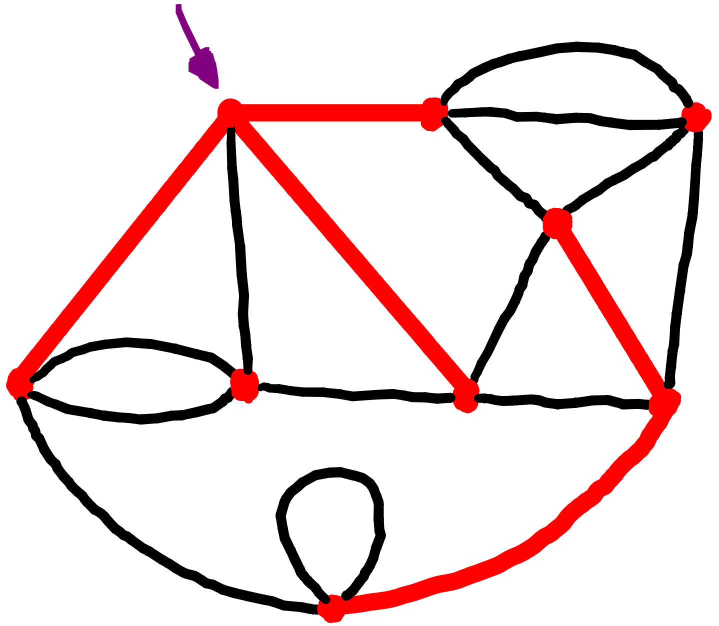
- $V(F) = V(M)$
- $E(F) \subseteq E(M)$ has no cycle.

Forested map (M,F) = Rooted map M with a spanning forest F.

Some other structures: Spanning trees, colourings, percolation, Ising/Potts model, self-avoiding walks... [Tutte, Mullin, Kazakov, Borot, Bouttier, Guitter, Sportiello, Eynard, Duplantier, Bousquet-Mélou, Schaeffer, Bernardi, Angel ...]

FORESTED MAPS : DEFINITION

Spanning forest of M =
graph F such that:
- V(F) = V(M)
- E(F) ⊆ E(M) has no cycle.

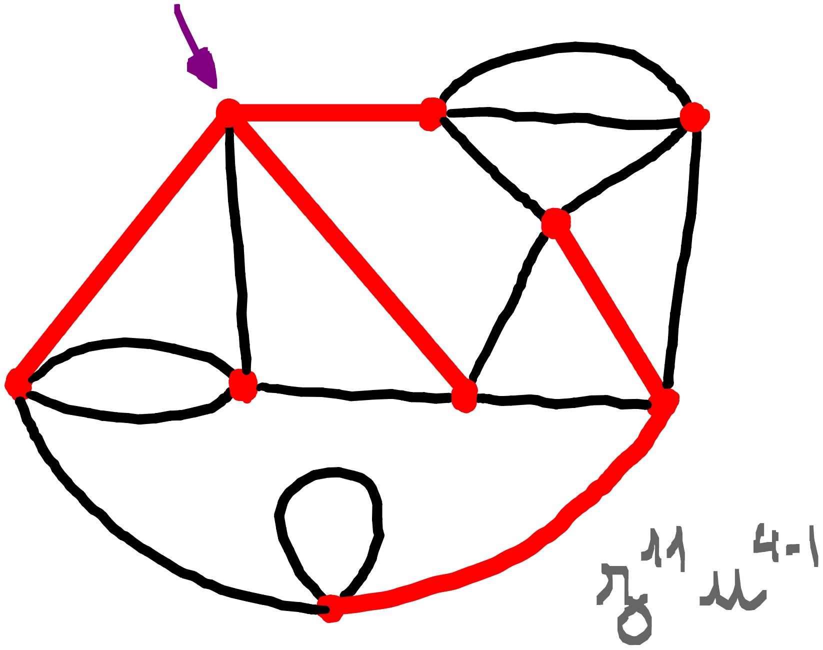


Forested map (M,F) = Rooted map M with a spanning forest F.

$$F(z,u) = \sum_{\substack{(M,F)\ \text{4-valent} \\ \text{forested map}}} z^{\#\text{faces}}\, u^{\#\text{components} - 1}$$

FORESTED MAPS : DEFINITION

Spanning forest of M =
graph F such that:
- $V(F) = V(M)$
- $E(F) \subseteq E(M)$ has no cycle.



Forested map (M,F) = Rooted map M with a spanning forest F.

$$F(z,u) = \sum_{\substack{(M,F)\ \text{4-valent} \\ \text{forested map}}} z^{\#\text{faces}} \, u^{\#\text{components} - 1}$$

SPECIAL VALUES OF u

$$F(z,u) = \sum_{\substack{(M,F) \text{ 4-valent} \\ \text{forested map}}} z^{\#\text{faces}}\, u^{\#\text{components} - 1}$$

- $u = 1$: spanning forests
- $u = 0$: spanning trees [Mullin, 1967]
- $u = -1$: root-connected acyclic orientations on (dual) quadrangulations. [Las Vergnas, 1984]

GENERIC VALUES OF u

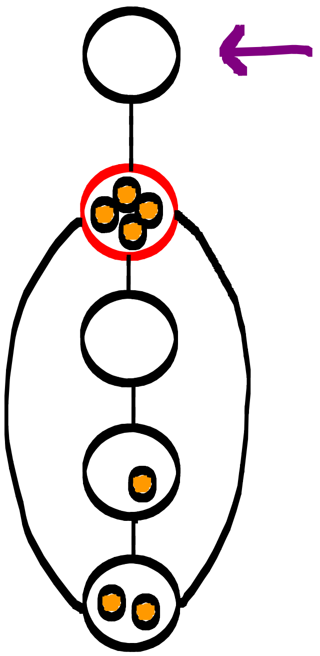
1) Connected subgraphs on quadrangulations (counted by cycles)
2) Tutte polynomial $T_M(u+1, 1)$
3) Sandpile model [Merino Lopez, Cori, Le Borgne]
4) Limit $q \to 0$ of the Potts model.

Natural domain $u \in [-1, +\infty)$

---

3)

$$F(z,u) = \sum_{\substack{\text{quadrangulation} \\ \text{with recurrent} \\ \text{configuration } C}} z^{\#\text{vertices}} (u+1)^{\text{level}(C)}$$



GENERIC VALUES OF μ

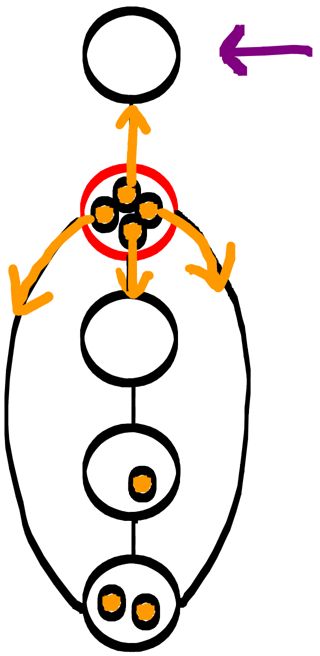
1) Connected subgraphs on quadrangulations (counted by cycles)
2) Tutte polynomial $T_M(\mu+1, 1)$
3) Sandpile model [Merino Lopez, Cori, Le Borgne]
4) Limit $q \to 0$ of the Potts model.

Natural domain $\mu \in [-1, +\infty)$

---

3)

$$F(z,\mu) = \sum_{\substack{\text{quadrangulation}\\ \text{with recurrent}\\ \text{configuration } C}} z^{\#\text{vertices}} (\mu+1)^{\text{level}(C)}$$



GENERIC VALUES OF μ

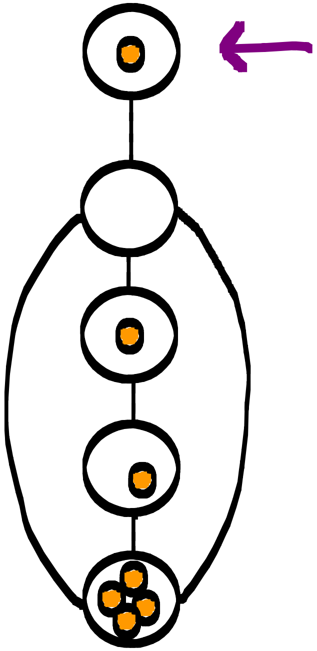
1) Connected subgraphs on quadrangulations (counted by cycles)
2) Tutte polynomial $T_M(\mu+1, 1)$
3) Sandpile model [Merino Lopez, Cori, Le Borgne]
4) Limit $q \to 0$ of the Potts model.

Natural domain $\mu \in [-1, +\infty)$

---

3)

$$F(z,\mu) = \sum_{\substack{\text{quadrangulation}\\ \text{with recurrent}\\ \text{configuration } C}} z^{\#\text{vertices}} (\mu+1)^{\text{level}(C)}$$



QUESTIONS

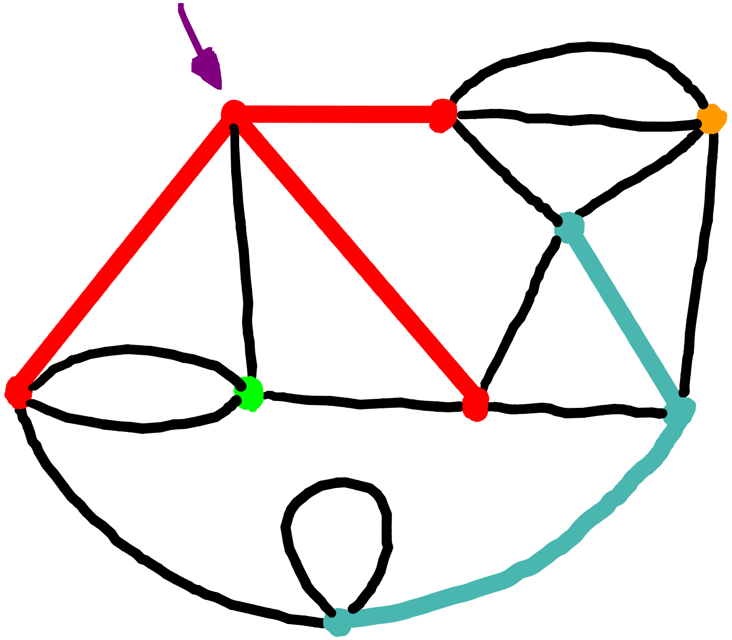
→ Characterization of F?

→ Asymptotic behaviour?

→ Nature of F?

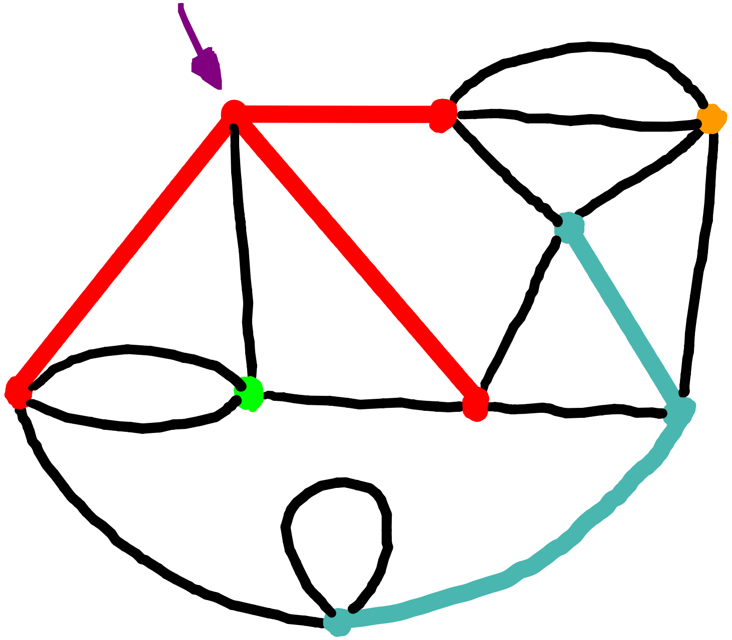
→ Statistical properties on large maps?

FROM FORESTED TO GENERAL MAPS

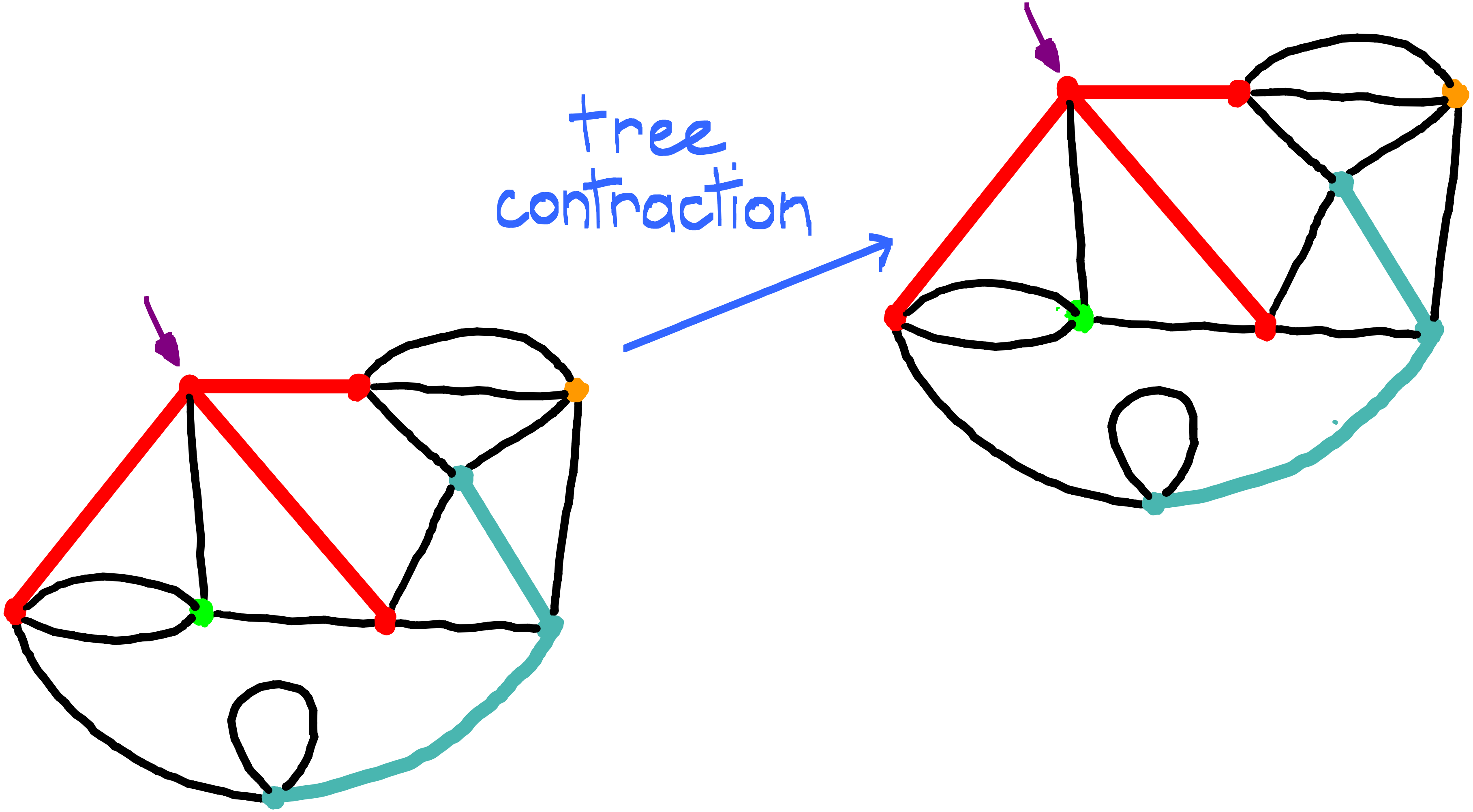


Idea from [Bouttier, Di Francesco, Guitter, 2008]

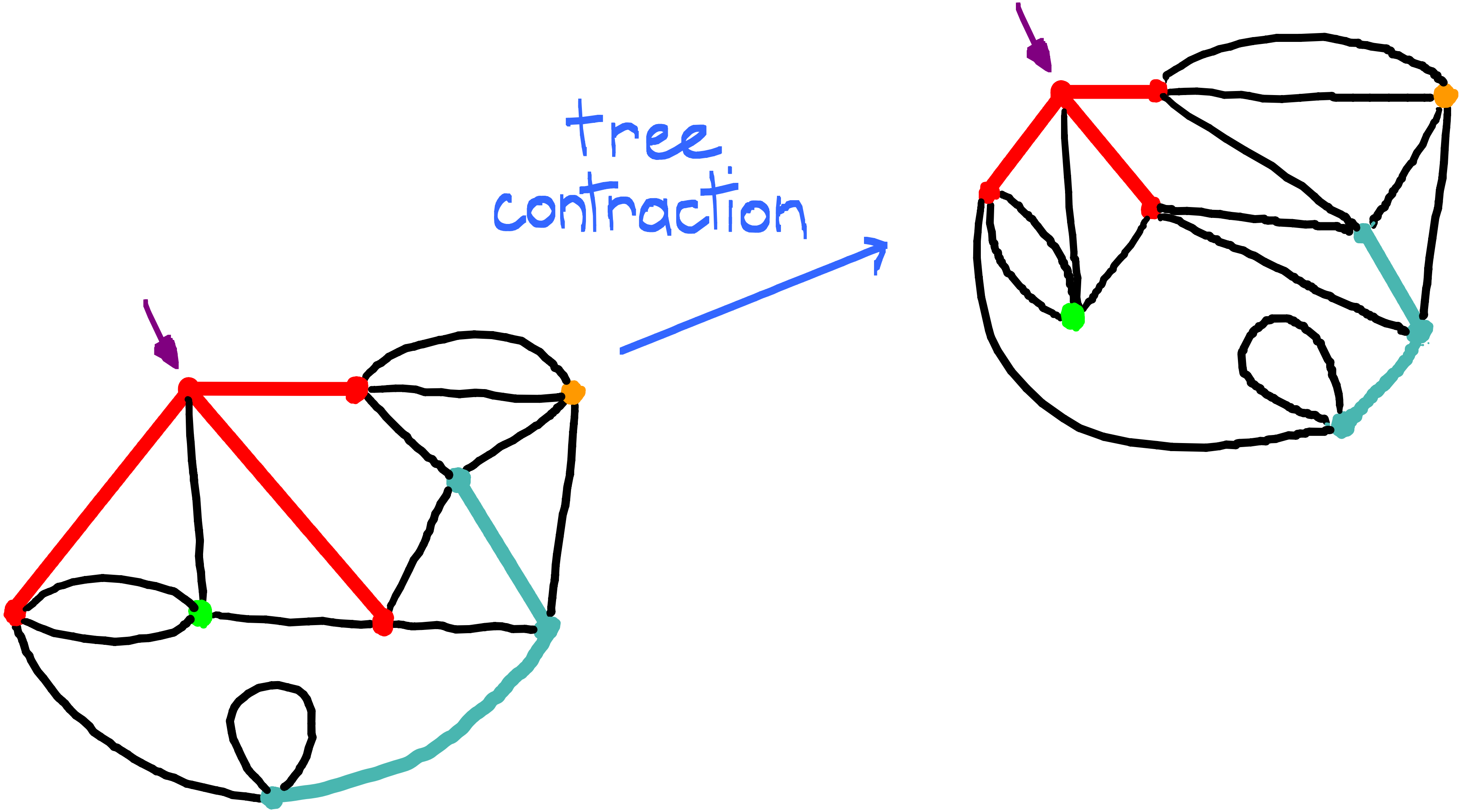
FROM FORESTED TO GENERAL MAPS



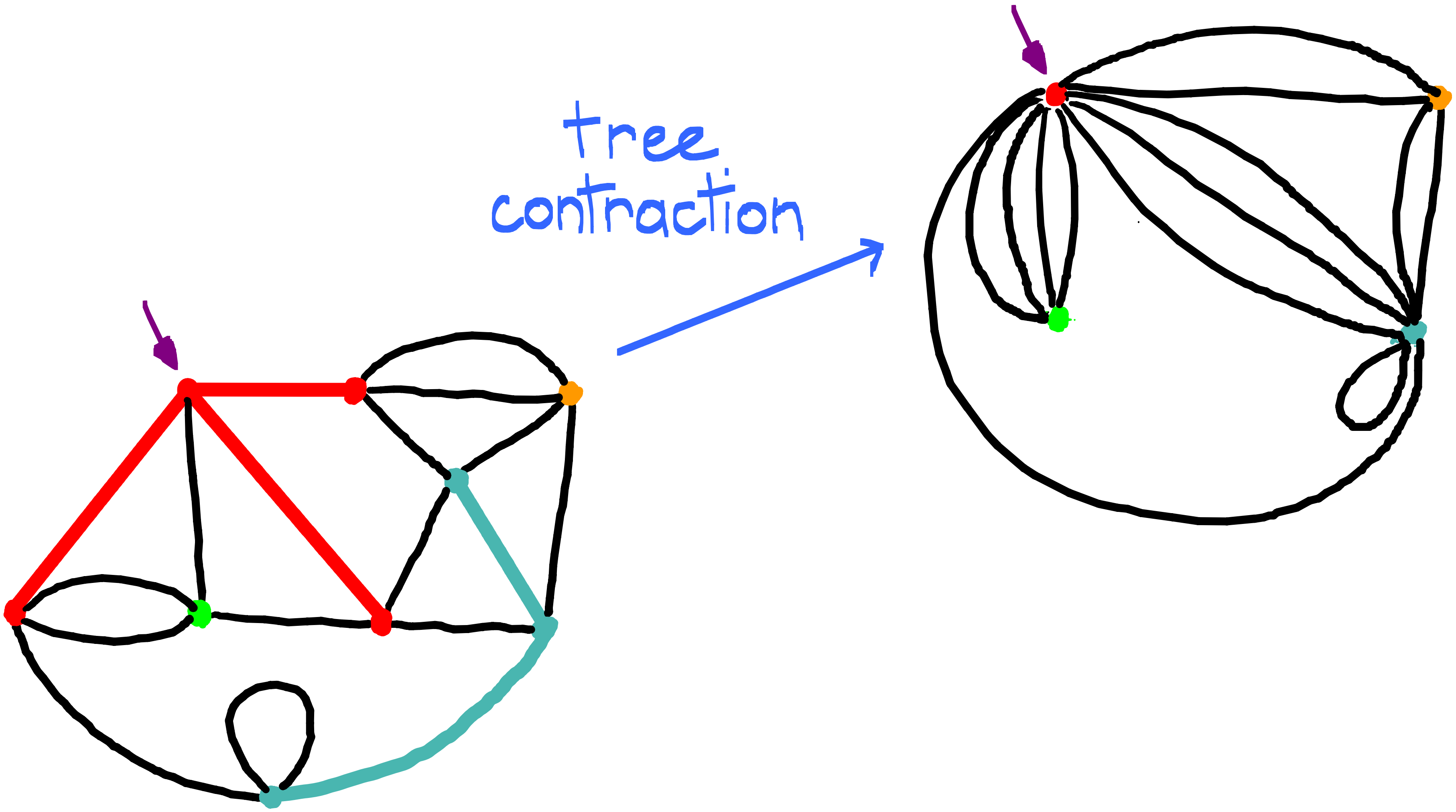
FROM FORESTED TO GENERAL MAPS



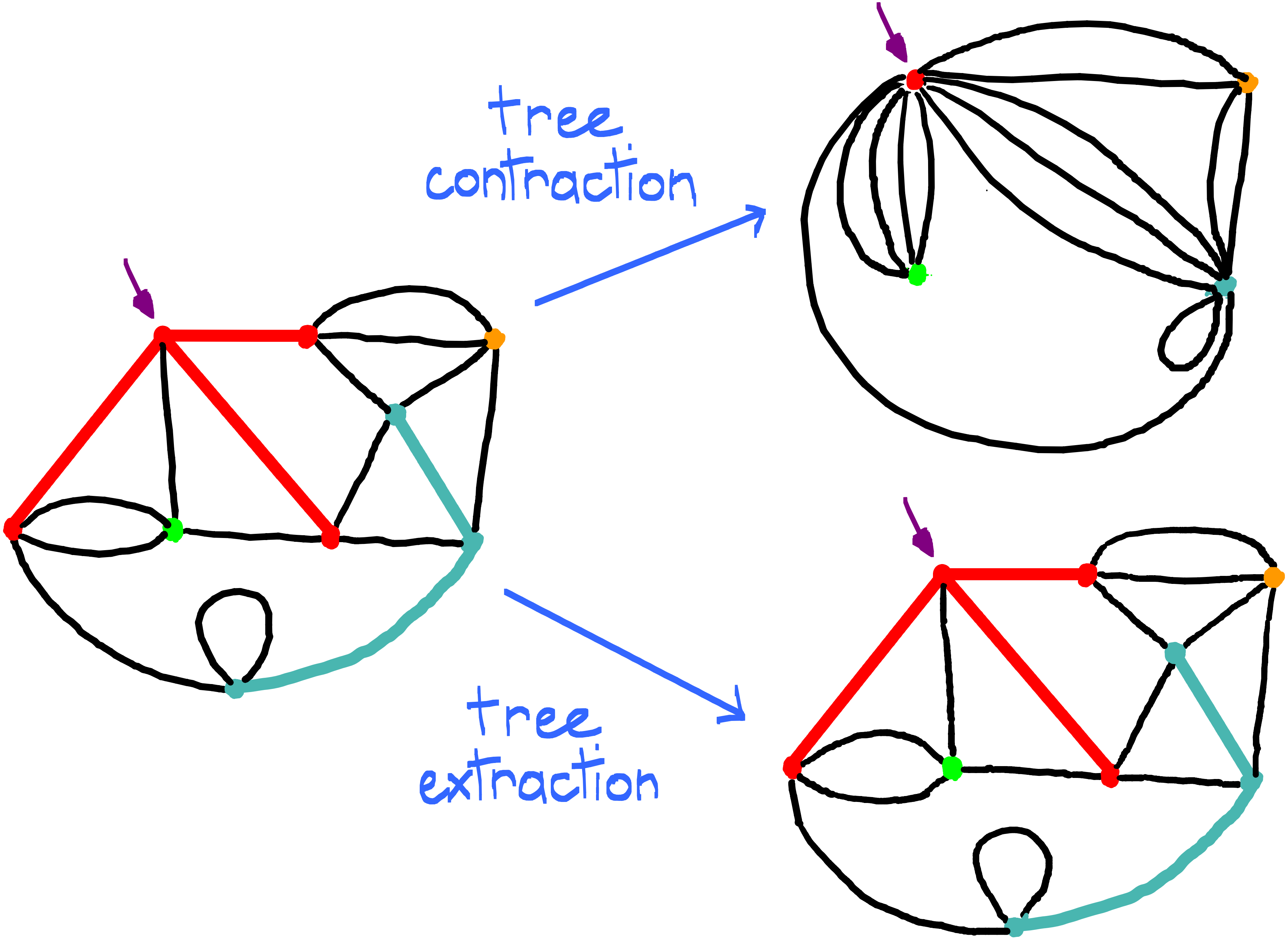
FROM FORESTED TO GENERAL MAPS



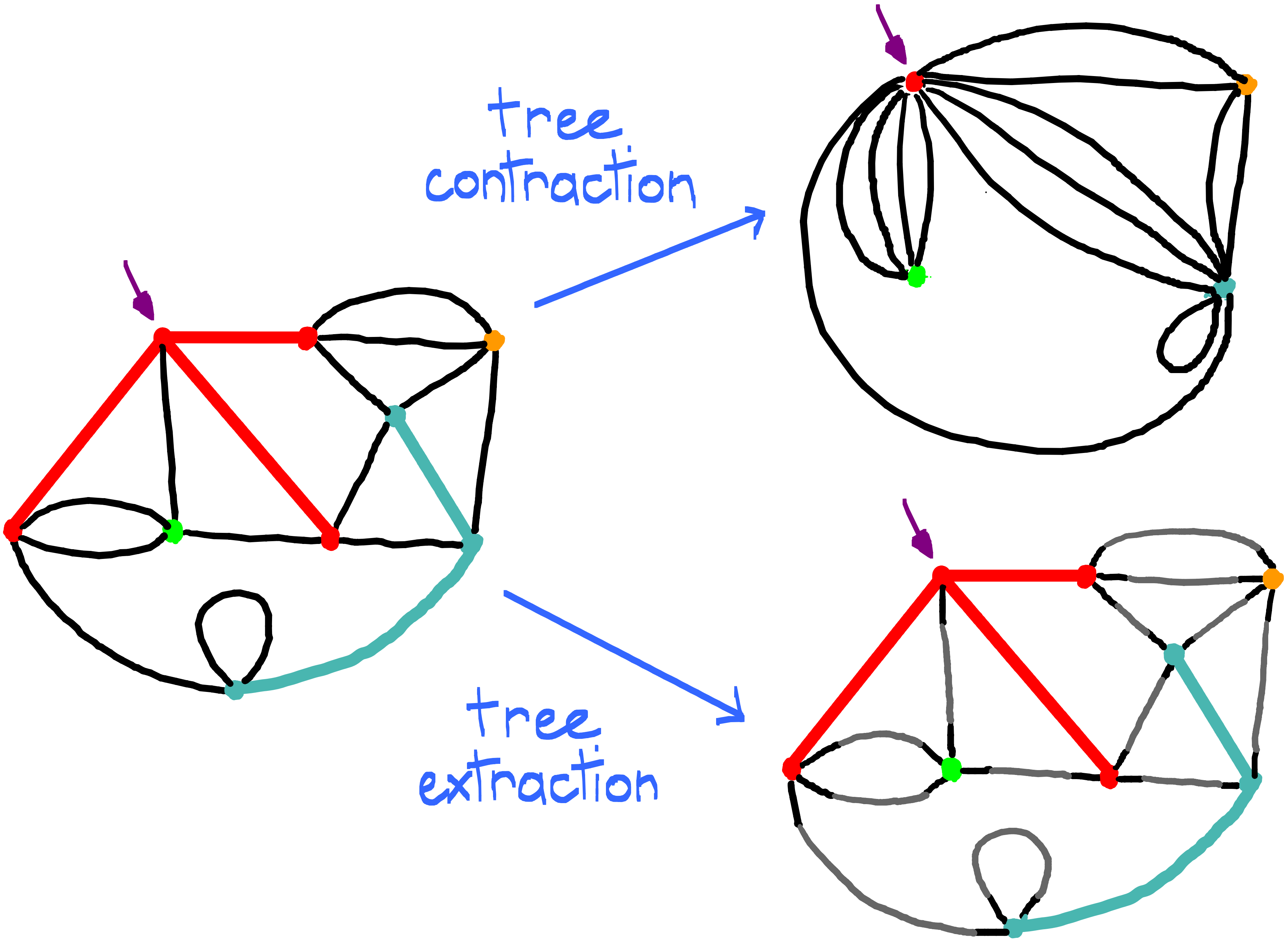
FROM FORESTED TO GENERAL MAPS



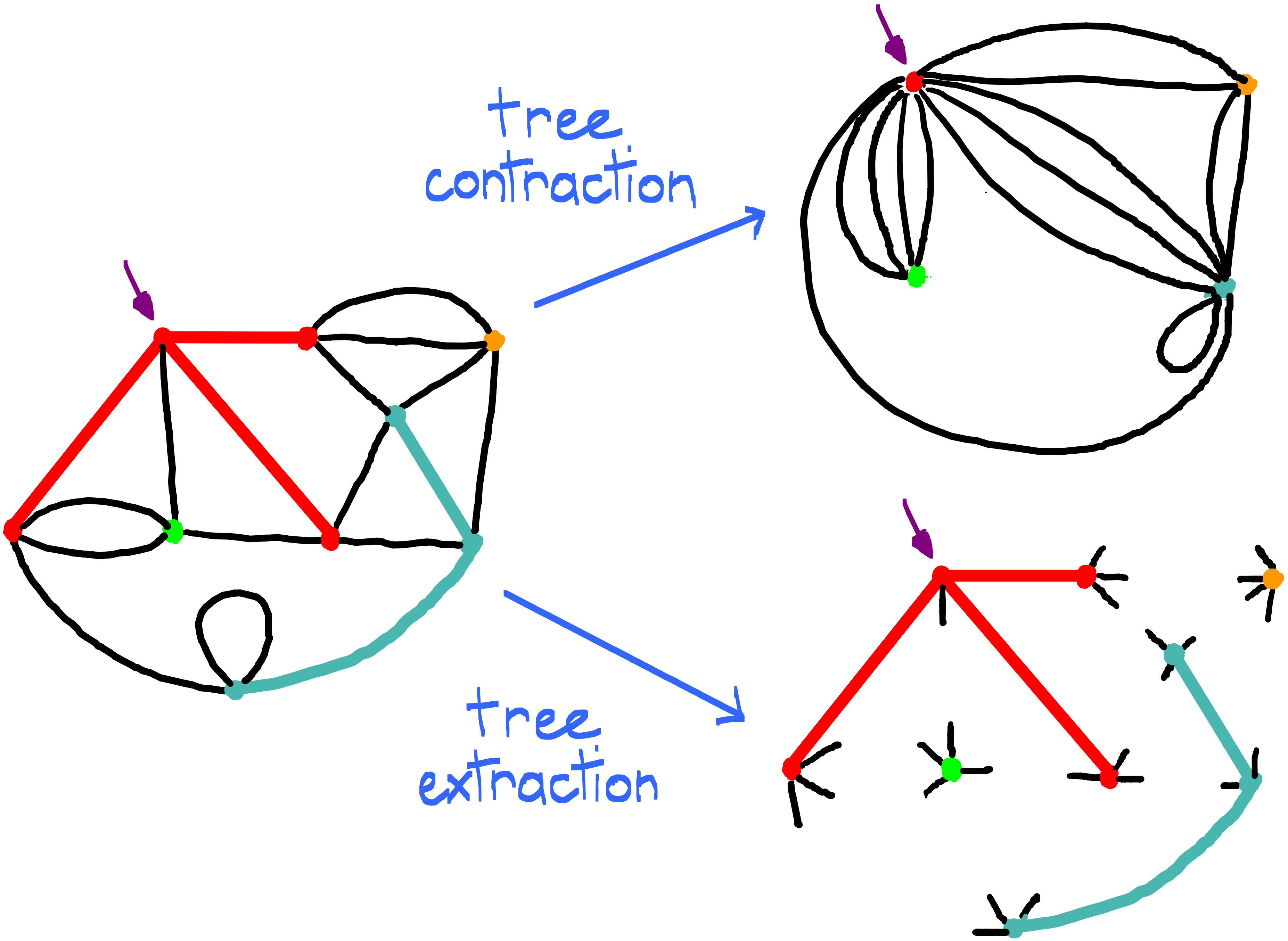
FROM FORESTED TO GENERAL MAPS



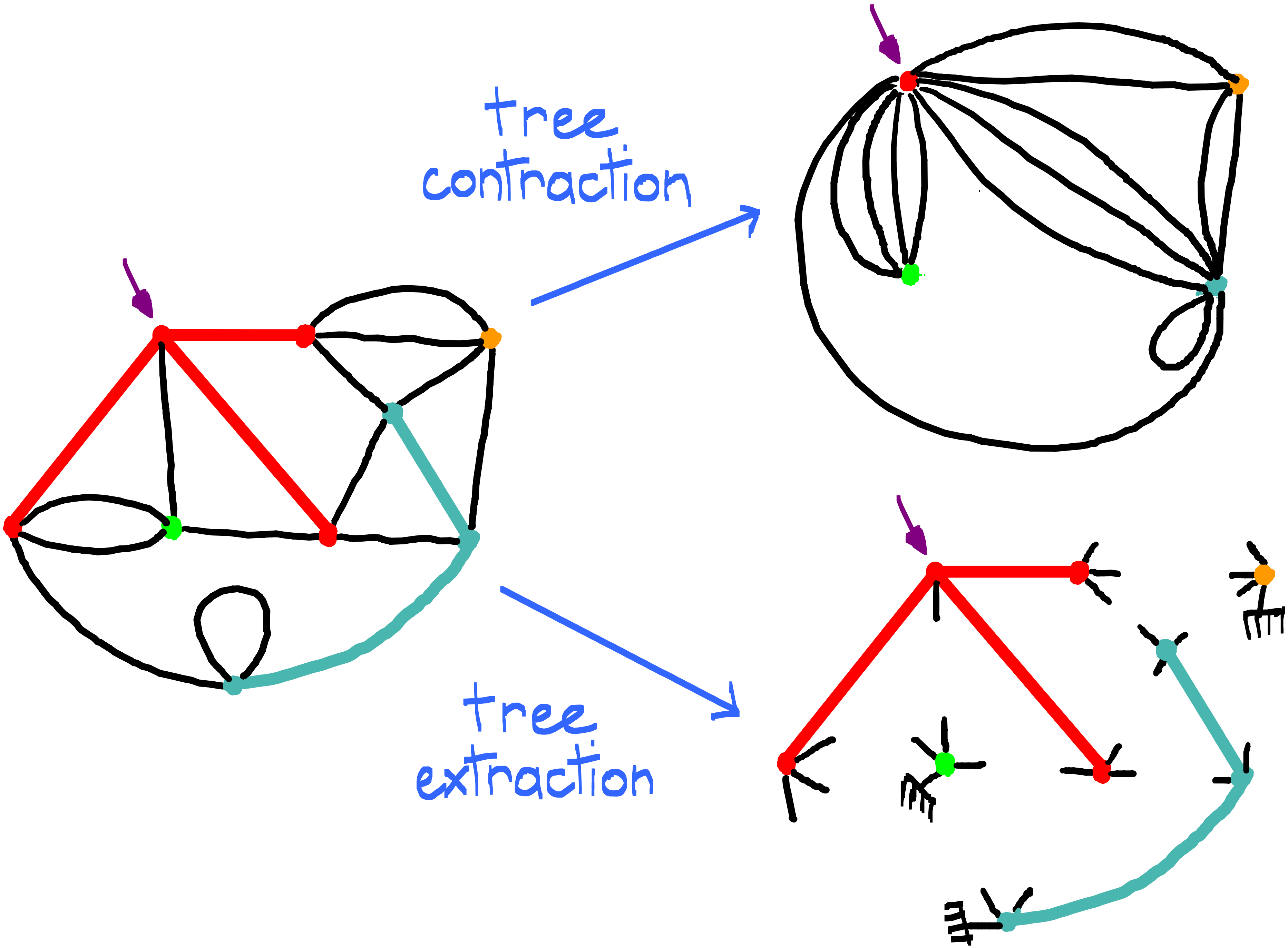
FROM FORESTED TO GENERAL MAPS



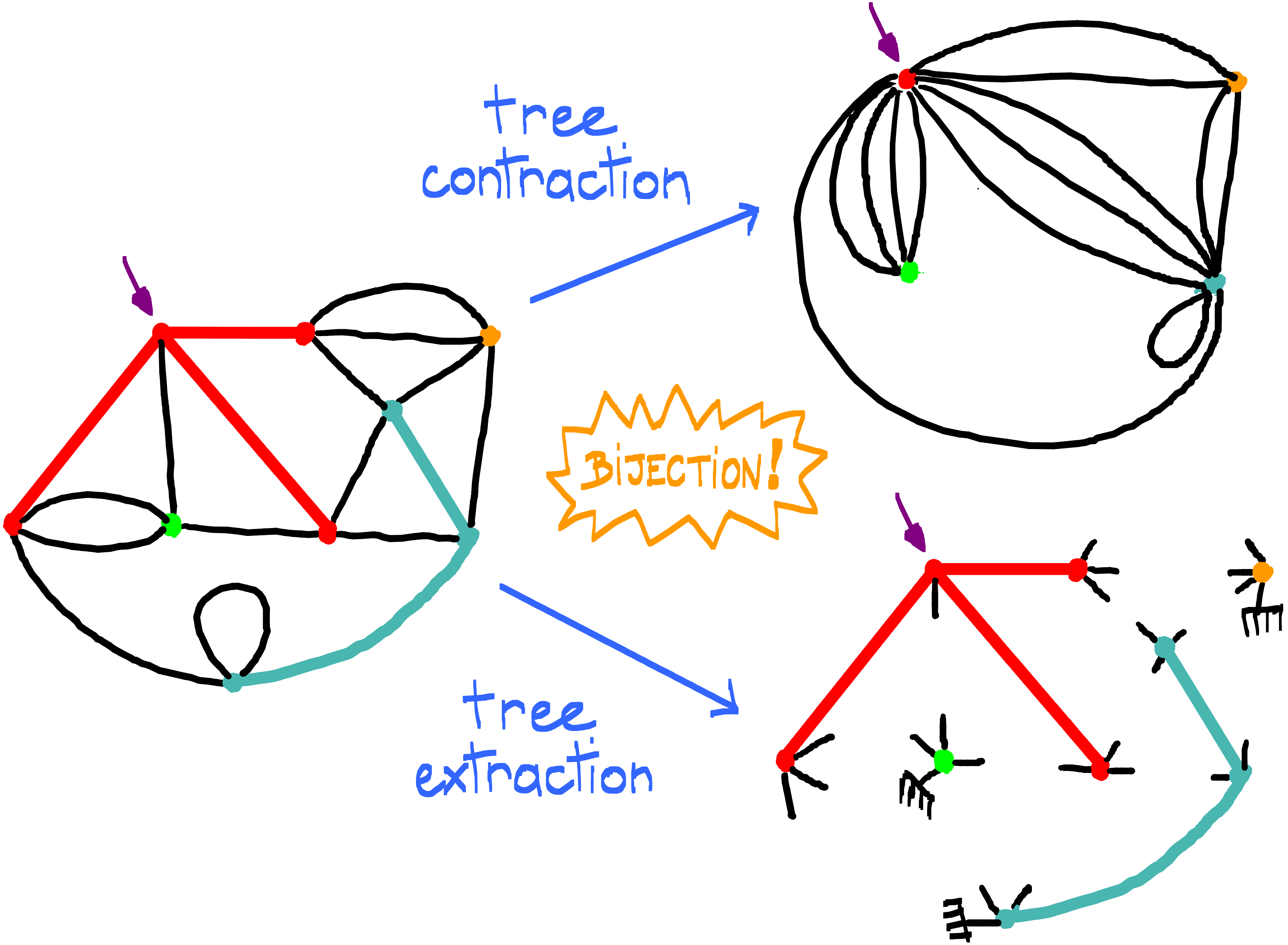
FROM FORESTED TO GENERAL MAPS



FROM FORESTED TO GENERAL MAPS



FROM FORESTED TO GENERAL MAPS



TRANSLATION INTO GENERATING FUNCTIONS

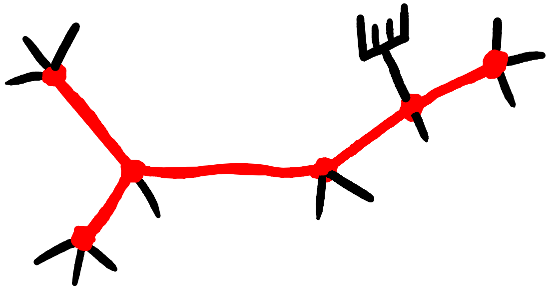
$M(z, u; g_1, g_2, g_3, \dots; h_1, h_2, h_3, \dots) =$

Generating function of rooted maps with a weight:

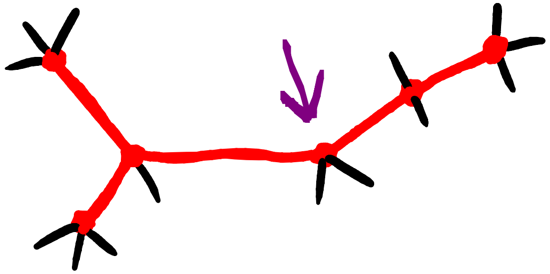
- z per face,
- $u\, g_k$ per non-root vertex of degree k,
- h_k if the root vertex has degree k.

$$\boxed{F(z, u) = M(z, u; t_1, t_2, t_3, \dots; t_1^c, t_2^c, t_3^c, \dots)}$$

t_k = # 4-valent leaf-rooted trees with k leaves



t_k^c = # 4-valent corner-rooted trees with k leaves



GENERATING FUNCTION FOR GENERAL MAPS

$M(z, u; g_1, g_2, g_3, \dots; h_1, h_2, h_3, \dots) =$

Generating function of rooted maps with a weight:

- z per face,
- $u \cdot g_k$ per non-root vertex of degree k,
- h_k if the root vertex has degree k.

This generating function is known.

cf [Bouttier - Guitter, 2012]

(M' is even nicer.)

Notation: $X' = \frac{\partial X}{\partial z}$

THE GENERATING FUNCTION OF FORESTED MAPS

Theorem

There exists a unique series R in z with coefficients in $\mathbb{Q}[u]$ such that

$$R = z + u \sum_{i \geq 2} \frac{(3i-3)!}{(i-1)!^2\, i!} R^i$$

Then :

$$F' = 4 \sum_{i \geq 2} \frac{(3i-2)!}{(i-2)!\; i!^2} R^i$$

THE GENERATING FUNCTION OF FORESTED MAPS

Theorem

There exists a unique series R in z with coefficients in $\mathbb{Q}[u]$ such that

$$R = z + u \sum_{i \geq 2} \frac{(3i-3)!}{(i-1)!^2 \, i!} R^i$$

Then :

$$F' = 4 \sum_{i \geq 2} \frac{(3i-2)!}{(i-2)! \, i!^2} R^i$$

For $u=0$, [Mullin]

$R = z$ and $F' = 4 \sum_{i \geq 2} \frac{(3i-2)!}{(i-2)! \, i!^2} z^i$ D-finite.

THE GENERATING FUNCTION OF FORESTED MAPS

Theorem

There exists a unique series R in z with coefficients in $\mathbb{Q}[u]$ such that

$$R = z + u \quad \phi(R)$$

Then :

$$F' = \quad \Theta(R)$$

where

$$\phi(x) = \sum_{i \geq 2} \frac{(3i-3)!}{(i-1)!^2 \, i!} x^i, \qquad \Theta(x) = 4 \sum_{i \geq 2} \frac{(3i-2)!}{(i-2)! \; i!^2} x^i .$$

A DIFFERENTIAL EQUATION FOR F

$$R = z + u\,\phi(R) \qquad F' = \Theta(R)$$

Prop F is D-algebraic.

(Fundamental reason : ϕ and Θ are D-finite.)

A DIFFERENTIAL EQUATION FOR F

$$R = z + u\,\phi(R) \qquad F' = \Theta(R)$$

Prop F is D-algebraic.

(Fundamental reason : ϕ and Θ are D-finite.)

cf Bernardi - Bousquet-Mélou 's result:

The Potts generating function of planar maps is D-algebraic.

(established in a more painful way.)

A DIFFERENTIAL EQUATION FOR F

$$R = z + u\,\phi(R) \qquad F' = \Theta(R)$$

Prop F is D-algebraic.

(Fundamental reason: ϕ and Θ are D-finite.)

Can a differential equation for F be explicitly computed?

A DIFFERENTIAL EQUATION FOR F

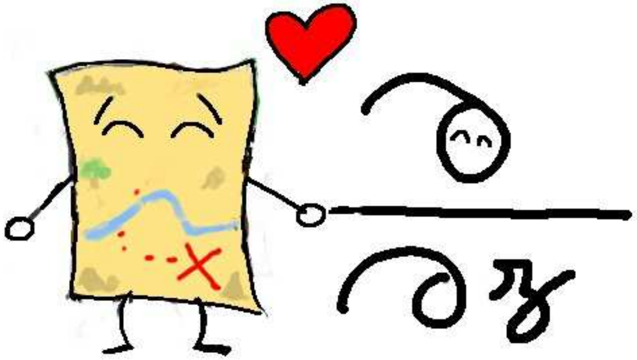
$$R = z + u\,\phi(R) \qquad F' = \Theta(R)$$

Prop F is D-algebraic.

(Fundamental reason : ϕ and Θ are D-finite.)

Can a differential equation for F be explicitly computed?

YES!



A DIFFERENTIAL EQUATION FOR F

$$\begin{aligned}
&9F'^2F''^5u^6 + 36F'^2F''^3F'''u^5z + 144F'^2F''^4u^5 - 12(21z-1)F'F''^5u^5 + 432F'^2F''^2F'''u^4z \\
&- 48(24z-1)F'F''^3F'''u^4z + 864F'^2F''^3u^4 - 96(27z-2)F'F''^4u^4 + 4(27z-1)(15z-1)F''^5u^4 \\
&+ 1728F'^2F''F'''u^3z - 288(21z-2)F'F''^2F'''u^3z + 10368F'F'''^2u^2z^3 + 16(27z-1)(21z-1)F''^3F'''u^3z \\
&+ 2304F'^2F''^2u^3 - 288(31z-4)F'F''^3u^3 - 64(6uz-162z^2+33z-1)F''^4u^3 + 2304F'^2F'''u^2z \\
&- 2304(6z-1)F'F''F'''u^2z - 192(8uz-54z^2+29z-1)F''^2F'''u^2z - 768(2u+189z-7)F'''^2uz^3 \\
&+ 2304F'^2F''u^2 - 3072(3z-1)F'F''^2u^2 - 192(24uz-27z^2+55z-2)F''^3u^2 - 1536(21z-2)F'F'''uz \\
&- 768(12uz+81z^2+24z-1)F''F'''uz + 1536(9z+2)F'F''u - 512(39uz+81z^2+51z-2)F''^2u \\
&+ 36864F'z - 1024(12uz-162z^2+33z-1)F'''z - 1024(36uz+27z-1)F'' - 24576z = 0.
\end{aligned}$$

Differential equation of order 2 in F' and degree 7.

(but not in F)

A DIFFERENTIAL EQUATION FOR F

$$\begin{aligned}
&9F'^2F''^5u^6 + 36F'^2F''^3F'''u^5z + 144F'^2F''^4u^5 - 12(21z-1)F'F''^5u^5 + 432F'^2F''^2F'''u^4z \\
&- 48(24z-1)F'F''^3F'''u^4z + 864F'^2F''^3u^4 - 96(27z-2)F'F''^4u^4 + 4(27z-1)(15z-1)F''^5u^4 \\
&+ 1728F'^2F''F'''u^3z - 288(21z-2)F'F''^2F'''u^3z + 10368F'F'''^2u^2z^3 + 16(27z-1)(21z-1)F''^3F'''u^3z \\
&+ 2304F'^2F''^2u^3 - 288(31z-4)F'F''^3u^3 - 64(6uz-162z^2+33z-1)F''^4u^3 + 2304F'^2F'''u^2z \\
&- 2304(6z-1)F'F''F'''u^2z - 192(8uz-54z^2+29z-1)F''^2F'''u^2z - 768(2u+189z-7)F'''^2uz^3 \\
&+ 2304F'^2F''u^2 - 3072(3z-1)F'F''^2u^2 - 192(24uz-27z^2+55z-2)F''^3u^2 - 1536(21z-2)F'F'''uz \\
&- 768(12uz+81z^2+24z-1)F''F'''uz + 1536(9z+2)F'F''u - 512(39uz+81z^2+51z-2)F''^2u \\
&+ 36864F'z - 1024(12uz-162z^2+33z-1)F'''z - 1024(36uz+27z-1)F'' - 24576z = 0.
\end{aligned}$$

Differential equation of order 2 in F' and degree 7.

(but not in F)

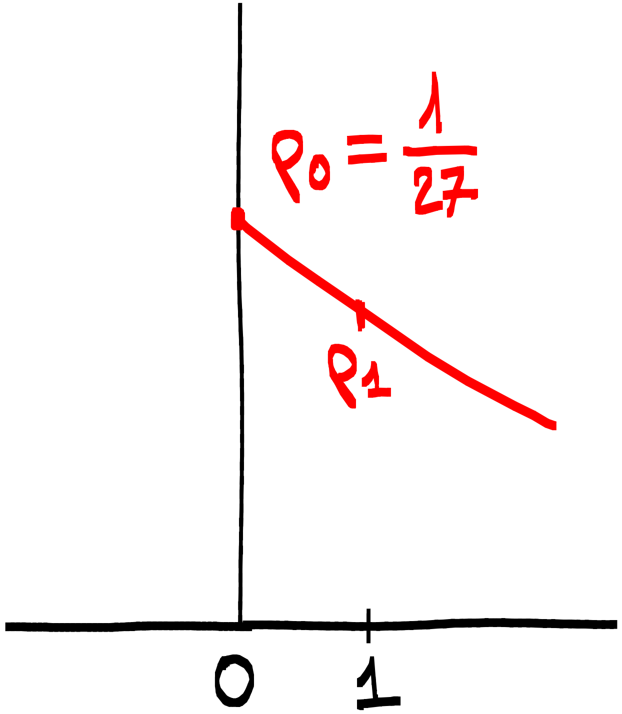
(Moreover, this is the equation with minimal order.)

↑ Thanks Alin Bostan, Bruno Salvy & Michael Singer!

RADIUS OF CONVERGENCE

Fix u,

ρ_u = radius of convergence of $F(z,u) = \sum_n f_n(u)\, z^n$.



$$\begin{cases} \rho_u = \tau_u - u\,\phi(\tau_u) \\ \phi'(\tau_u) = \frac{1}{u} \end{cases}$$

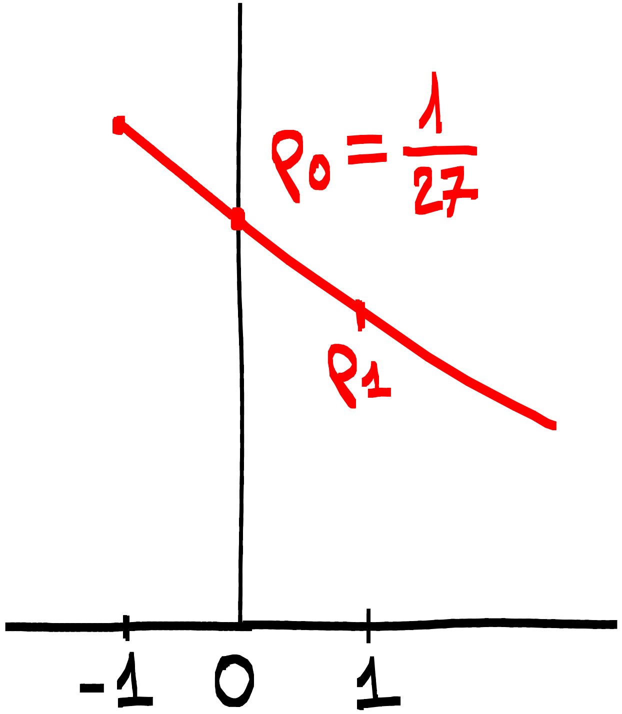
$(u > 0)$

RADIUS OF CONVERGENCE

Fix u in $[-1,+\infty)$,

ρ_u = radius of convergence of $F(z,u)=\sum_n f_n(u)\, z^n$.

ρ_u is affine on $[-1,0]$!



$$\begin{cases} \rho_u = \tau_u - u\,\phi(\tau_u) \\ \phi'(\tau_u) = \frac{1}{u} \end{cases}$$

$(u>0)$

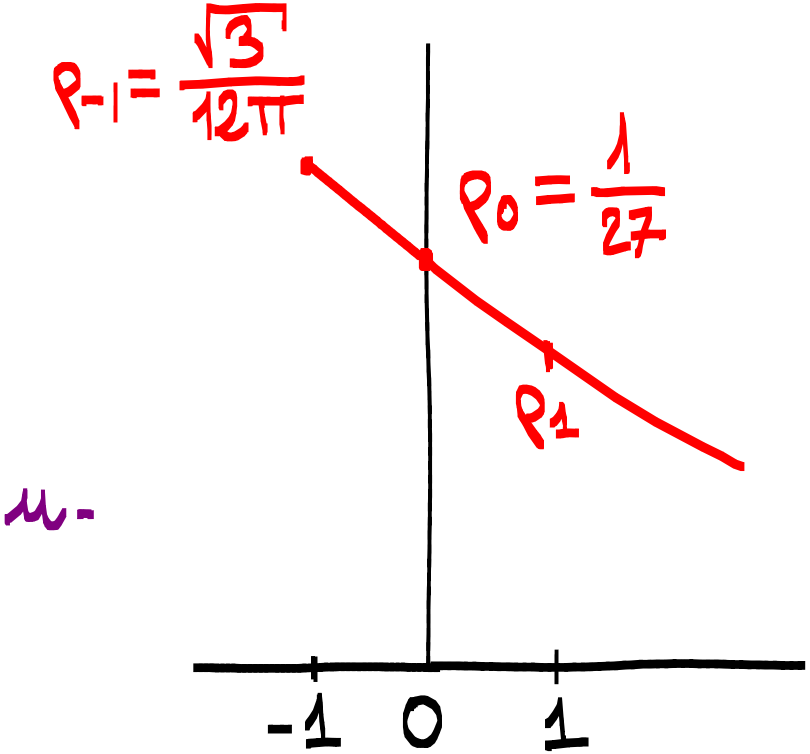
RADIUS OF CONVERGENCE

Fix u in $[-1,+\infty)$,

ρ_u = radius of convergence of $F(z,u)=\sum_n f_n(u)\, z^n$.

ρ_u is affine on $[-1,0]$!

$$\rho_u = \frac{1}{27}(1+u) - \frac{\sqrt{3}}{12\pi}u.$$



$$\begin{cases} \rho_u = \tau_u - u\,\phi(\tau_u) \\ \phi'(\tau_u) = \frac{1}{u} \end{cases}$$

$(u>0)$

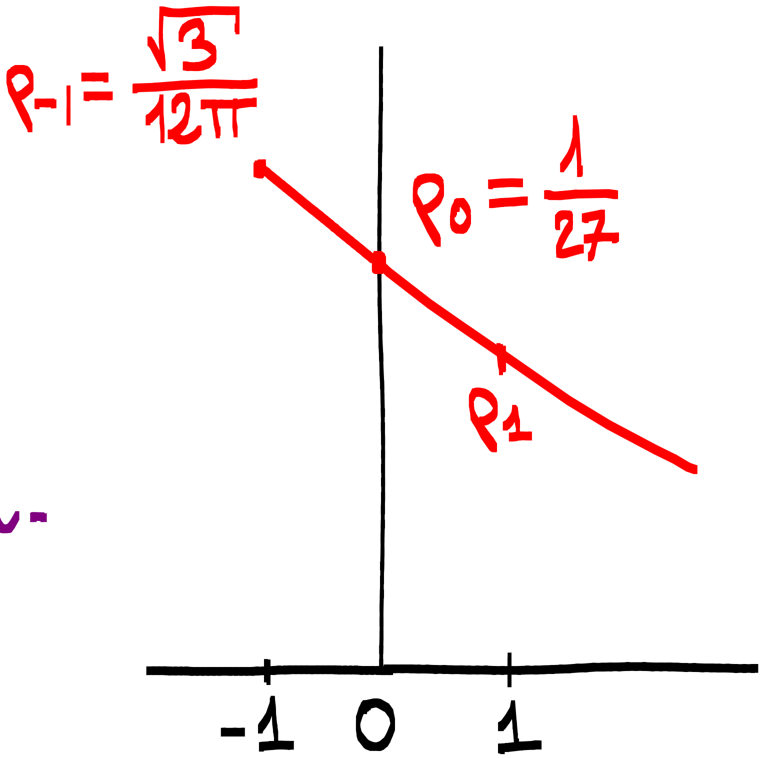
RADIUS OF CONVERGENCE

Fix u in $[-1,+\infty)$,

ρ_u = radius of convergence of $F(z,u)=\sum_n f_n(u)\, z^n$.

ρ_u is affine on $[-1,0]$!

$$\rho_u = \frac{1}{27}(1+u) - \frac{\sqrt{3}}{12\pi}u.$$



$$\begin{cases}\rho_u = \tau_u - u\,\phi(\tau_u)\\ \phi'(\tau_u)=\frac{1}{u}\end{cases}$$

$(u>0)$

Cor ρ_{-1} is transcendental:
$F(z,-1)$ is not D-finite.

PHASE TRANSITION AT 0

$f_n(u) = [z^n]\, F(z, u)$

$-1 \leqslant u < 0$

$$f_n(u) \sim \frac{c_u\, \rho_u^{-n}}{n^3 \ln^2 n}$$

New "Universality class" for maps

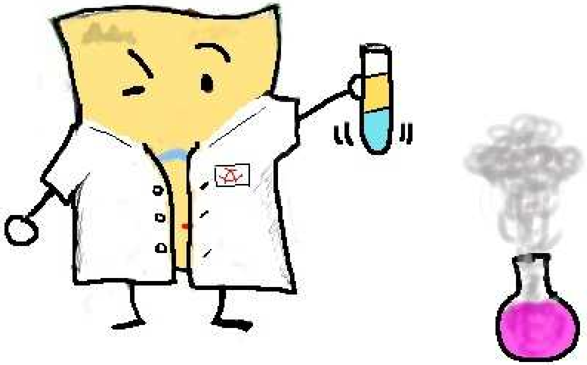
$u = 0$

$$f_n(u) \sim \frac{c_u\, \rho_u^{-n}}{n^3}$$

maps with a spanning tree

$0 < u$

$$f_n(u) \sim \frac{c_u\, \rho_u^{-n}}{n^{\underbrace{5/2}_{\text{standard}}}}$$



PHASE TRANSITION AT 0

$$f_n(u) = [z^n]\, F(z,u)$$

$-1 \leqslant u < 0$	$u = 0$	$0 < u$
$f_n(u) \sim \dfrac{c_u\, \rho_u^{-n}}{n^3 \ln^2 n}$	$f_n(u) \sim \dfrac{c_u\, \rho_u^{-n}}{n^3}$	$f_n(u) \sim \dfrac{c_u\, \rho_u^{-n}}{n^{\underbrace{5/2}}}$
New "Universality class" for maps	maps with a spanning tree	standard

D-finite

Cor For $u \in [-1,0)$, $F(z,u)$ is not D-finite.

IDEA OF THE PROOF

Singularity analysis [Flajolet - Odlyzko]

A link between the singular behaviour of $F(z,u)$ near ρ_u and the asymptotic behaviour of $f_n(u)$.

IDEA OF THE PROOF

Singularity analysis [Flajolet - Odlyzko]

A link between the singular behaviour of $F(z,u)$ near ρ_u and the asymptotic behaviour of $f_n(u)$.

$R = z + u\,\phi(R)$ radius of convergence of $\phi = \frac{1}{27}$

IDEA OF THE PROOF

Singularity analysis [Flajolet - Odlyzko]

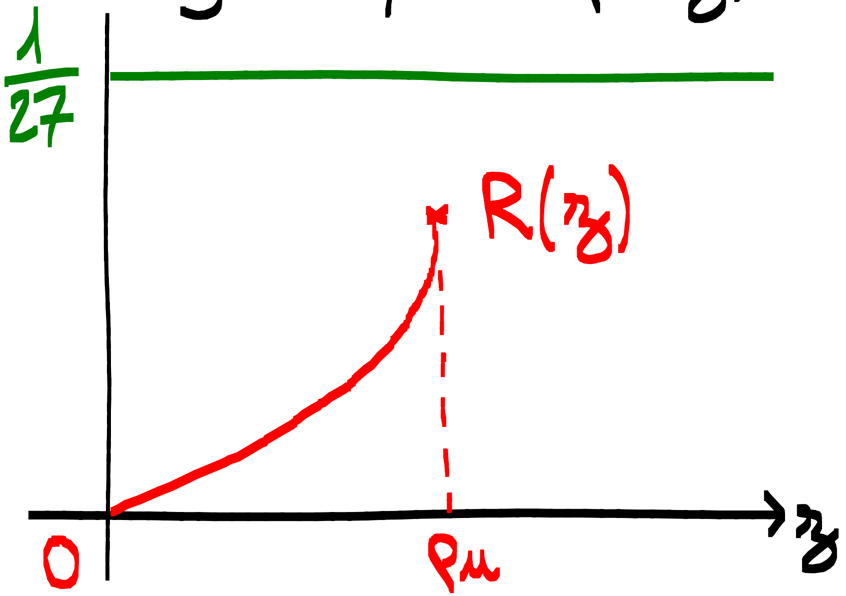
A link between the singular behaviour of $F(z,u)$ near ρ_u and the asymptotic behaviour of $f_n(u)$.

$R = z + u\,\phi(R)$

radius of convergence of $\phi = \frac{1}{27}$

$\boxed{u > 0}$

singularity in $(\rho_u - z)^{\frac{1}{2}}$



IDEA OF THE PROOF

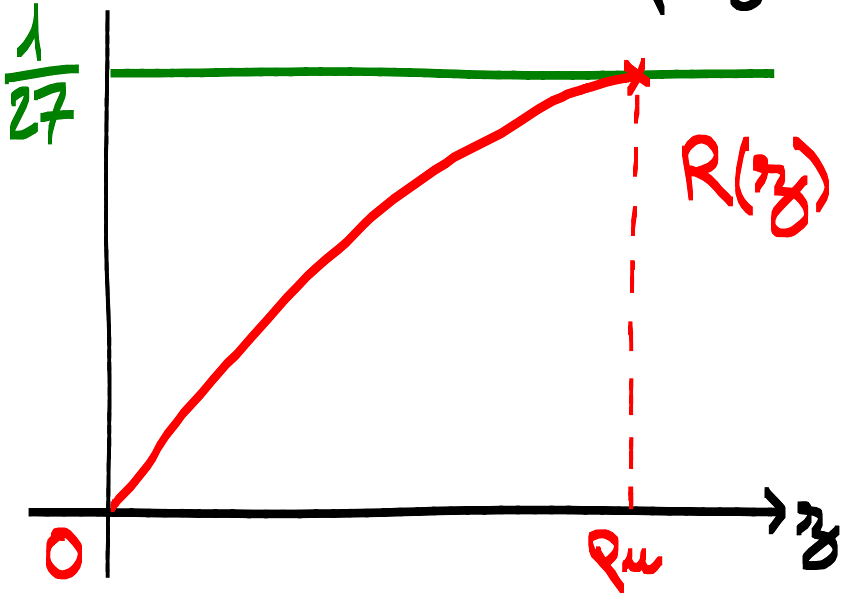
Singularity analysis [Flajolet - Odlyzko]

A link between the singular behaviour of $F(z,u)$ near ρ_u and the asymptotic behaviour of $f_n(u)$.

$R = z + u\,\phi(R)$ radius of convergence of $\phi = \frac{1}{27}$

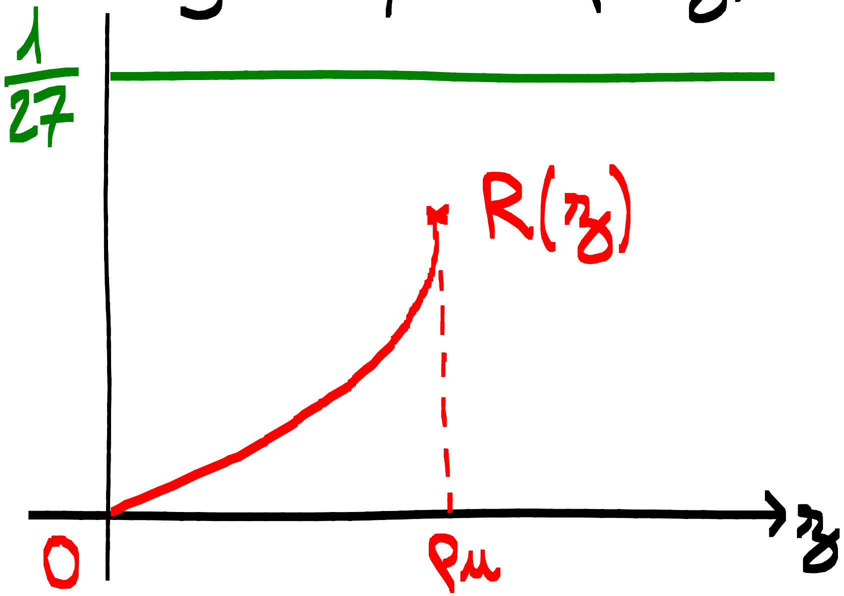
$\boxed{u<0}$

singularity in $\frac{\rho_u - z}{\ln(\rho_u - z)}$



$\boxed{u>0}$

singularity in $(\rho_u - z)^{\frac{1}{2}}$



IDEA OF THE PROOF

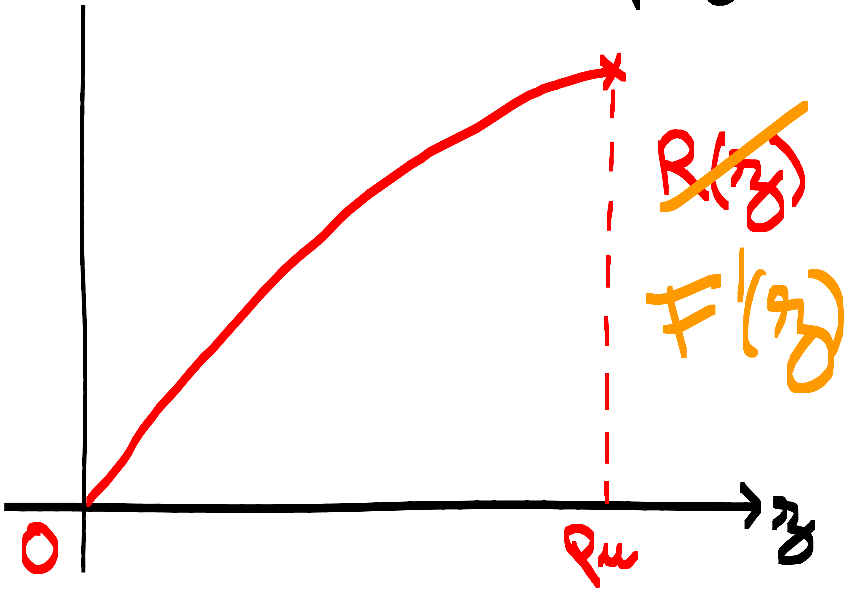
Singularity analysis [Flajolet - Odlyzko]

A link between the singular behaviour of $F(z,u)$ near ρ_u and the asymptotic behaviour of $f_n(u)$.

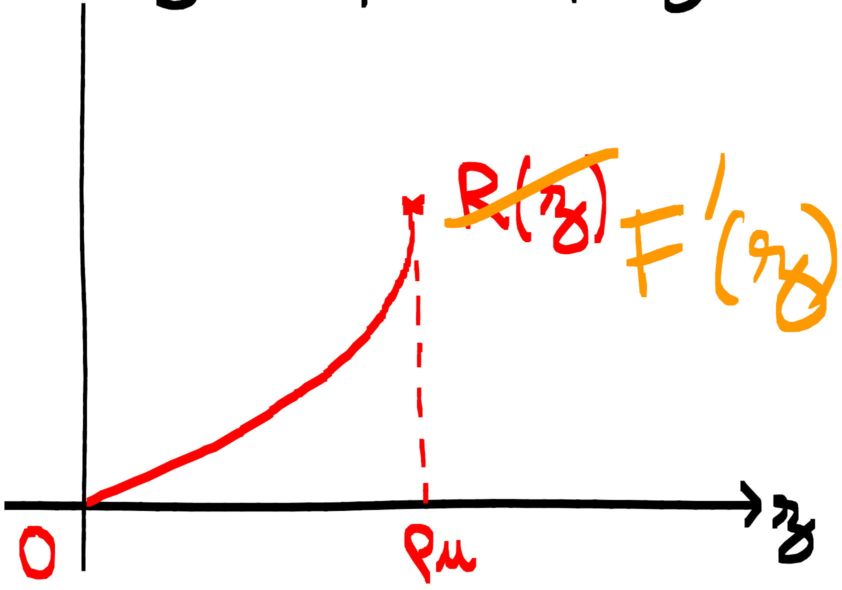
$R = z + u\phi(R)$ radius of convergence of $\phi = \frac{1}{27}$

$\boxed{u<0}$ $\boxed{u>0}$

singularity in $\frac{\rho_u - z}{\ln(\rho_u - z)}$

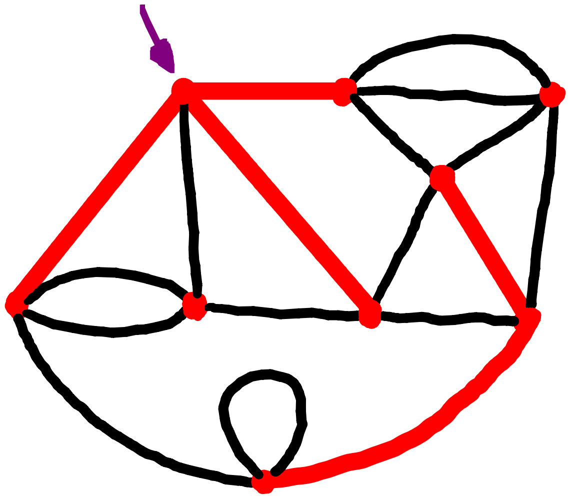


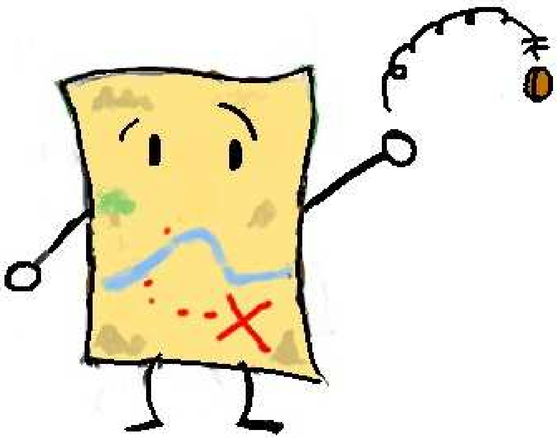
singularity in $(\rho_u - z)^{\frac{1}{2}}$



SOME PROBABILITY RESULTS

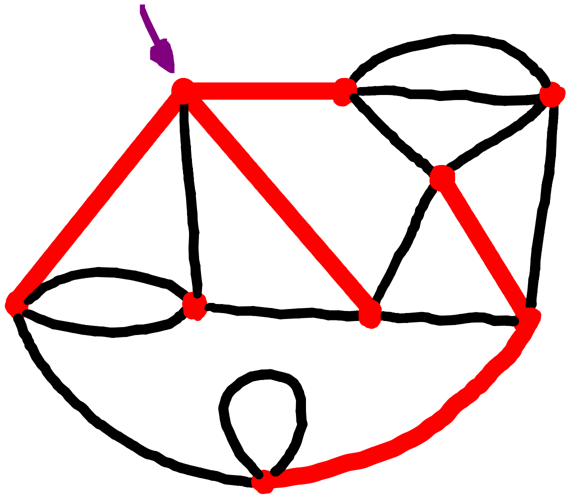
Fix $n \in \mathbb{N}$,
consider a random forested
map with n faces.
(under uniform distribution)





SOME PROBABILITY RESULTS

Fix $n \in \mathbb{N}$,
consider a random forested map with n faces.
(under uniform distribution)

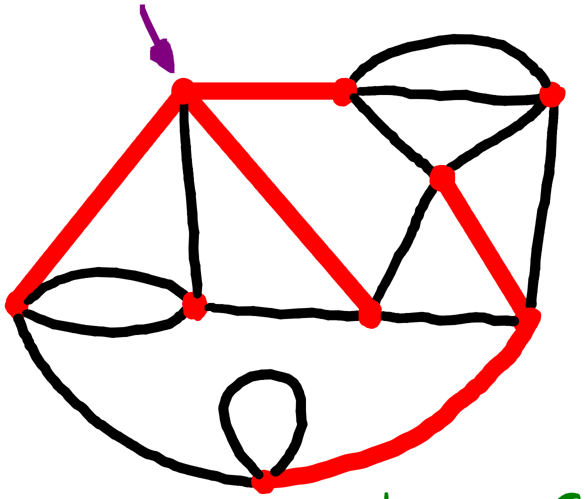


C_n = r.v that counts the number of components

Th $C_n \xrightarrow[\text{distribution}]{}$ gaussian law with linear mean & linear variance.

SOME PROBABILITY RESULTS

Fix $n \in \mathbb{N}$, consider a random forested map with n faces. (under uniform distribution)



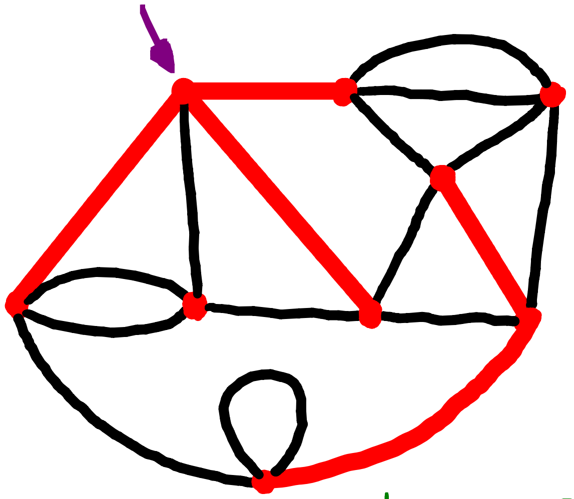
Here $C_n = 4$

C_n = r.v that counts the number of components

Th $C_n \xrightarrow[\text{distribution}]{}$ gaussian law with linear mean & linear variance.

SOME PROBABILITY RESULTS

Fix $n \in \mathbb{N}$,
consider a random forested
map with n faces.
(under uniform distribution)



Here $S_n = 4$

S_n = size of the root component (number of vertices)

Th

$$\lim_{n \to +\infty} \mathbb{P}_n (S_n = k) = \frac{4\,(3k)!}{(k-1)!\,k!\,(k+1)!} \frac{\tau_1^k}{\phi'(\tau_1)}$$

EXTENSION OF THE RESULTS

Fix a set of permitted vertex degrees -

<table>
<tr><th></th><th colspan="2">Eulerian</th><th colspan="2">not Eulerian</th></tr>
<tr><td>Functional system</td><td colspan="2">$R = z + u\,\phi(R)$
$F' = \Theta(R)$</td><td colspan="2">$R = z + u\,\phi_1(R,S)$
$S = u\,\phi_2(R,S)$
$F' = \Theta(R,S)$</td></tr>
<tr><td>Nature of $F(z,u)$</td><td colspan="4">D-algebraic if the set of permitted degrees is a finite union of arithmetic progressions.</td></tr>
<tr><td></td><td>aperiodic</td><td>periodic</td><td>aperiodic</td><td>periodic</td></tr>
<tr><td>Asymptotic behaviour</td><td>4-valent ✓
Eulerian general maps ✓</td><td>(2ℓ)-valent
$\ell \geq 3$ ✓</td><td>cubic ✓
general maps ?</td><td>$(2\ell+1)$-valent, $\ell \geq 2$?</td></tr>
</table>

EXTENSION OF THE RESULTS

Fix a set of permitted vertex degrees –

coupled equations ☹

	Eulerian		not Eulerian	
Functional system	$R = z + u\,\phi(R)$ $F' = \Theta(R)$		$R = z + u\,\phi_1(R,S)$ $S = u\,\phi_2(R,S)$ $F' = \Theta(R,S)$	
Nature of $F(z,u)$	D-algebraic if the set of permitted degrees is a finite union of arithmetic progressions.			
	aperiodic	periodic	aperiodic	periodic
Asymptotic behaviour	4-valent ✓ Eulerian general maps ✓	(2ℓ)-valent $\ell \geq 3$ ✓	cubic ✓ general maps ?	$(2\ell+1)$-valent, $\ell \geq 2$?

EXTENSION OF THE RESULTS

Fix a set of permitted vertex degrees –

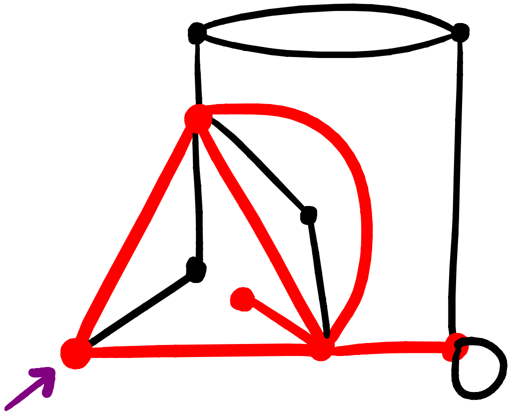
coupled equations ☹

	Eulerian		not Eulerian	
Functional system	$R = z + u\,\phi(R)$ $F' = \theta(R)$		$R = z + u\,\phi_1(R,S)$ $S = u\,\phi_2(R,S)$ $F' = \theta(R,S)$	
Nature of $F(z,u)$	D-algebraic if the set of permitted degrees is a finite union of arithmetic progressions.			
	aperiodic	periodic	aperiodic	periodic
Asymptotic behaviour	4-valent ✓ Eulerian general maps ✓	(2ℓ)-valent $\ell \geq 3$ ✓	cubic ✓ general maps ?	$(2\ell+1)$-valent, $\ell \geq 2$?
			Prospects	

OTHER PROSPECTS

→ Go further into probability results.

→ Maps equipped with a bond animal.

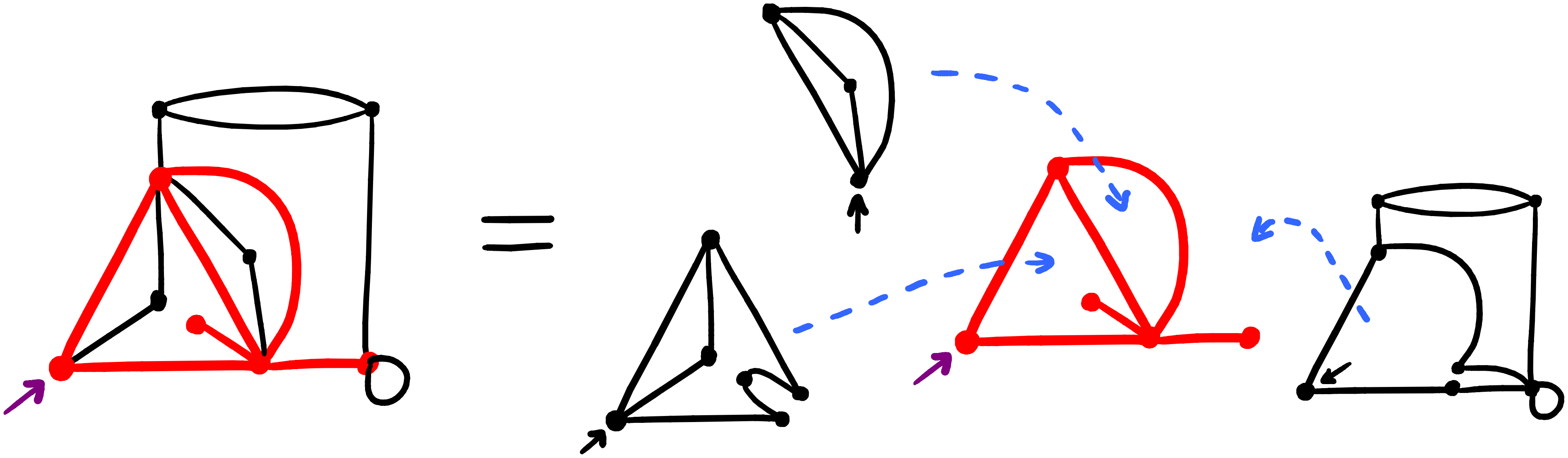


Objective: Bond percolation on maps.

OTHER PROSPECTS

→ Go further into probability results.

→ Maps equipped with a bond animal.



Objective: Bond percolation on maps.

THANK YOU! AND

HAPPY HALLOWEEN!