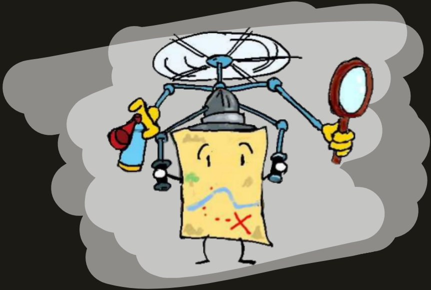
# ANALYSIS OF PARAMETERS FOR LARGE COMBINATORIAL MAPS

Julien **COURTIEL** (LIPN, Paris 13)



WORK IN PROGRESS

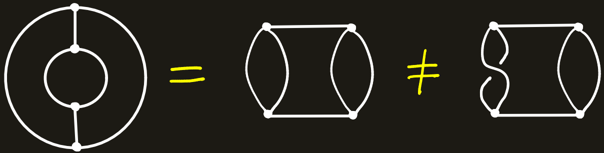
ALEA Young Researcher 2017

Co-authors: Olivier BODINI (Paris 13), Hsien-Kuei HWANG (Taïwan)

# DEFINITION

**combinatorial map** = connected graph where we have cyclically ordered the half-edges around each vertex.

Examples:



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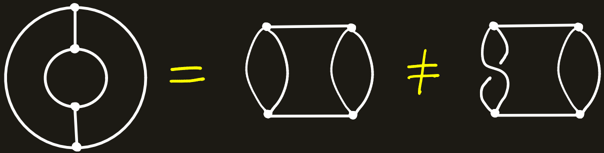
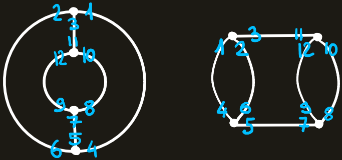


Why is  the same as  ?



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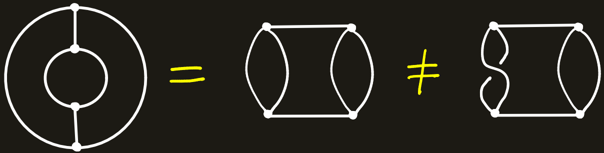

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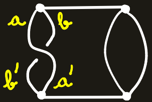


Absent pattern in  :

# DEFINITION

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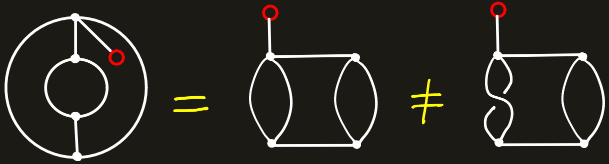

We root every map on a leaf.

# DEFINITION

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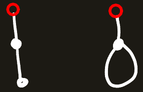
1 edge

(1)

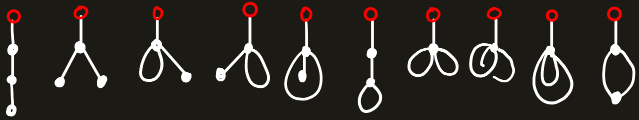
2 edges

(2)



3 edges

(10)



# RECURRENCE FORMULA

$c_n$ = number of combinatorial maps with $n$ edges

Recurrence formula : [Arquès Béraud]

$$c_1 = 1 \qquad c_n = \sum_{k=1}^{n-1} c_k c_{n-k} + (2n-3)\, c_{n-1}$$

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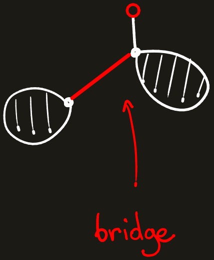
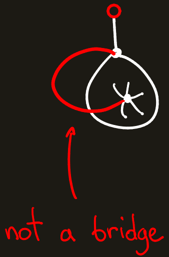
map =  or 

or 

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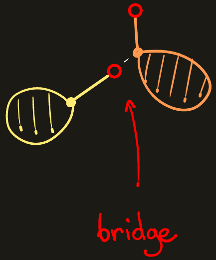
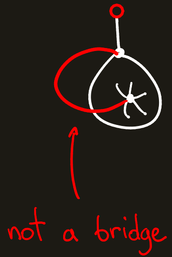
map =  or 

or 

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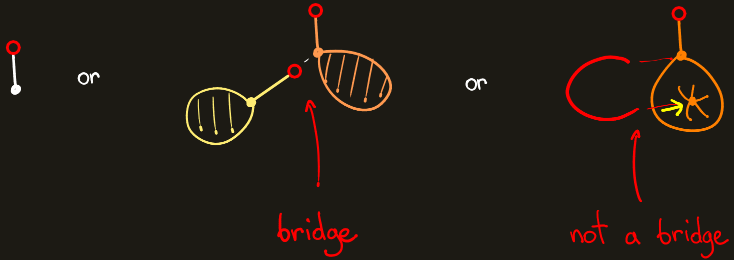

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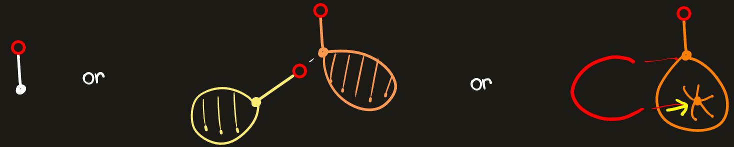
map = 

Generating function : $C(z) = \sum_{n \geq 0} c_n z^n$

$$C(z) = z + C(z)^2 + z\left(2z\frac{\partial C(z)}{\partial z} - C(z)\right)$$

# WHY COUNTING MAPS WITH NO CONSIDERATION FOR GENUS?

→ Good framework to study parametric Riccati equations.

→ Connections with other combinatorial families.

- indecomposable chord diagrams (link with the Quantum Fields Theory)
- lambda-terms

# WHY COUNTING MAPS WITH NO CONSIDERATION FOR GENUS?

Part 2

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Part 1

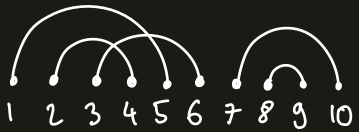
PART I

# Connections with other combinatorial families

# CHORD DIAGRAMS

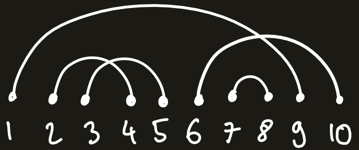
**diagram of $n$ chords**

= matching of the set $\{1, \ldots, 2n\}$




**indecomposable diagram**

= diagram that is not the concatenation of two diagrams.



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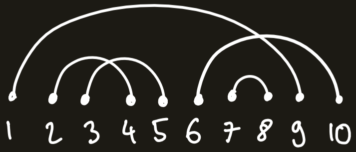
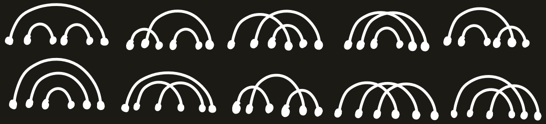


# CHORD DIAGRAMS

1 chord  (1)

2 chords (2)

3 chords (10)



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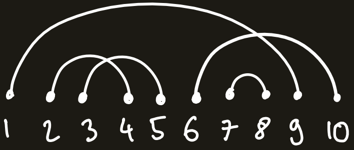

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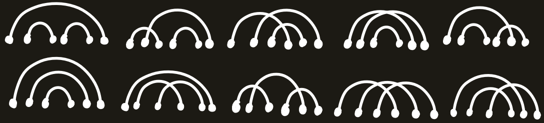

**Proposition** [Cvitanović Lautrup Pearson, Ossona de Mendez Rosenstiehl, Cori]

number of combinatorial maps with $n$ edges

= number of indecomposable diagrams with $n$ chords

# RECURRENCE FORMULA : THE COMEBACK

$c_n$ = number of indecomposable diagrams with $n$ chords

Recurrence formula :

$$c_1 = 1 \qquad c_n = \sum_{k=1}^{n-1} c_k c_{n-k} + (2n-3)\, c_{n-1}$$

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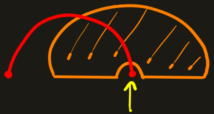
indecomposable diagram =  ou  ou 

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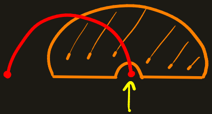
indecomposable diagram =  ou  ou 

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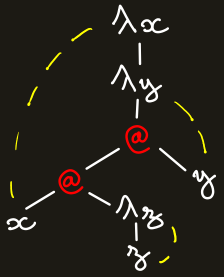
# LINEAR LAMBDA-TERMS

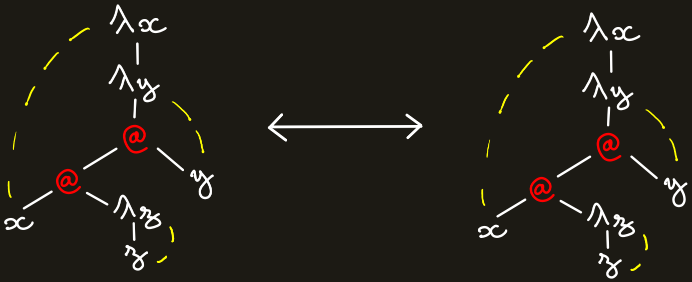



$\lambda x. \lambda y ((x \;\; \lambda z. z)\; y)$

linear lambda-term =
Motzkin tree where each leaf is bound by a unary vertex and each vertex binds exactly one leaf.

**Theorem** [Bodini Gardy Gittenberg Jacquot]
linear lambda-terms $\longleftrightarrow$ trivalent maps

# LINEAR LAMBDA-TERMS

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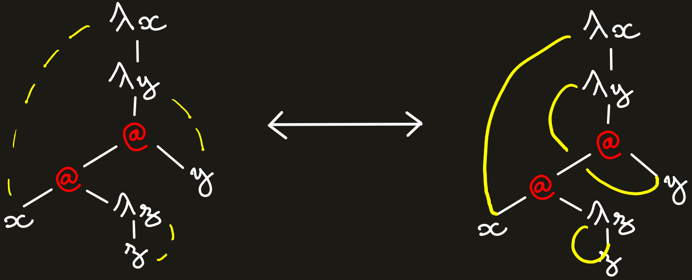


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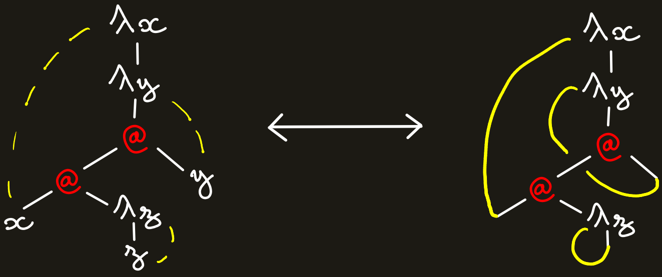


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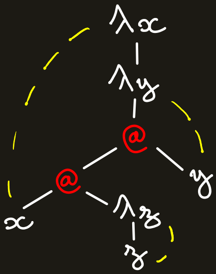


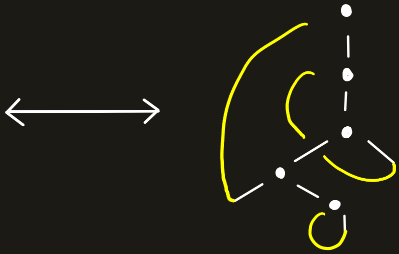

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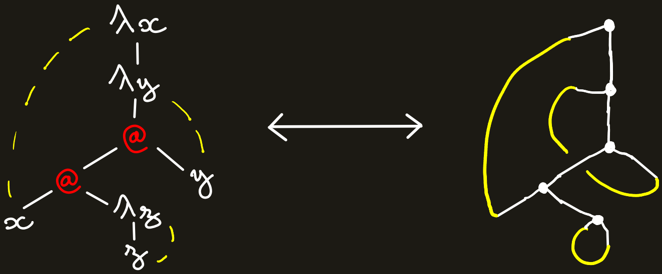


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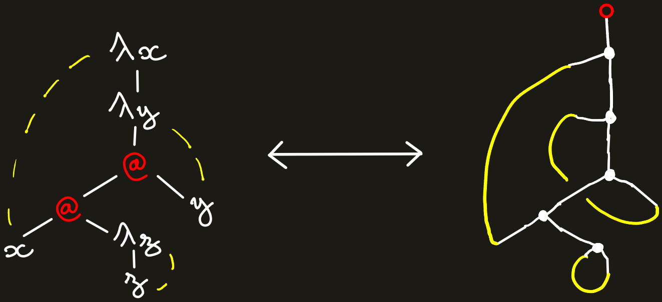


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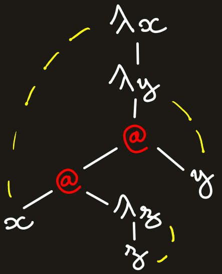


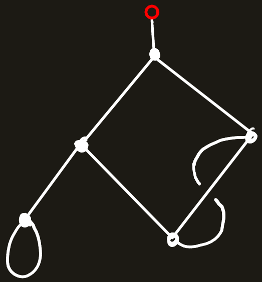

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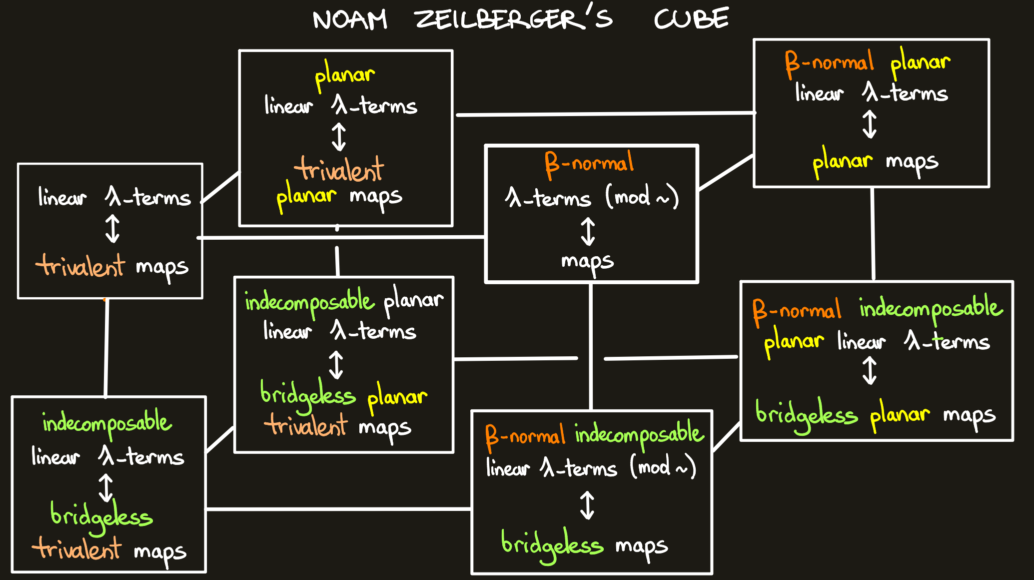

NOAM ZEILBERGER'S CUBE
planar linear λ-terms ↕ trivalent planar maps
β-normal planar linear λ-terms ↕ planar maps
linear λ-terms ↕ trivalent maps
β-normal λ-terms (mod ~) ↕ maps
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# PART II

Asymptotic analysis of statistics on maps

# ASYMPTOTIC NUMBER OF MAPS

$c_n$ = number of combinatorial maps with $n$ edges

Recurrence formula :

$c_1 = 1$

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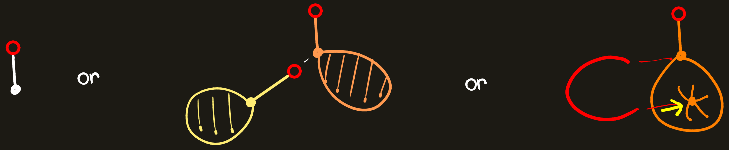
map =



Question 0 : Asymptotic estimate of $c_n$ ?

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Idea: (Formally) solve it !

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Riccati :( ⟶ linear :)

$$C(z) = z + 2z^2 \frac{\phi'(z)}{\phi(z)}$$

MAGIC TRICK!

$$2z^2\phi''(z) + (5z-1)\phi'(z) + \phi(z) = 0$$

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$(2n-1)!! = (2n-1) \times (2n-3) \times \dots \times 1$

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$$C(z) = z + 2z^2 \frac{\phi'(z)}{\phi(z)} \iff c_{n+1} = 2n\,\phi_n - \sum_{k=1}^{n-1} c_k\,\phi_{n-k}$$

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$$C(z) = z + 2z^2 \frac{\phi'(z)}{\phi(z)} \iff c_{n+1} = 2n\,\phi_n - \sum_{k=1}^{n-1} c_n \phi_{n-k}$$

By some bootstrapping, $c_n \sim \phi_n \left(2n - 1 - \frac{3}{2} n^{-1} - \frac{19}{4} n^{-2} + O(n^{-3})\right)$

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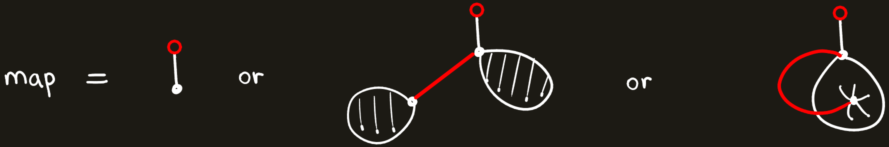
# NUMBER OF VERTICES

$C(z)$ = generating function of maps where $z$ counts the edges

**Equation**

$$C = z + C^2 + 2z^2\frac{\partial C}{\partial z} - zC$$



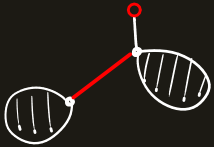
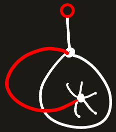

**Question 1:** behaviour of the number of vertices ?

# NUMBER OF VERTICES

$C(z,u)$ = generating function of maps where $z$ counts the edges and $u$ counts the vertices

**Equation**

$$C = z u + C^2 + 2 z^2 \frac{\partial C}{\partial z} - z C$$

map =  or  or 

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Solution: $\phi(z,u) = 1 + \frac{u(u+1)}{2}z + \frac{u(u+1)(u+2)(u+3)}{2^2 \times 2!}z^2 + \dots + \frac{u(u+1)\dots(u+2n-1)}{2^n \times n!}z^n + \dots$

# NUMBER OF VERTICES

Fact: $\phi(z,u)$ behaves like $C(z,u)$

**Theorem:**

For the uniform distribution of combinatorial maps,

Number of vertices $\xrightarrow[\text{law}]{}$ Gaussian law

mean $\sim \ln(n) + \gamma + \dots$

variance $\sim \ln(n) + \gamma - \frac{\pi^2}{12} + \dots$

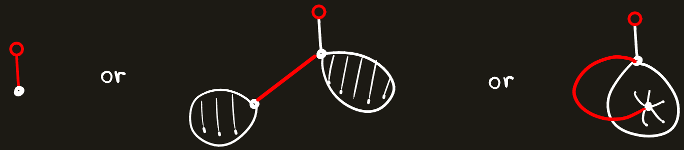
$$\phi(z,u) = 1 + \frac{u(u+1)}{2} z + \frac{u(u+1)(u+2)(u+3)}{2^2 \times 2!} z^2 + \dots + \frac{u(u+1)\dots(u+2n-1)}{2^n \times n!} z^n + \dots$$

# NUMBER OF EDGES INCIDENT TO THE ROOT

$C(z,u)$ = generating function of maps where $z$ counts the edges and $u$ counts the number of edges incident to the root vertex.

Equation:

$$C(z,u) = zu + uC(z,u)C(z,1) + u\left(2z^2\frac{\partial C}{\partial z} - zC\right)$$

map = 

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$$C(z,u) = z u + u\, C(z,u)\, C(z,1) + u\left(2 z^2 \frac{\partial C}{\partial z} - z C\right)$$


$$C(z,1) = z + 2z^2 \frac{\phi'(z,1)}{\phi(z,1)}$$

$$2uz^2 C'(z,u)\phi(z,1) + 2uz^2 C(z,u)\phi'(z,1) = (1-2uz)C(z,u)\phi(z,1) - \phi(z,1)$$

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$$P(z,u) = \phi(z,u)\,\phi(z,1)$$

$$2uz^2\,P'(z,u) = (1-2uz)\,P(z,u) - \phi(z,1)$$

almost linear!

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$$C(z,u) = zu + u\,C(z,u)\,C(z,1) + u\left(2z^2\frac{\partial C}{\partial z} - zC\right)$$

**Theorem :**

For the uniform distribution of combinatorial maps,

Number of edges incident to the root $\xrightarrow[\text{law}]{}$

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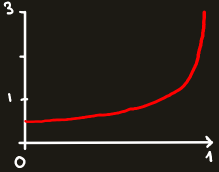
**Theorem :**

For the uniform distribution of combinatorial maps,

Number of edges incident to the root divided by $n$ $\xrightarrow{\text{law}}$ Beta-law density $\frac{1}{2}(1-t)^{-\frac{1}{2}}$ sur $[0,1)$

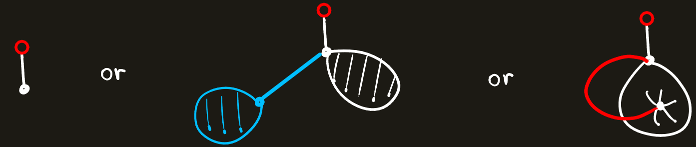


# NUMBER OF COMPONENTS ATTACHED TO THE ROOT

$C(z,u)$ = generating function of maps where $z$ counts the edges and $u$ counts the number of connected components attached to the root vertex.

Equation:

$$C(z,u) = z + u\,C(z,u)\,C(z,1) + \left(2z^2\frac{\partial C}{\partial z} - zC\right)$$

map = 

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Theorem:

Number of connected components attached to the root vertex $\xrightarrow{\text{law}}$ Geometric law of parameter $1/2$.

# PART III

Lessons to learn

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It will be on blackboard, sorry :(

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1

2

3

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1. Know your bijections
2. 
3. 

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It will be on blackboard, sorry :(

1. Know your bijections
2. Think bigger
3. 

# PART III Lessons to learn

It will be on blackboard, sorry :(

1. Know your bijections
2. Think bigger
3. Be humble and work, grasshopper.

# CONCLUSION

→ Wide range of limits laws for combinatorial maps:
towards a taxonomy of possible laws?

→ Understand the operation $C = z + K z^2 \frac{\Phi'}{\Phi}$

→ Extension to other families of maps?
to other combinatorial families?

# THANK YOU!