Julien COURTIEL (LIPN, Paris 13)

LET'S COUNT THE CONNECTED CHORD DIAGRAMS

Séminaire Algo (LIGM)

Coauthors : Karen YEATS (Waterloo), Noam ZEILBERGER (Birmingham)

Part I

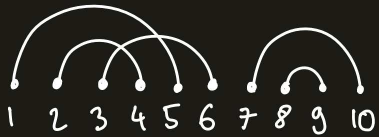
Interesting things to know about chord diagrams

DEFINITIONS

diagram with n chords

= matching of

the set $\{1, \ldots, 2n\}$



number of diagrams with n chords =

DEFINITIONS

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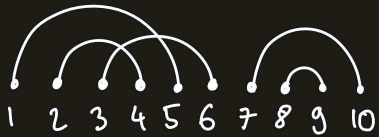

number of diagrams with n chords $= (2n-1)!!$

$$= (2n-1) \times (2n-3) \times \ldots \times 3 \times 1$$

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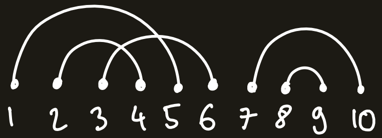

computer science

theoretical physics

cumulants

(more refined)
enumeration of
chord diagrams

knot theory

bioinformatics

...

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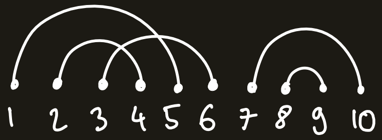

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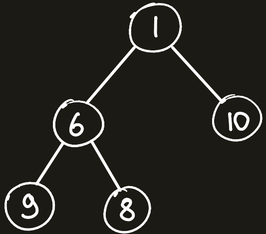
WHY COUNTING DIAGRAMS ... IN COMPUTER SCIENCE ?

Application 1

Queue priority (heap)



Operations:

$+_k$ Add an element with value k

$-_{min}$ Remove the smallest element

History: $+_8$ $+_3$ $+_2$ $+_{10}$ $-_{min}$ $+_9$ $-_{min}$ $+_6$ $+_1$

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3

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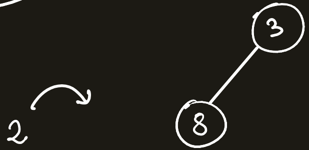

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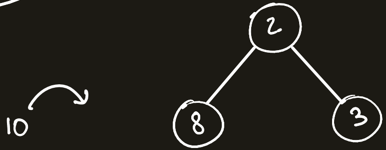

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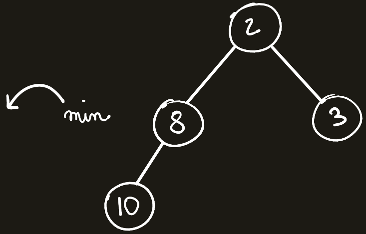

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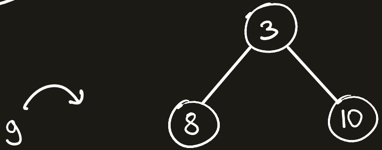

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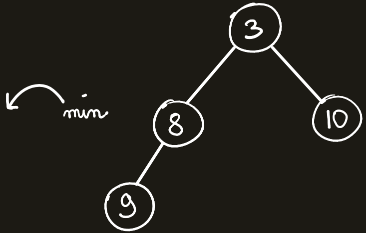

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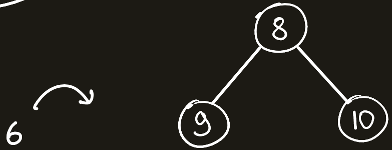

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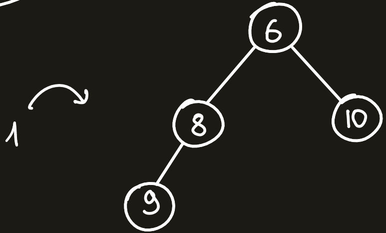

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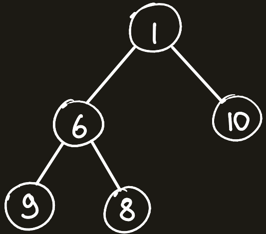

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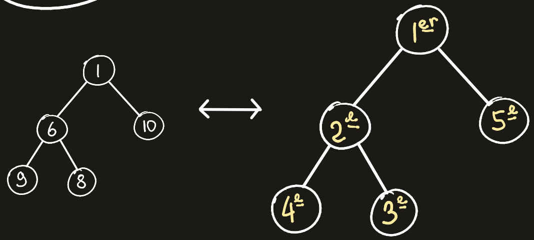

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Now we focus on relative positions

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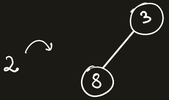

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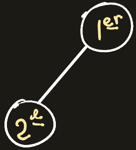

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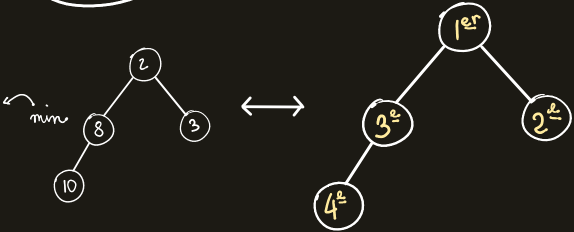

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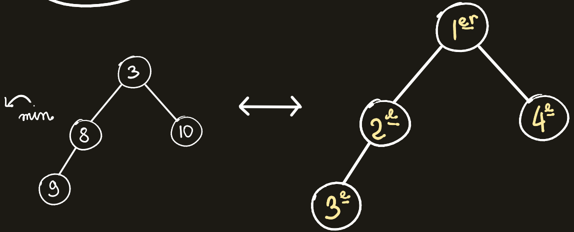

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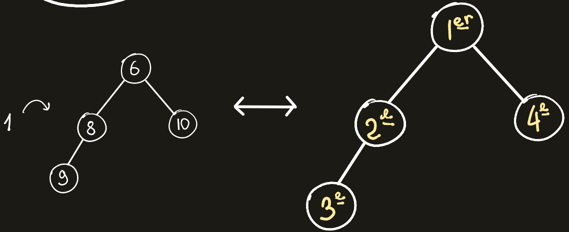

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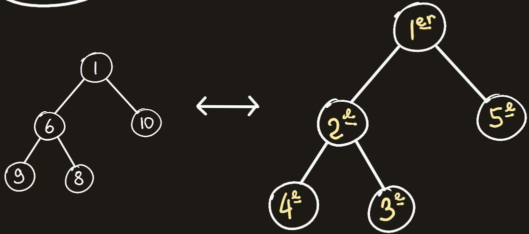

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WHY COUNTING DIAGRAMS ... IN COMPUTER SCIENCE ?

Application 1 Queue priority (heap)

[Flajolet Françon Vuillemin] Histories ending at $0 \longleftrightarrow$ Chord diagrams

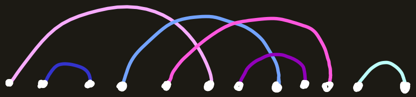
Diagram : 

History : $+_1 \; +_1 \; +_1 \; +_4 \; -_{min} \; +_3 \; -_{min} \; +_1 \; +_1 \; -_{min} \; -_{min} \; -_{min} \; -_{min} \; -_{min}$

WHY COUNTING DIAGRAMS ... IN COMPUTER SCIENCE ?

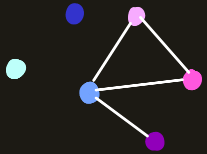
Application 2 Biaised model $G(n, m)$ [Acan Pittel]

Random diagram → Intersection graph



n = number of chords

m = number of crossings

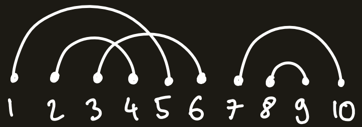


n = number of vertices

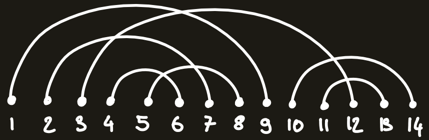
m = number of edges

DEFINITIONS

diagram with n chords $=$ matching of the set $\{1, \ldots, 2n\}$



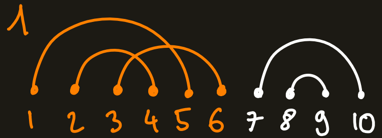
connected diagram $=$ "everything holds in one block."



DEFINITIONS

diagram with n chords $=$ matching of the set $\{1, \ldots, 2n\}$

3 connected components



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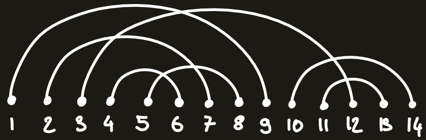

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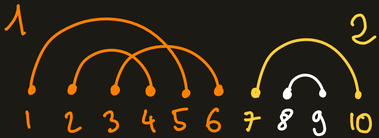

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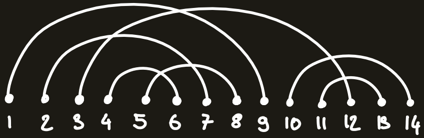

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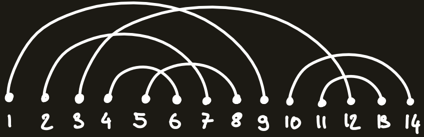

DEFINITIONS

Enumeration of connected diagrams :

PREHISTORY — 1952 Touchard

EXPLICIT FORMULAS — 1978 Stein Everett

ANALYTIC COMBINATORICS — 2000 Flajolet Noy

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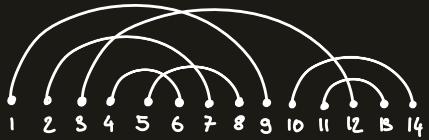

ELEMENTARY ENUMERATION

number of connected diagrams with n chord $= c_n$

$c_1 = 1 \quad c_2 = 1 \quad c_3 = 4 \quad c_4 = 27 \quad c_5 = 248$

For $n = 3$:

<u>Induction formula</u> [Stein] $c_n = (n-1) \times \sum_{k=1}^{n-1} c_k \times c_{n-k}$

ELEMENTARY ENUMERATION

Proof of $$c_n = (n-1) \times \sum_{k=1}^{n-1} c_k \times c_{n-k} \quad ?$$

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Proof of $\quad c_n = \sum_{k=1}^{n-1} (2k-1) \times c_k \times c_{n-k}$?

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$k \leftarrow n-k$

$$c_n = \sum_{k=1}^{n-1} (2(n-k)-1) \times c_{n-k} \times c_k$$

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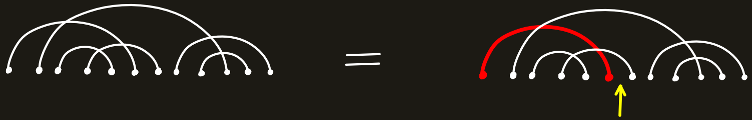

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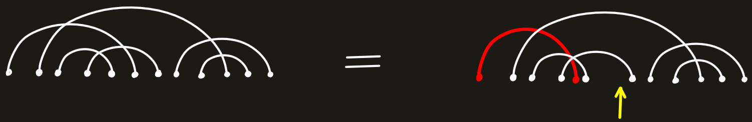

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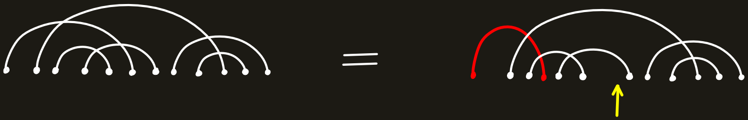

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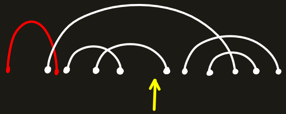

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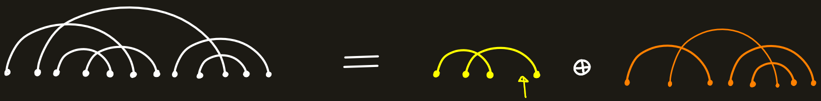

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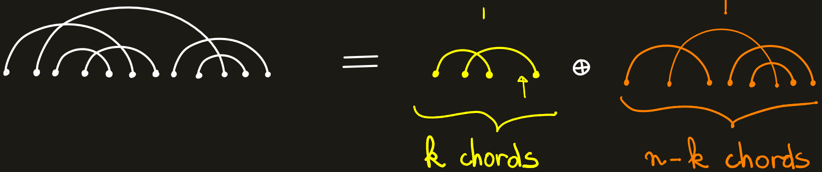

ASYMPTOTIC BEHAVIOR

$$c_n \sim \frac{1}{e} \times (2n-1)!! \qquad \text{[Stein Everett]}$$

Corollary : $\mathbb{P}(\text{connected diagram}) = \dfrac{c_n}{(2n-1)!!} \to \dfrac{1}{e}$

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Better [Flajolet Noy]

- number of connected components $\sim$ Poisson(1)
- $|C|$ – size of the largest component $\sim$ Poisson(1)

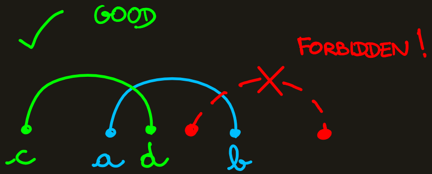
Part II

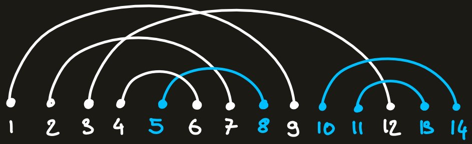
Analytic side: terminal chords

with Karen Yeats

DEFINITION

terminal chord =
chord (a,b) such that
if (c,d) intersects (a,b),
then $c < a$.





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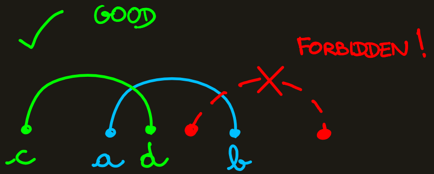

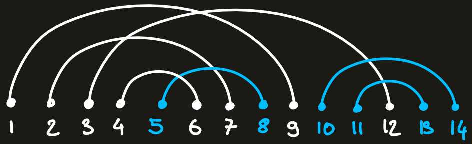

<u>Question</u> : Average number of terminal chords in a connected chord diagram ?

CONTEXT

Theorem [Marie, Yeats]

BIG FORMULA!

The Dyson-Schwinger equation

$$G(x,L) = 1 - x\, G\left(x, \frac{\partial}{\partial(-\rho)}\right)^{-1} \left(e^{-L\rho} - 1\right) F(\rho)\Big|_{\rho=0}$$

has for solution:

$$G(x,L) = 1 - \sum_{i \geq 1} \sum_{C} \frac{L^i}{i!}\, x^{|C|}\, f_0^{|C|-k}\, f_{t_1-i}\, f_{t_2-t_1}\, f_{t_3-t_2} \cdots f_{t_k-t_{k-1}}$$

C connected chord diagram such that $t_1 \geq i$ where $t_1 < t_2 < \ldots < t_k$ denote the positions of terminal chords of C for the intersection order.

CONTEXT

BIG FORMULA!

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WHY IT IS HARD

The generating function $C(z,u) = \sum_{\substack{n \geq 0 \\ k \geq 0}} c_{n,k} \, z^n u^k$ where

$c_{n,k}$ = number of connected diagrams with n chords and k terminal chords

satisfies

$$C(z,u) = zu + \frac{z\left(2z\frac{\partial C}{\partial z}(z,u) - C(z,u)\right)}{1 - 2z\frac{\partial C}{\partial z}(z,u) + C(z,u)}.$$

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non-linear differential equation :(

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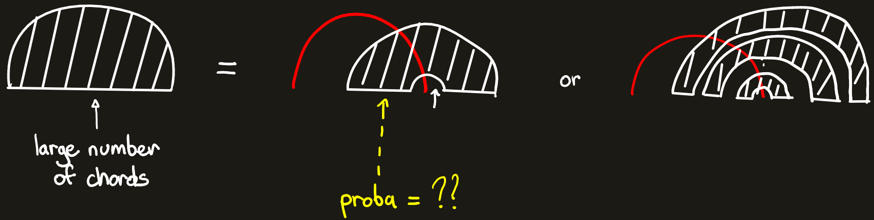
non-linear differential equation

$$c_n = (n-1)\sum_{k=1}^{n-1} c_k\, c_{n-k} \quad \Rightarrow \quad c_n \geq n!$$

AVERAGE NUMBER OF TERMINAL CHORDS

Idea :



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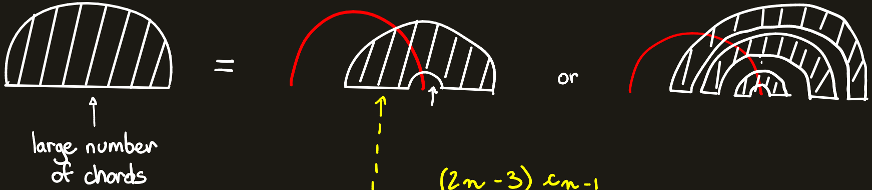

$$\text{proba} = \frac{(2n-3)\, c_{n-1}}{c_n}$$

AVERAGE NUMBER OF TERMINAL CHORDS

Idea:

large number of chords

$$\text{proba} = \frac{(2n-3)\,c_{n-1}}{c_n} \longrightarrow 1$$

$$\text{proba} \to 0$$

Almost surely, removing the root chord does not disconnect the diagram.

Interesting but not enough!

AVERAGE NUMBER OF TERMINAL CHORDS

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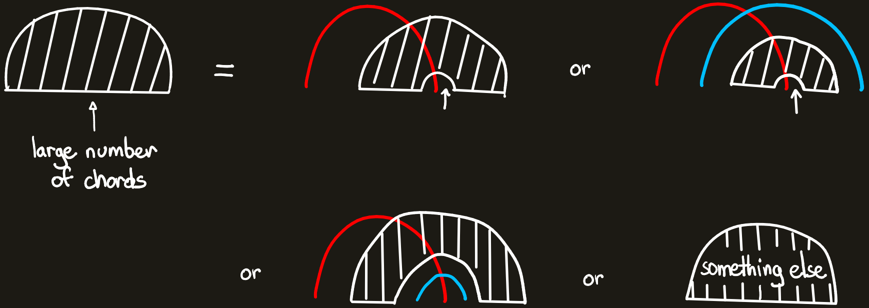

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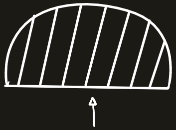

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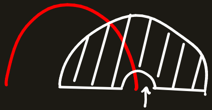
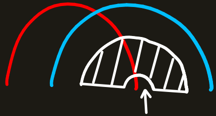


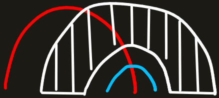
$$\text{proba} = (2n-3)\,\frac{c_{n-1}}{c_n} = 1 - \frac{1}{n} + o\left(\frac{1}{n}\right)$$

or



$$\text{proba} = (2n-5)\,\frac{c_{n-2}}{c_n} \sim \frac{1}{2n}$$

or



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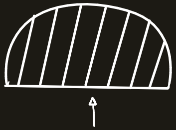

large number of chords

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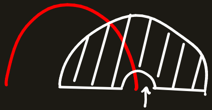

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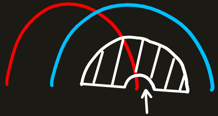

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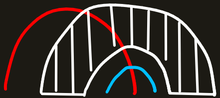

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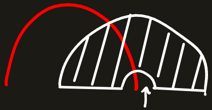

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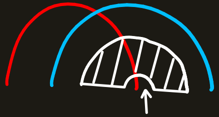

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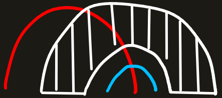

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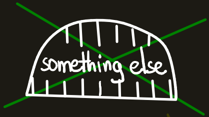


$$\text{proba} = o\left(\frac{1}{n}\right)$$

It works!

AVERAGE NUMBER OF TERMINAL CHORDS

Let X_n be the random variable such that

$$X_n = \begin{cases} X_{n-1} & \text{with proba } 1 - \frac{1}{n} \\ X_{n-2} + 1 & \text{with proba } \frac{1}{n} \end{cases}$$

$\boxed{\text{Idea n° 1}}$

For every initial conditions,

$X_n \longrightarrow$ Gaussian law.

$\boxed{\text{Idea n° 2}}$

Number of terminal chords '$\sim$' X_n.

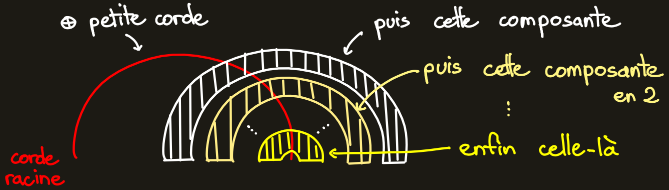
AVERAGE NUMBER OF TERMINAL CHORDS

Theorem For the uniform distribution,

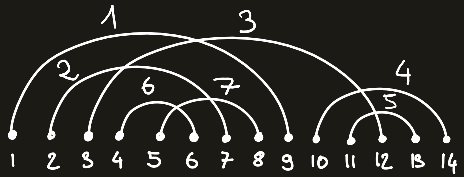
$$\text{Number of terminal chords} \xrightarrow[\text{distribution}]{} \begin{array}{l} \text{Gaussian law} \\ \text{mean} \sim \ln(n) \\ \text{variance} \sim \ln(n) \end{array}$$

ORDRE D'INTERSECTION

Règle :

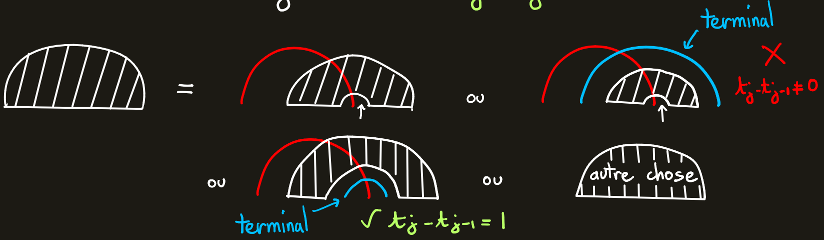


Exemple :



NOMBRE MOYEN DE CORDES TERMINALES CONSÉCUTIVES

Si les cordes terminales sont en position $t_1 < t_2 < \ldots < t_k$, combien de j satisfont $t_j - t_{j-1} = 1$?



<u>Théorème</u> Pour la distribution uniforme,

$$\text{Nombre de cordes terminales consécutives} \xrightarrow[\text{loi}]{} \text{Loi gaussienne}$$

moyenne $\sim \frac{1}{2} \ln(n)$

variance $\sim \frac{1}{2} \ln(n)$

CURRENT WORK

→ Extension to other Dyson-Schwinger equations

notion of decorated diagram

with Yeats

→ Generalization of the method to other combinatorial families

when the generating function is not analytic
and satisfies a non linear differential equation

with Bodini & Dougal.

Part III

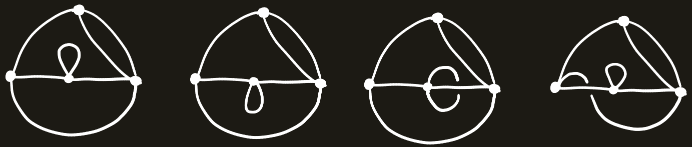
Bijective side: combinatorial maps

with Karen Yeats and Noam Zeilberger

AN OTHER FAMILY OF OBJECTS

combinatorial map = graph where we have cyclically ordered the 1/2-edges around each vertex.

Example. Four different maps:



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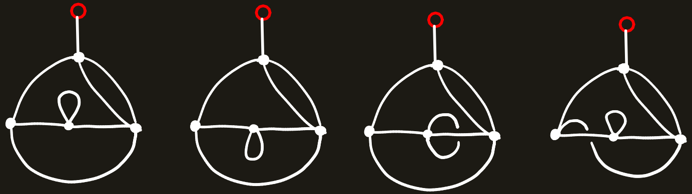

We root a map by marking a leaf.

AN OTHER FAMILY OF OBJECTS

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1 edge

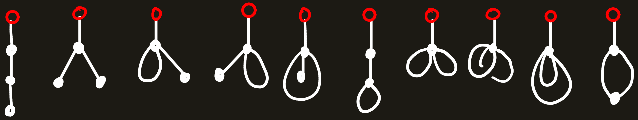
1

2 edges

2

3 edges

10



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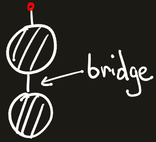

bridge = edge $\neq$ whose deletion disconnects the map.

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2 edges

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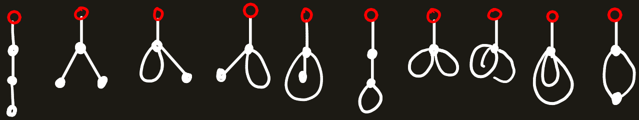

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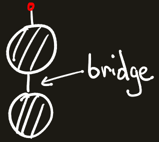

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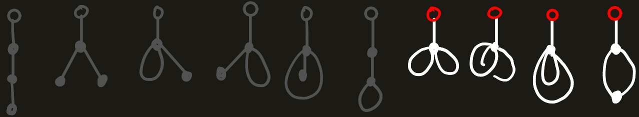

4

NOT COINCIDENTAL

Theorem

number of bridgeless maps with n edges

=

number of connected diagrams with n chords

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Reminder $c_n = \sum_{k=1}^{n-1} (2k-1) c_k c_{n-k}$

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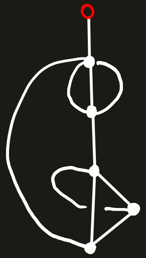

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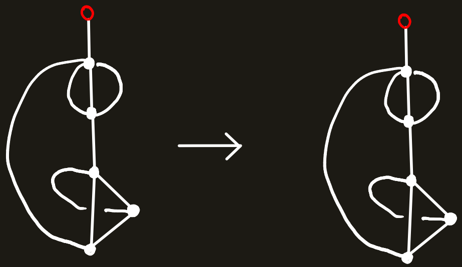

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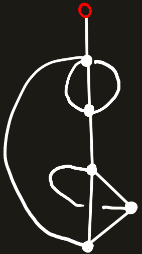
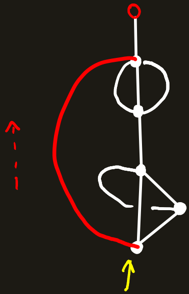


$\longrightarrow$



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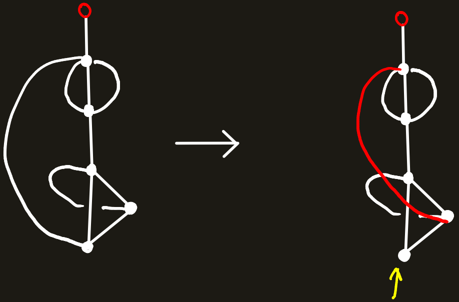

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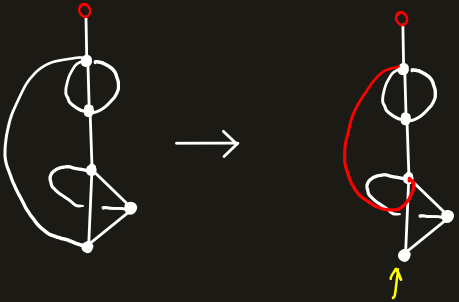

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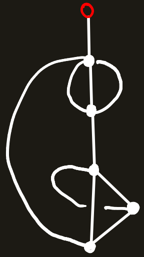

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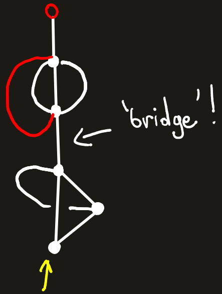

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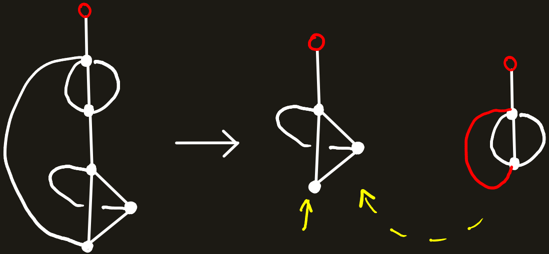

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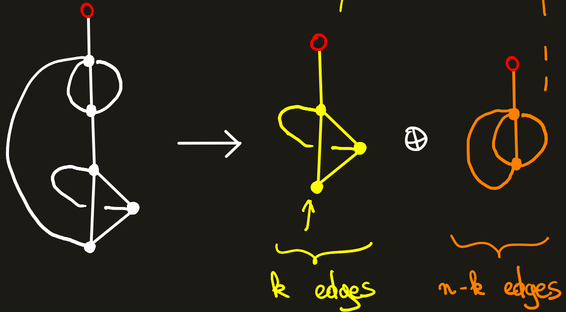

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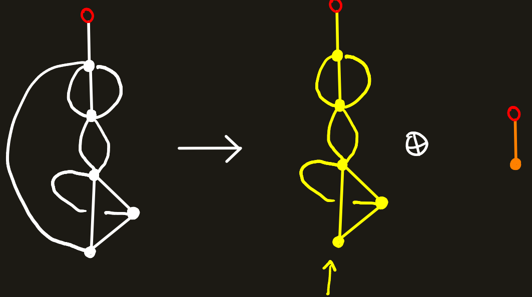

(A SAMPLE OF THE) BIJECTION PROPERTIES

We can describe an explicit bijection between bridgeless maps and connected diagrams.

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We can describe an explicit bijection between bridgeless maps and connected diagrams.

(1) Can be extended in a bijection

maps (with bridges, or not)

$\updownarrow$

irreducible diagrams

already known by [Ossana de Mendez Rosenstiehl, Cori]

reducible diagram

irreducible diagram

(A SAMPLE OF THE) BIJECTION PROPERTIES

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① Can be extended in a bijection

maps (with bridges, or not)

$\updownarrow$

irreducible diagrams

already known by [Ossana de Mendez, Rosenstiehl, Cori]

reducible diagram

irreducible diagram

② Planarity characterization

<u>Theo</u> A map is planar iff its image under the bijection avoids the pattern

BACK TO THE TERMINAL CHORDS

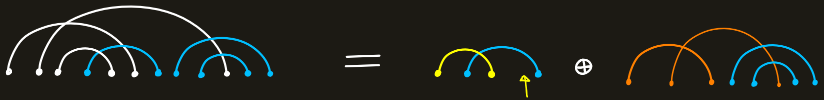
$$\text{diagrams} \to \text{maps}$$
$$\text{terminal chords} \mapsto ???$$

BACK TO THE TERMINAL CHORDS

diagrams $\rightarrow$ maps

terminal chords $\mapsto$???

Decomposition:



$$\#\,\text{terminal}\,(C_1 \oplus C_2) = \#\,\text{terminal}\,(C_1) + \#\,\text{terminal}\,(C_2)$$

BACK TO THE TERMINAL CHORDS

diagrams $\rightarrow$ maps

terminal chords $\mapsto$???

DIAGRAMS

Decomposition:

 $=$  $\oplus$ 

$$\#\,\text{terminal}\,(C_1 \oplus C_2) = \#\,\text{terminal}\,(C_1) + \#\,\text{terminal}\,(C_2)$$

if $C_2 \neq$ 

 $=$  $\oplus$ 

$$\#\,\text{terminal}\,(C_1 \oplus C_2) = \#\,\text{terminal}\,(C_1)$$

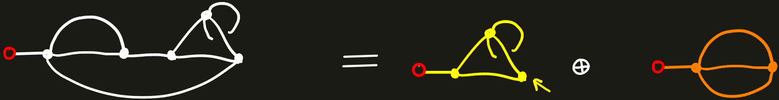
if $C_2 =$ 

BACK TO THE TERMINAL CHORDS

diagrams $\to$ maps

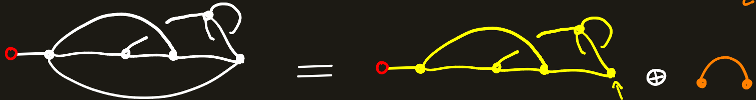
terminal chords $\mapsto$???

Decomposition:



$\#$?????? $(C_1 \oplus C_2)$ $=$ $\#$?????? (C_1) $+$ $\#$?????? (C_2)

if $C_2 \neq$ 



$\#$?????? $(C_1 \oplus C_2)$ $=$ $\#$?????? (C_1) if $C_2 =$ 

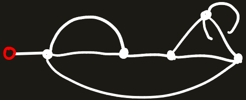
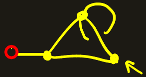
BACK TO THE TERMINAL CHORDS

diagrams $\rightarrow$ maps

terminal chords $\mapsto$???

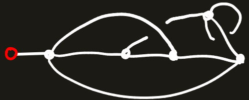
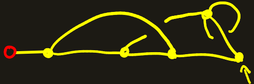
MAPS

<u>Decomposition:</u>

 $=$  $\oplus$ 

$$\#\ \text{vertices}\ (C_1 \oplus C_2) = \#\ \text{vertices}\ (C_1) + \#\ \text{vertices}\ (C_2)$$

if $C_2 \neq$ 

 $=$  $\oplus$ 

$$\#\ \text{vertices}\ (C_1 \oplus C_2) = \#\ \text{vertices}\ (C_1)$$

if $C_2 =$ 

NOT COINCIDENTAL 2

Theorem

number of bridgeless maps
with n edges
and k vertices
=
number of connected diagrams
with n chords
and k terminal chords

NOT COINCIDENTAL 2

Theorem

number of bridgeless maps
with n edges
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Corollary

For the uniform distribution on bridgeless combinatorial maps,

$$\text{Number of vertices} \xrightarrow[\text{distribution}]{} \text{Gaussian law} \quad \begin{array}{l} \text{mean} \sim \ln(n) \\ \text{variance} \sim \ln(n) \end{array}$$

PROSPECTS

→ A new bijection, mother lode of new properties
Many open questions !

→ Combinatorial interpretation of the works of
[Marie Yeats, Hihn Yeats]

→ Connection with lambda-calculus ?

THANK YOU!