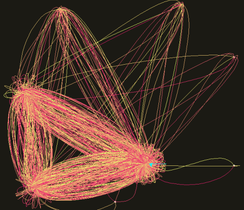
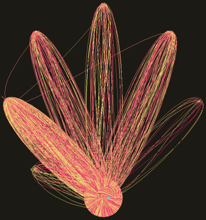
ASYMPTOTIC DISTRIBUTION OF PARAMETERS IN RANDOM MAPS

Julien COURTIEL (Caen, France)

Co-authors: Olivier BODINI (Paris 13), Sergey DOVGAL (Paris 13), Hsien-Kuei HWANG (Acad. Sinica)



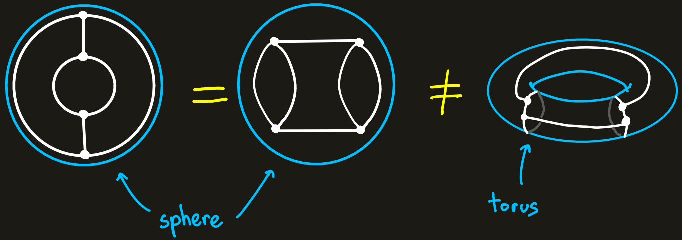


Analysis of Algorithms 2018, Uppsala

DEFINITION

(unrooted) map = cellular embedding of a connected graph onto an oriented surface.

Examples:



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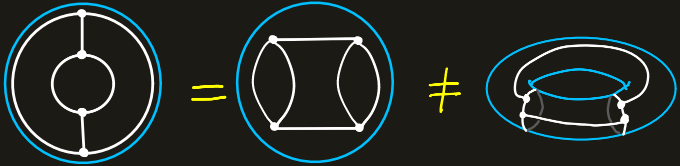

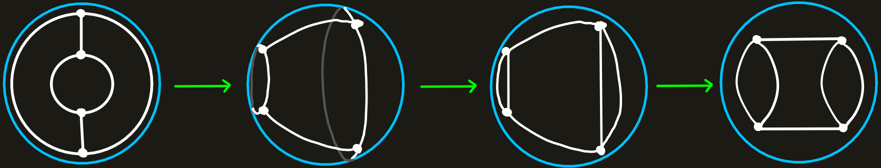

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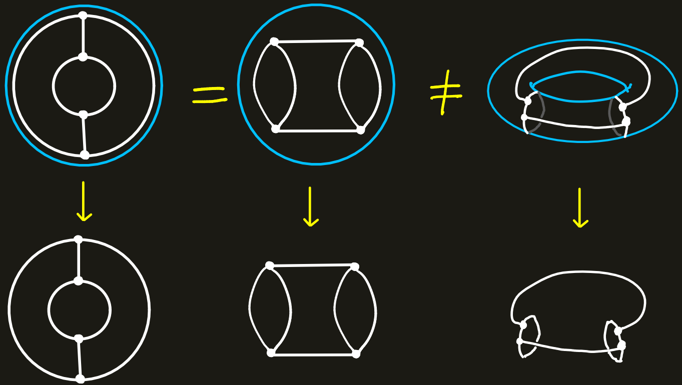

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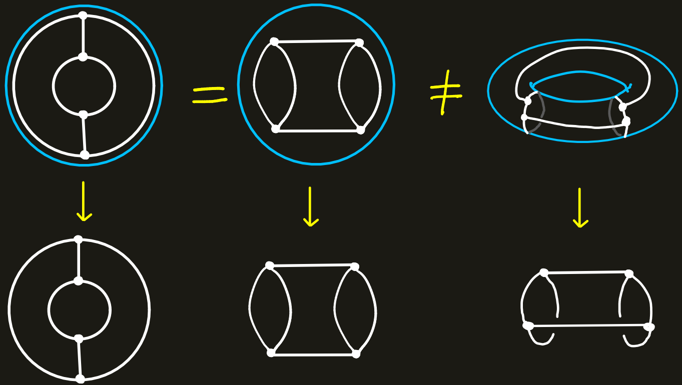

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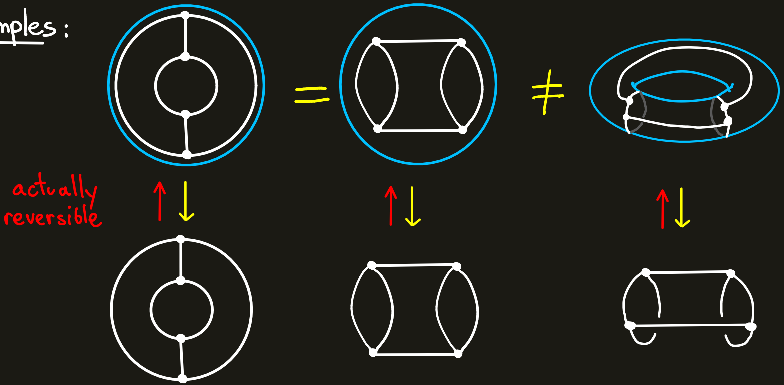
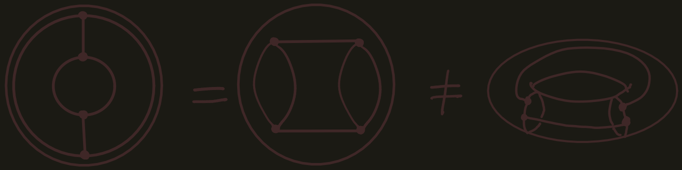


DEFINITION

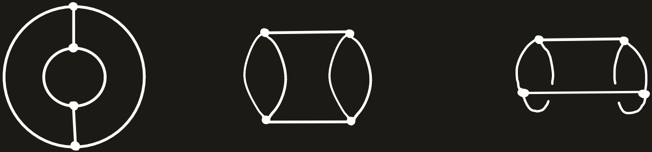
FORGET IT!

(unrooted) **map** = cellular embedding of a connected graph onto an oriented surface.

Examples:



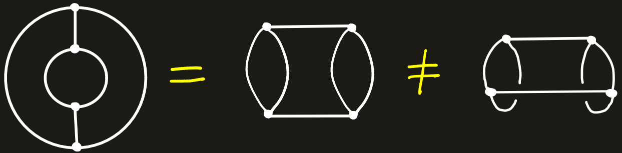
actually reversible



DEFINITION

(unrooted) map = connected graph where we have cyclically ordered the half-edges around each vertex.

Examples:



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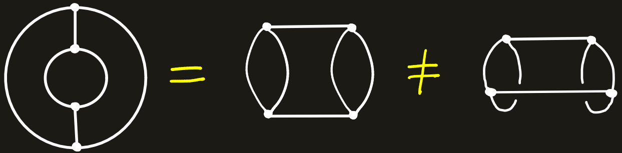
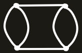
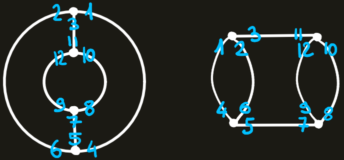


Why is  the same as  ?



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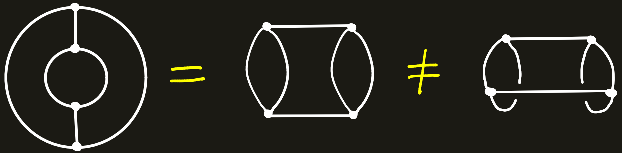
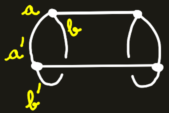


Why is  different from  ?



Absent pattern in  :

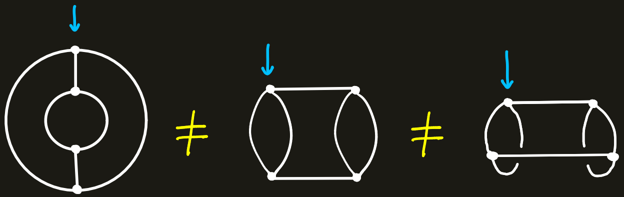
$a \leftrightarrow a'$ $\quad a \curvearrowright b$

$b \leftrightarrow b'$ $\quad a' \curvearrowright b'$

DEFINITION

rooted map = connected graph where we have cyclically ordered the half-edges around each vertex.

Examples:



We root every map on a half-edge.

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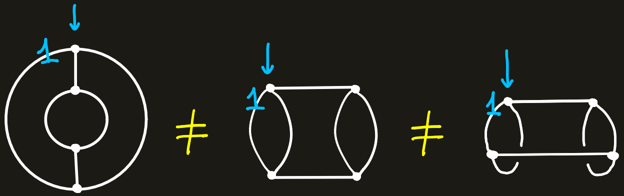

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DEFINITION

rooted map = connected graph where we have cyclically ordered the half-edges around each vertex.

0 edge

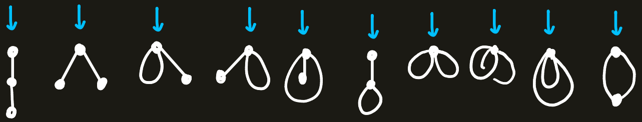
1

1 edge

2

2 edges

10



RECURRENCE FORMULA

c_n = number of rooted maps with n edges

Recurrence formula : [Arquès Béraud]

$$c_0 = 1 \qquad c_n = \sum_{k=0}^{n-1} c_k c_{n-k-1} + (2n-1)\, c_{n-1}$$

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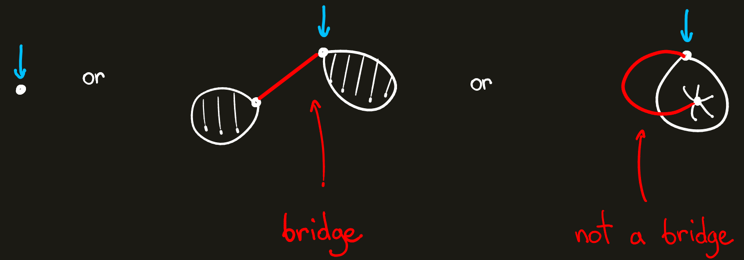
map =



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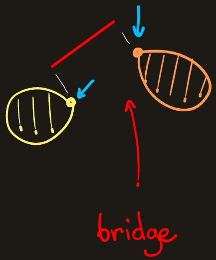
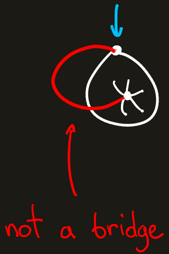
map =  or  or 

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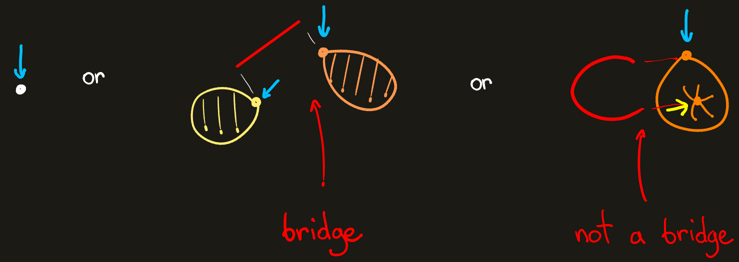
map = 

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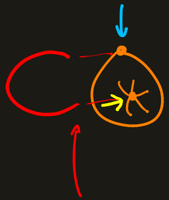
e.g:



not a bridge

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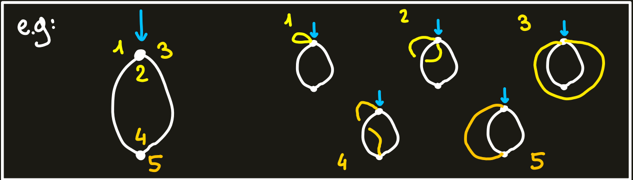

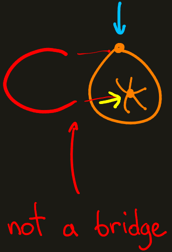

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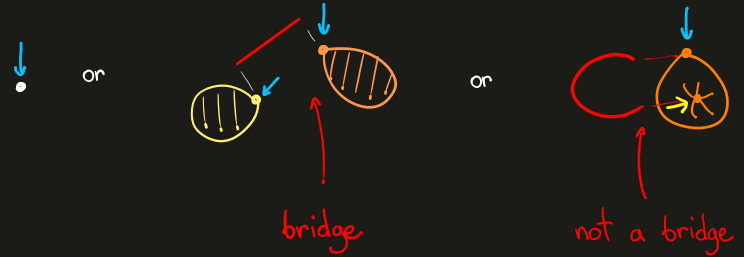
map = 

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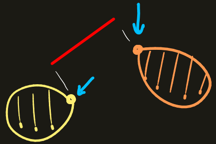
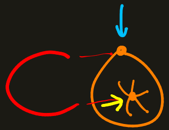
map =  or  or 

Generating function : $C(z) = \sum_{n \geq 0} c_n z^n$

$$C(z) = 1 + z\,C(z)^2 + z\left(2z\frac{\partial C(z)}{\partial z} + C(z)\right)$$

WHY COUNTING MAPS WITH NO CONSIDERATION FOR GENUS?

→ Good framework to study parametric Riccati equations.

→ Connections with other combinatorial families.

- indecomposable chord diagrams (link with the Quantum Fields Theory)
- lambda-terms
- symmetric polynomials

GOAL

Obtain limit laws for many map parameters:

vertices, root vertex, ...

ASYMPTOTIC NUMBER OF MAPS

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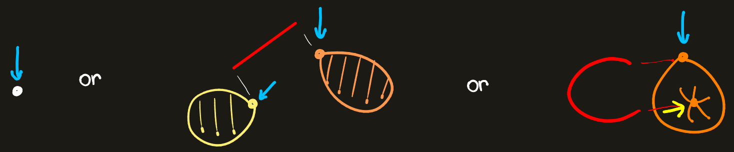

Question 0 : Asymptotic estimate of c_n ?

ASYMPTOTIC NUMBER OF MAPS

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<u>Idea:</u> (Formally) solve it!

ASYMPTOTIC NUMBER OF MAPS

Generating function : $C(z) = \sum_{n \geq 0} c_n z^n$

$$C(z) = 1 + z\,C(z)^2 + z\left(2z\frac{\partial C(z)}{\partial z} - C(z)\right)$$

Riccati :(

$$C(z) = 1 + 2z\frac{\phi'(z)}{\phi(z)}$$

MAGIC TRICK!

linear :)

$$2z^2\phi''(z) + (5z-1)\phi'(z) + \phi(z) = 0$$

ASYMPTOTIC NUMBER OF MAPS

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Solution : $\phi(z) = \sum_{n \geq 0} (2n-1)!!\, z^n$

$(2n-1)!! = (2n-1) \times (2n-3) \times \ldots \times 1$

ASYMPTOTIC NUMBER OF MAPS

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ASYMPTOTIC NUMBER OF MAPS

$$C(z) = 1 + 2z \frac{\phi'(z)}{\phi(z)} \iff c_n = 2n\,\phi_n - \sum_{k=1}^{n-1} c_n \phi_{n-k}$$

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ASYMPTOTIC NUMBER OF MAPS

$$C(z) = 1 + 2z\,\frac{\phi'(z)}{\phi(z)} \iff c_n = 2n\,\phi_n - \sum_{k=1}^{n-1} c_n\,\phi_{n-k}$$

By some bootstrapping, $\boxed{c_n \sim \phi_n\left(2n - 1 - \frac{3}{2}n^{-1} - \frac{19}{4}n^{-2} + O(n^{-3})\right)}$

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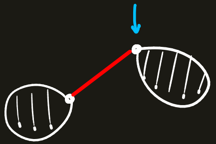
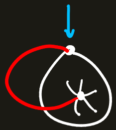
$(2n-1)!! = (2n-1)\times(2n-3)\times\ldots\times 1$

NUMBER OF VERTICES

$C(z)$ = generating function of maps where z counts the edges

<u>Equation</u>

$$C = 1 + z\,C^2 + 2z^2\frac{\partial C}{\partial z} + z\,C$$

map =  or  or 

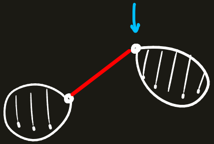
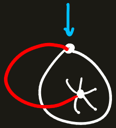
<u>Question</u> 1: behaviour of the number of vertices ?

NUMBER OF VERTICES

$C(z,u)$ = generating function of maps where z counts the edges
and u counts the vertices

<u>Equation</u>

$$C = u + z\,C^2 + 2z^2\frac{\partial C}{\partial z} + z\,C$$

map =  or  or 

<u>Question</u> 1: behaviour of the number of vertices ?

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$$C(z,u) = u + 2z \frac{\phi'(z,u)}{\phi(z,u)}$$

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$$2z^2 \phi''(z,u) + (3z + 2zu - 1)\,\phi'(z,u) + \frac{1+u}{2}\,\phi(z,u) = 0$$

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<u>Solution</u>: $\phi(z,u) = 1 + \frac{u(u+1)}{2}z + \frac{u(u+1)(u+2)(u+3)}{2^2\times 2!}z^2 + \dots + \frac{u(u+1)\dots(u+2n-1)}{2^n\times n!}z^n + \dots$

NUMBER OF VERTICES

Fact: $\phi(z,u)$ behaves like $C(z,u)$

Theorem:

For the uniform distribution of rooted maps,

$$\text{Number of vertices} \xrightarrow[\text{law}]{} \text{Gaussian law} \quad \begin{array}{l} \text{mean} \sim \ln(n) + \gamma + \dots \\ \text{variance} \sim \ln(n) + \gamma - \frac{\pi^2}{12} + \dots \end{array}$$

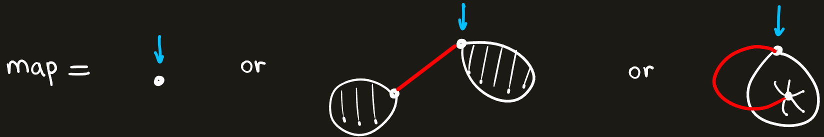
$$\phi(z,u) = 1 + \frac{u(u+1)}{2} z + \frac{u(u+1)(u+2)(u+3)}{2^2 \times 2!} z^2 + \dots + \frac{u(u+1)\dots(u+2n-1)}{2^n \times n!} z^n + \dots$$

NUMBER OF EDGES INCIDENT TO THE ROOT

$C(z,u) =$ generating function of maps where z counts the edges and u counts the number of edges incident to the root vertex.

Equation:

$$C(z,u) = 1 + zu\,C(z,u)\,C(z,1) + u\left(2z^2\frac{\partial C}{\partial z} + zC\right)$$



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MAGIC TRICK!

$$C(z,1) = 1 + 2z\,\frac{\phi'(z,1)}{\phi(z,1)}$$

$$2uz^2\,C'(z,u)\,\phi(z,1) + 2uz^2\,C(z,u)\,\phi'(z,1) = (1-2uz)\,C(z,u)\,\phi(z,1) - \phi(z,1)$$

NUMBER OF EDGES INCIDENT TO THE ROOT

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MAGIC TRICK!

$$C(z,1) = 1 + 2z\,\frac{\phi'(z,1)}{\phi(z,1)}$$

$$2uz^2\,C'(z,u)\,\phi(z,1) + 2uz^2\,C(z,u)\,\phi'(z,1) = (1-2uz)\,C(z,u)\,\phi(z,1) - \phi(z,1)$$

$$P(z,u) = \phi(z,u)\,\phi(z,1)$$

$$2uz^2\,P'(z,u) = (1-2uz)\,P(z,u) - \phi(z,1)$$

almost linear!

NUMBER OF EDGES INCIDENT TO THE ROOT

$$C(z,u) = 1 + z\,u\,C(z,u)\,C(z,1) + u\left(2z^2\frac{\partial C}{\partial z} + zC\right)$$

Theorem: For the uniform distribution of rooted maps,

Number of edges incident to the root $\xrightarrow[\text{law}]{}$

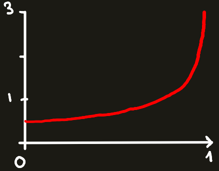
NUMBER OF EDGES INCIDENT TO THE ROOT

$$C(z,u) = 1 + z\,u\,C(z,u)\,C(z,1) + u\left(2z^2\frac{\partial C}{\partial z} + zC\right)$$

Theorem :

For the uniform distribution of rooted maps,

Number of edges incident to the root divided by n $\xrightarrow{\text{law}}$ Beta-law density $\frac{1}{2}(1-t)^{-\frac{1}{2}}$ sur $[0,1)$



NUMBER OF COMPONENTS ATTACHED TO THE ROOT

$C(z,u) =$ generating function of maps where z counts the edges and u counts the number of connected components attached to the root vertex.

Equation:

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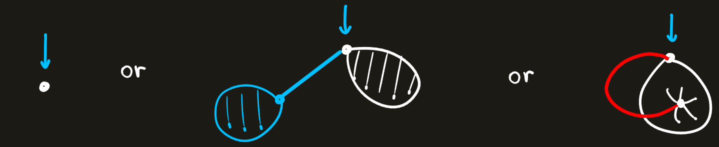
map = 

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$$C(z,u) = 1 + z\,u\,C(z,u)\,C(z,1) + \left(2z^2\frac{\partial C}{\partial z} + zC\right)$$

Theorem:

$$\boxed{\text{Number of connected components attached to the root vertex.} \xrightarrow[\text{law}]{} \quad}$$

NUMBER OF COMPONENTS ATTACHED TO THE ROOT

$C(z,u)$ = generating function of maps where z counts the edges and u counts the number of connected components attached to the root vertex.

Equation:

$$C(z,u) = 1 + zu\,C(z,u)\,C(z,1) + \left(2z^2\frac{\partial C}{\partial z} + zC\right)$$

Theorem:

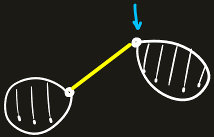
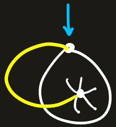
Number of connected components attached to the root vertex. $\xrightarrow[\text{law}]{}$ Geometric law of parameter $1/2$.

ROOT VERTEX DEGREE

$C(z,u) =$ generating function of maps where z counts the edges and u counts the degree of the root vertex

Equation:

$$C(z,u) = 1 + z\,u\,C(z,u)\,C(z,1) + u\left(2z^2\frac{\partial C}{\partial z} + zC\right) + (u^2 - u)\frac{\partial C}{\partial u}$$

map =  or  or 

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Theorem:

Degree of the root vertex $\xrightarrow{\text{law}}$

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$C(z,u)$ = generating function of maps where z counts the edges and u counts the degree of the root vertex

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Theorem:

Degree of the root vertex divided by n $\xrightarrow[\text{law}]{}$ uniform law on $[0,1]$

NUMBER OF LOOPS

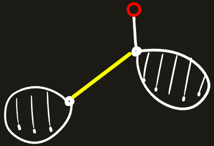
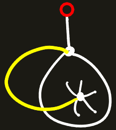
$C(z,u,l)$ = generating function of maps where z counts the edges

u counts the degree of the root vertex

and l counts the number of loops.

<u>Equation</u>:

$$C(z,u) = 1 + z\,u\,C(z,u)\,C(z,1) + u\left(2z^2\frac{\partial C}{\partial z} + zC\right) + (u^2 l - u)\frac{\partial C}{\partial u}$$

map =  or  or 

NUMBER OF LOOPS

$C(z,u,l)$ = generating function of maps where z counts the edges

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Equation:

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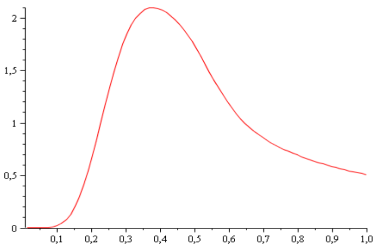
Theorem

number of loops divided by n $\xrightarrow[\text{law}]{}$



→ Why does the operation $C = 1 + K z \frac{\phi'}{\phi}$ work?

→ Why does the operation $C = 1 + K_{\delta} \frac{\phi'}{\phi}$ work?

We're working on it!

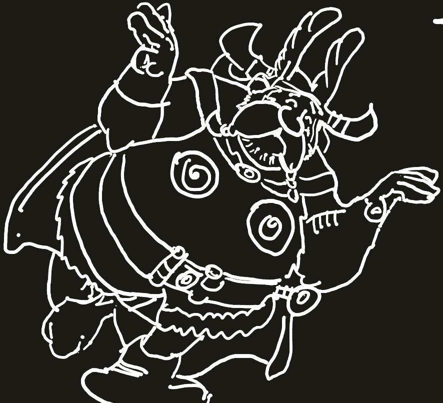
Perspectives

→ Why does the operation $C = 1 + K z \frac{\phi'}{\phi}$ work?

We're working on it!

→ Wide range of limits laws for Riccati equations:
towards a taxonomy of possible laws?

→ Extension to other families of maps?
to other combinatorial families?
(lambda-terms)



THANK YOU!