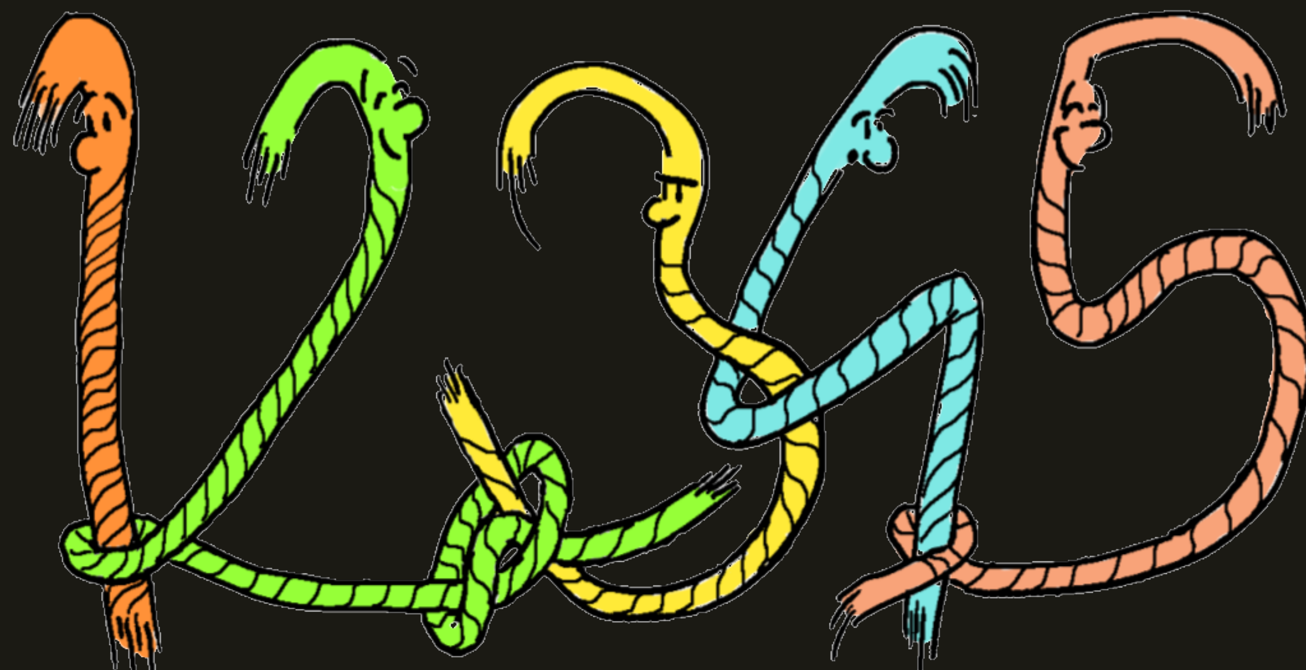
UNDERSTANDING THE DYSON-SCHWINGER EQUATIONS VIA CHORD DIAGRAMS

Berlin, February 26th 2019



Julien COURTIEL (Caen, France) with Karen YEATS (Waterloo, Canada)

PART I

THE BEGINNING: KAREN'S AND NICOLAS' PAPER

PHYSICAL BACKGROUND

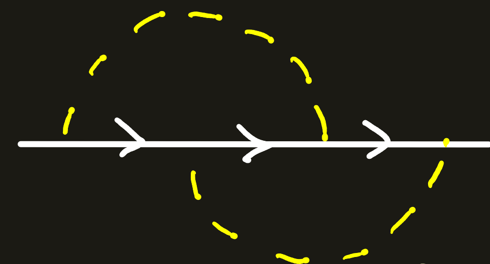
Yukawa theory

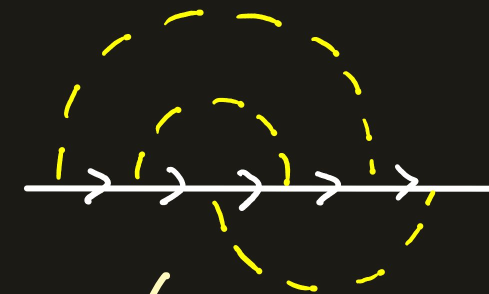
fermion

meson

PHYSICAL BACKGROUND

Yukawa theory





...

renormalized Feynman integrals

All contribute to a Green function $G(x, L)$

fermion

meson

THE BACKGROUND OF THIS TALK

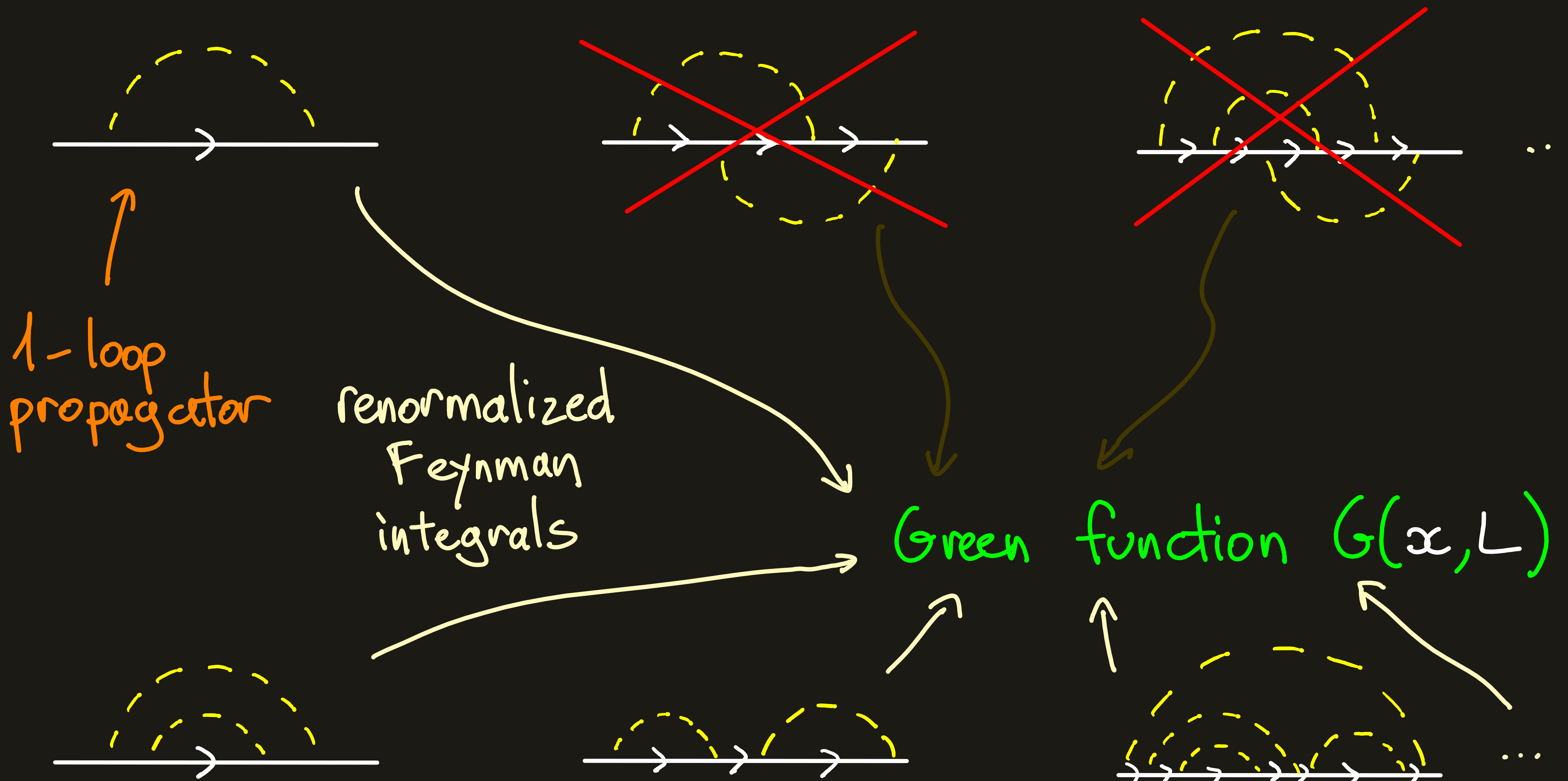
Successive insertions of the 1-loop propagator

renormalized Feynman integrals

Some contribute to a Green function $G(x,L)$

THE BACKGROUND OF THIS TALK

Successive insertions of the 1-loop propagator



THE BACKGROUND OF THIS TALK

Successive insertions of the 1-loop propagator

The corresponding Green function $G(x,L)$ is solution to the Dyson-Schwinger equation

$$G(x,L) = 1 - x\, G\left(x, \frac{\partial}{\partial(-p)}\right)^{-1} \left(e^{-Lp} - 1\right) F(p) \Big|_{p=0}$$

where $F(p)$ = regularized Feynman integral of the one-loop graph

= contribution of 

SOLUTION OF THE EQUATION OF DYSON-SCHWINGER

Theorem [Marie, Yeats]

The solution of the previous equation is :

$$G(x,L) = 1 - \sum_{\substack{C \text{ connected} \\ \text{chord diagram} \\ \text{such that } t_1 < t_2 < \dots < t_k \\ \text{denote the positions} \\ \text{of the terminal chords of } C}} \left(\sum_{i=1}^{t_1} f_{t_1 - i} \frac{(-L)^i}{i!} \right) x^{|C|} f_0^{|C|-k} f_{t_2 - t_1} f_{t_3 - t_2} \cdots f_{t_k - t_{k-1}}$$

où $F(\rho) = \dfrac{f_0}{\rho} + f_1 + f_2 \rho + f_3 \rho^2 + \dots$ = regularized Feynman integral of the one-loop graph

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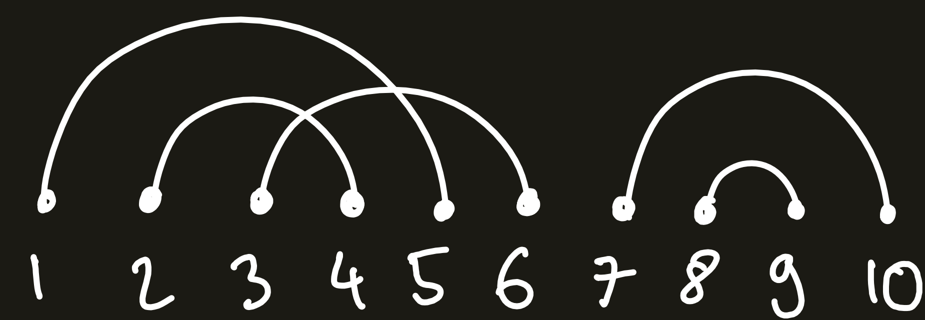
DEFINITIONS

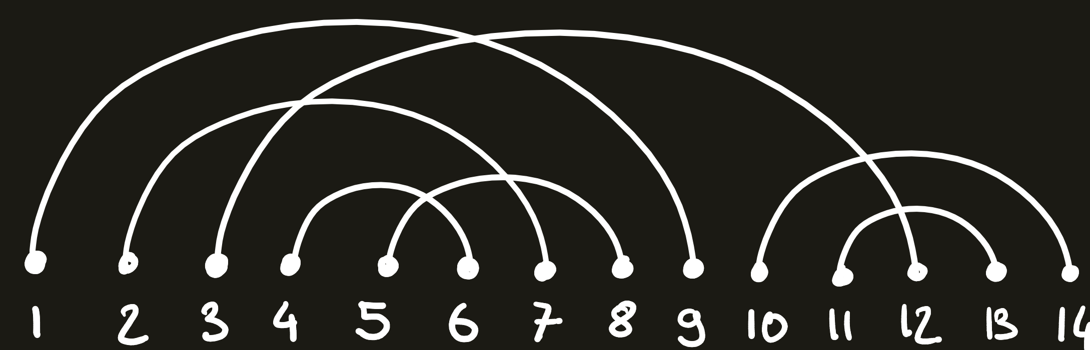
diagram with n chords

= perfect matching of the set $\{1, \ldots, 2n\}$



connected diagram =

"everything is one block."



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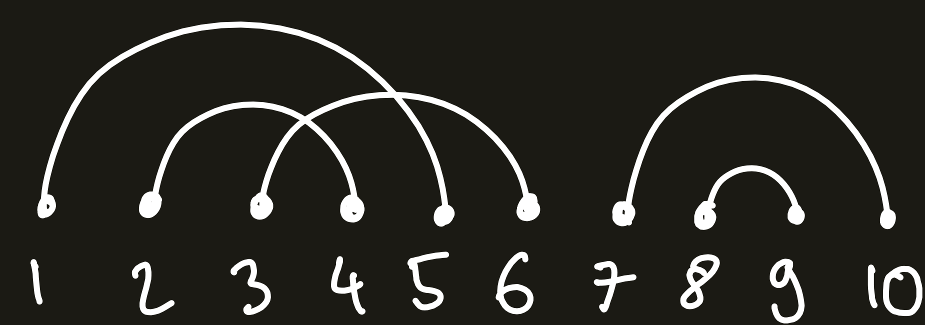

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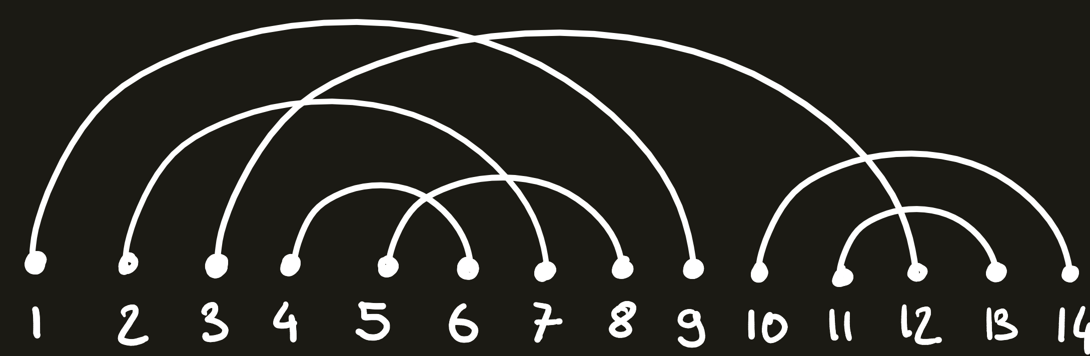

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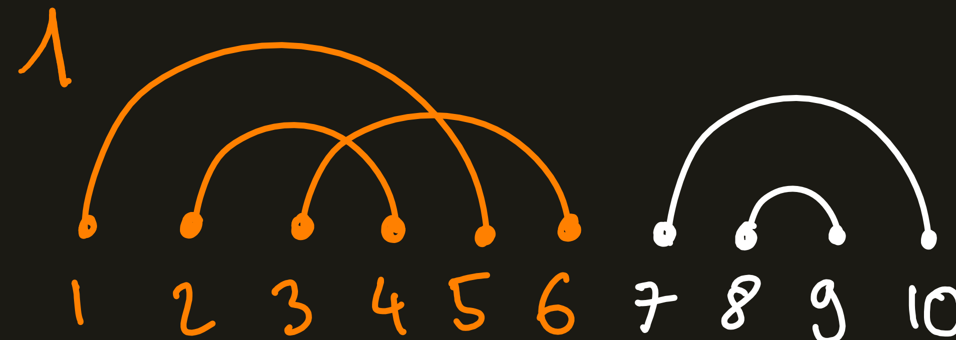

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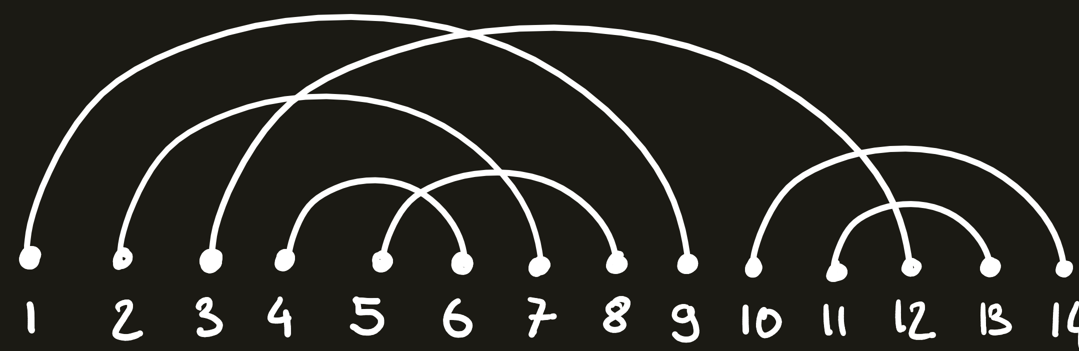

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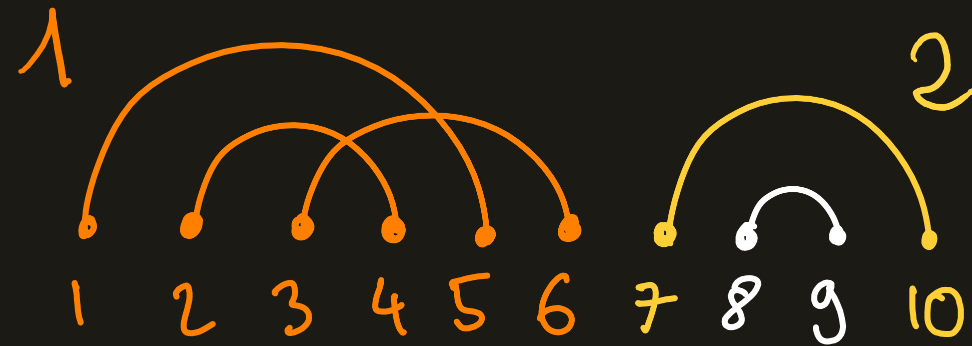

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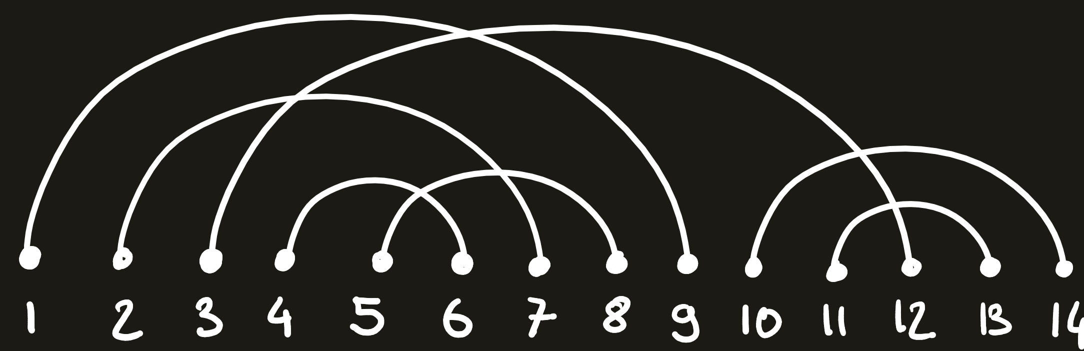

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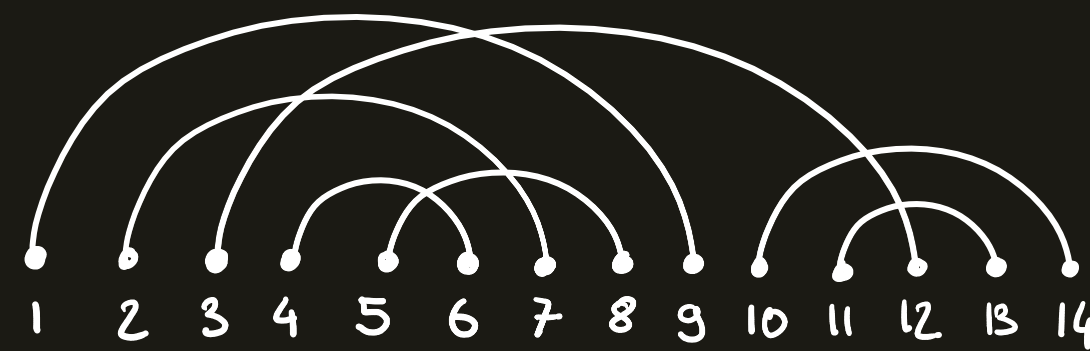

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such that $t_1 < t_2 < \ldots < t_k$ denote the positions of the terminal chords of C

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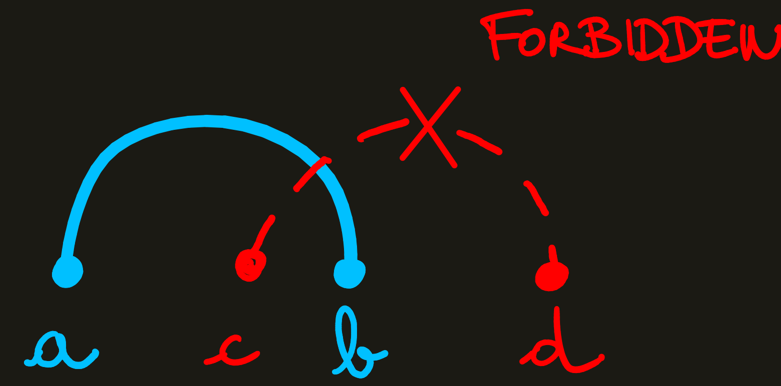

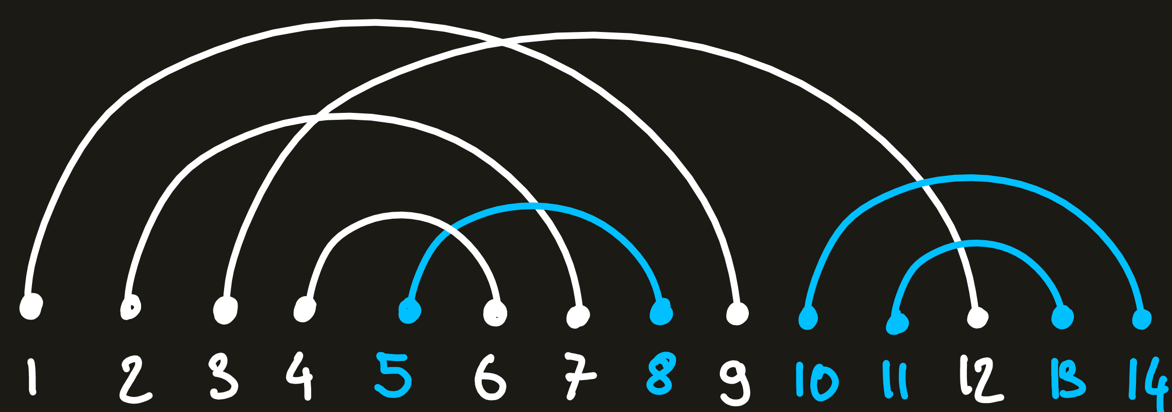

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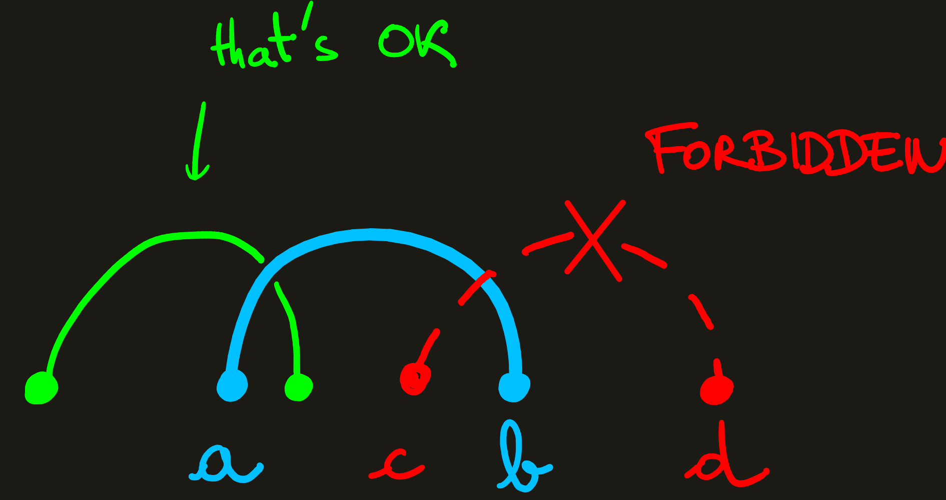

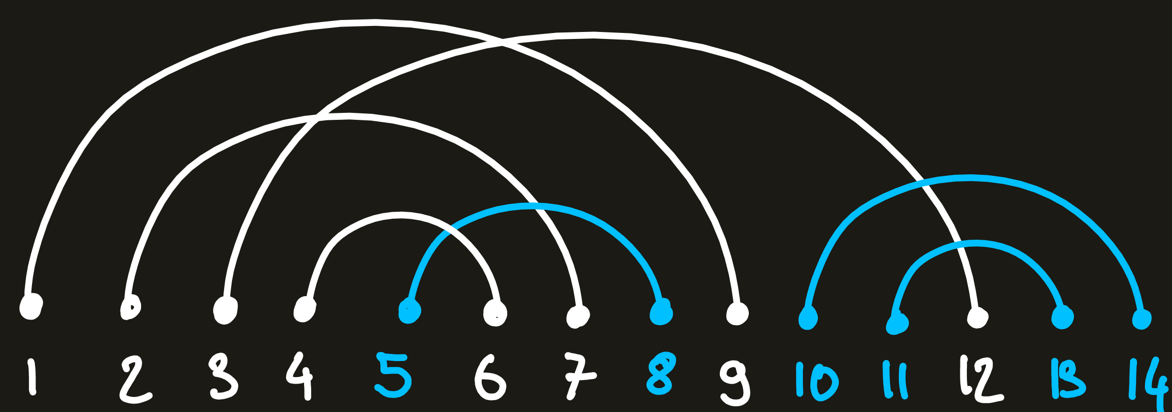

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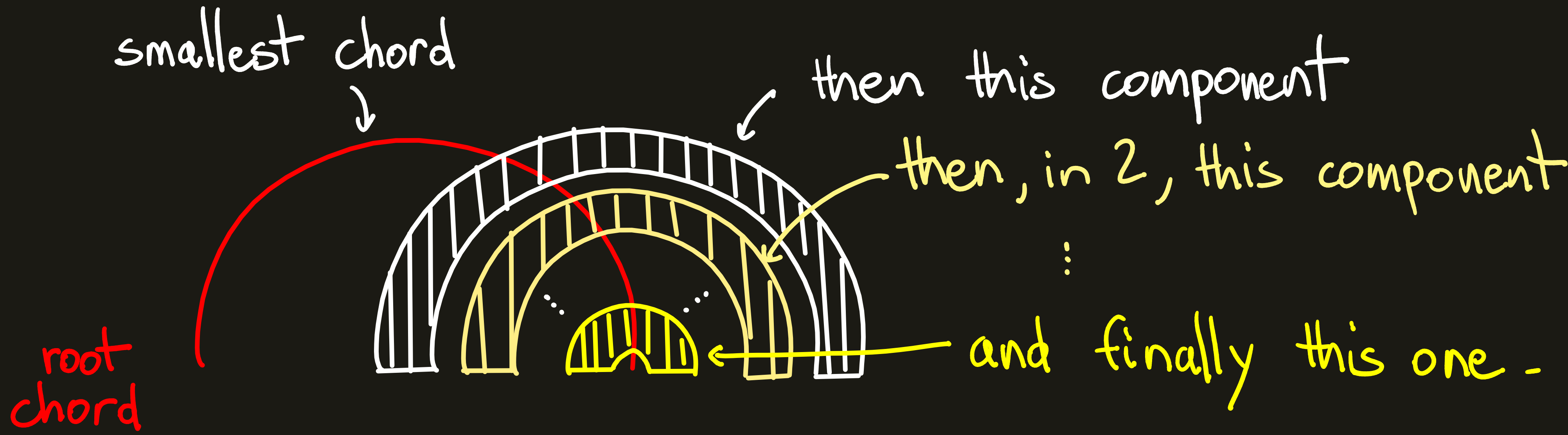
INTERSECTION ORDER

Rule:



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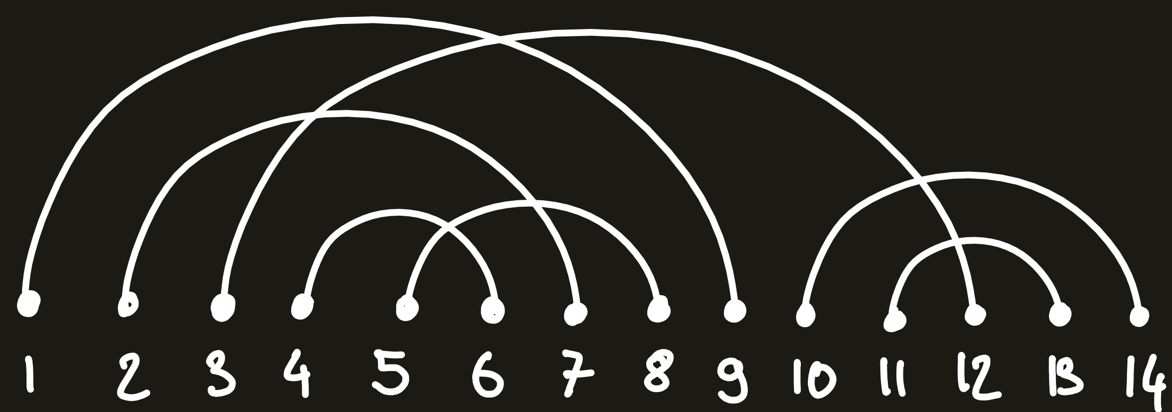

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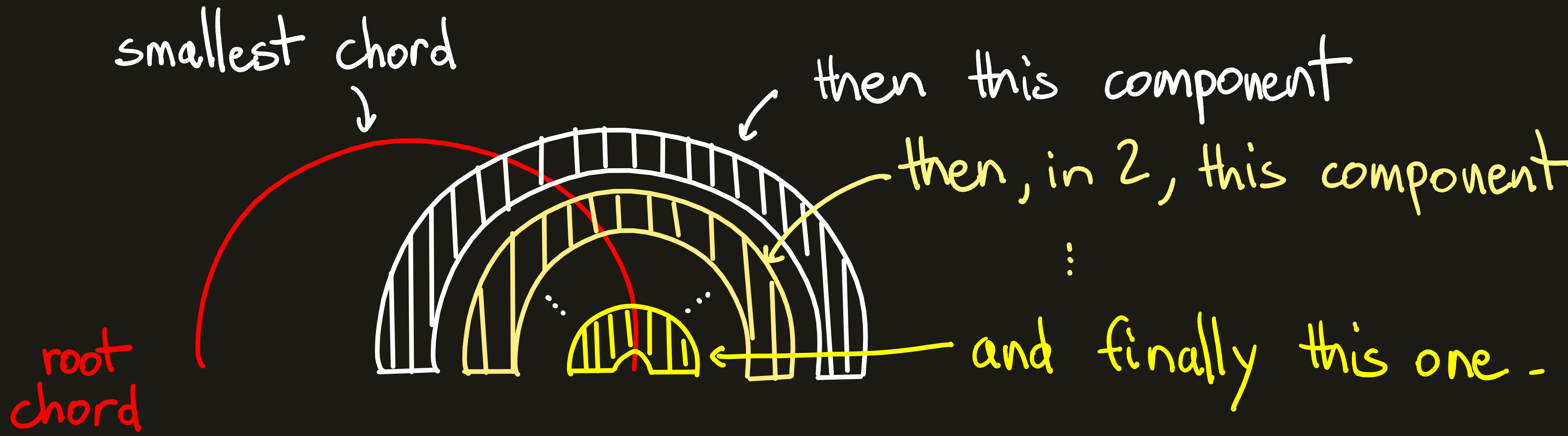

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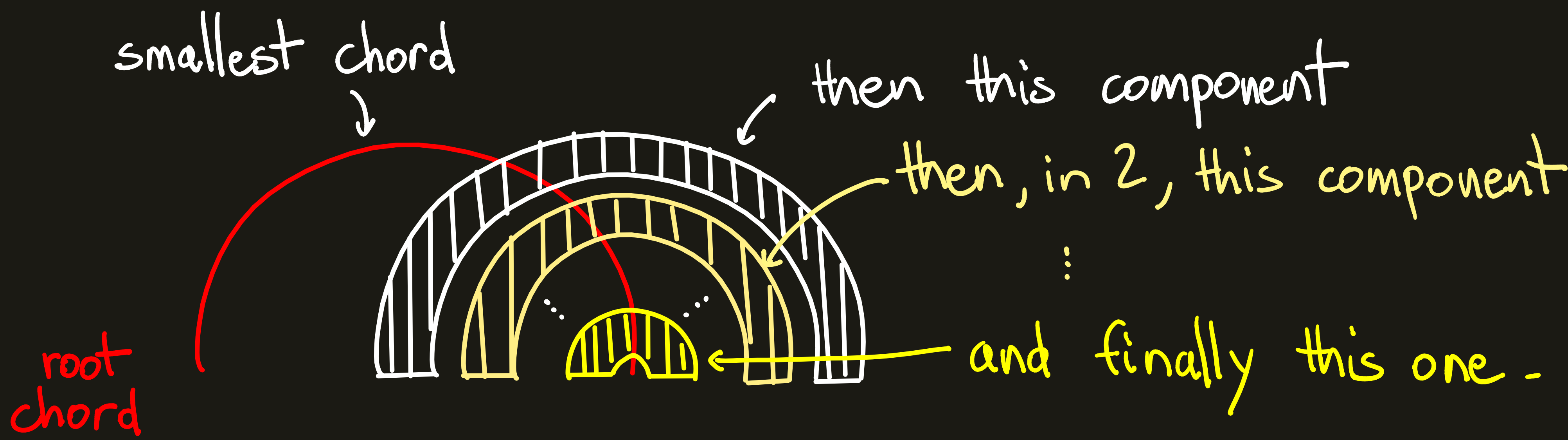

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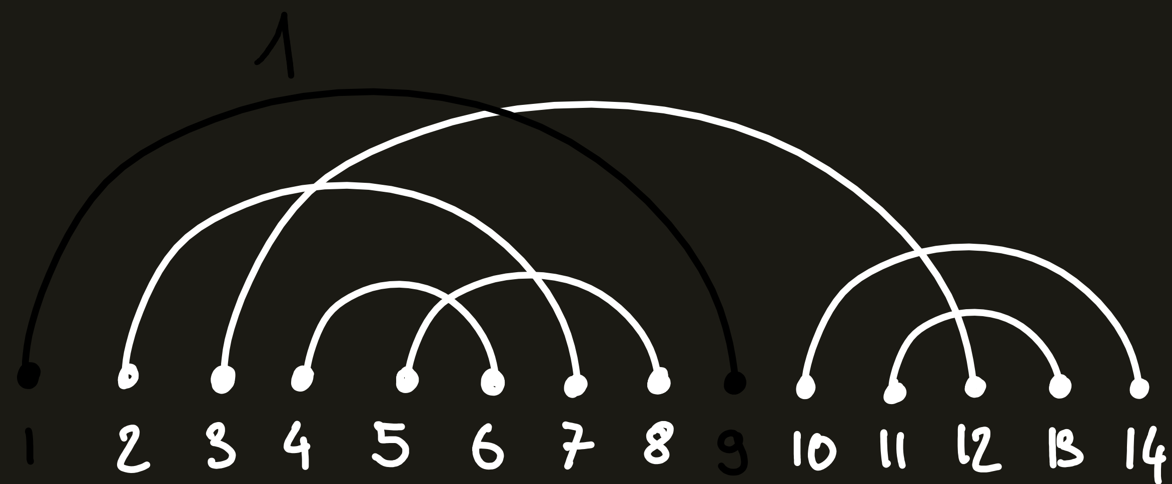

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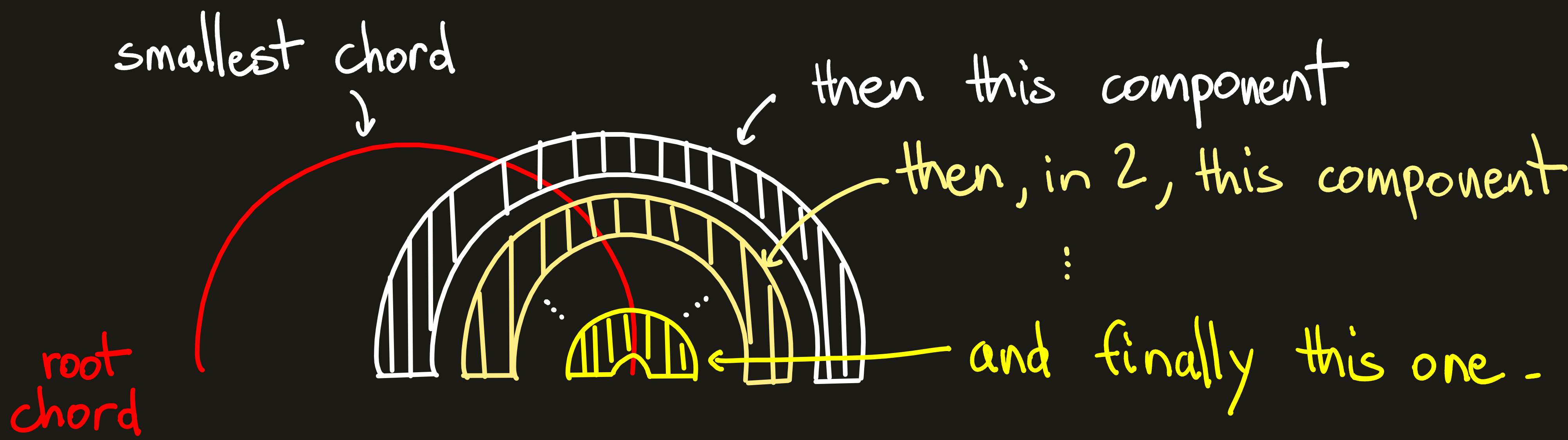

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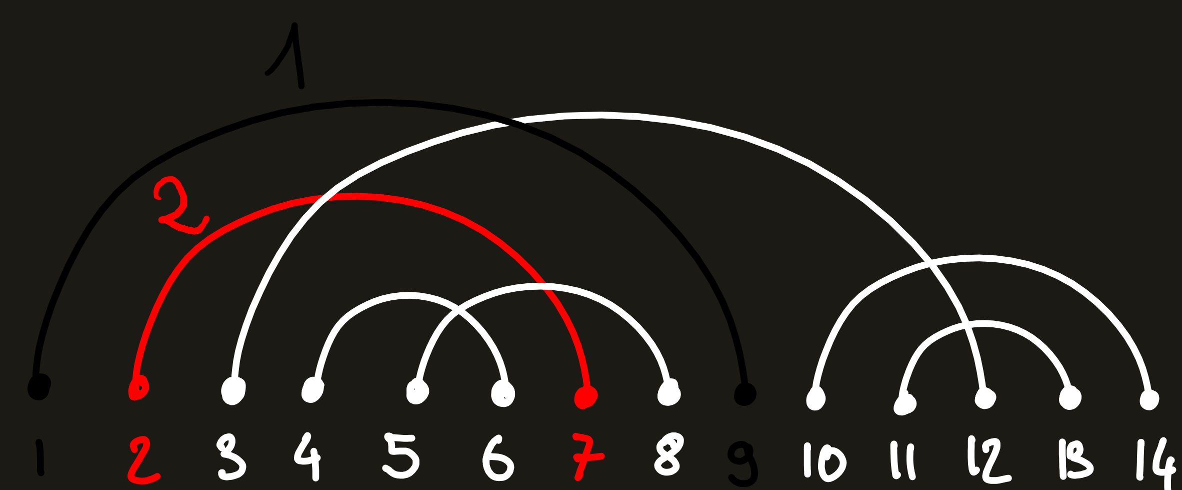

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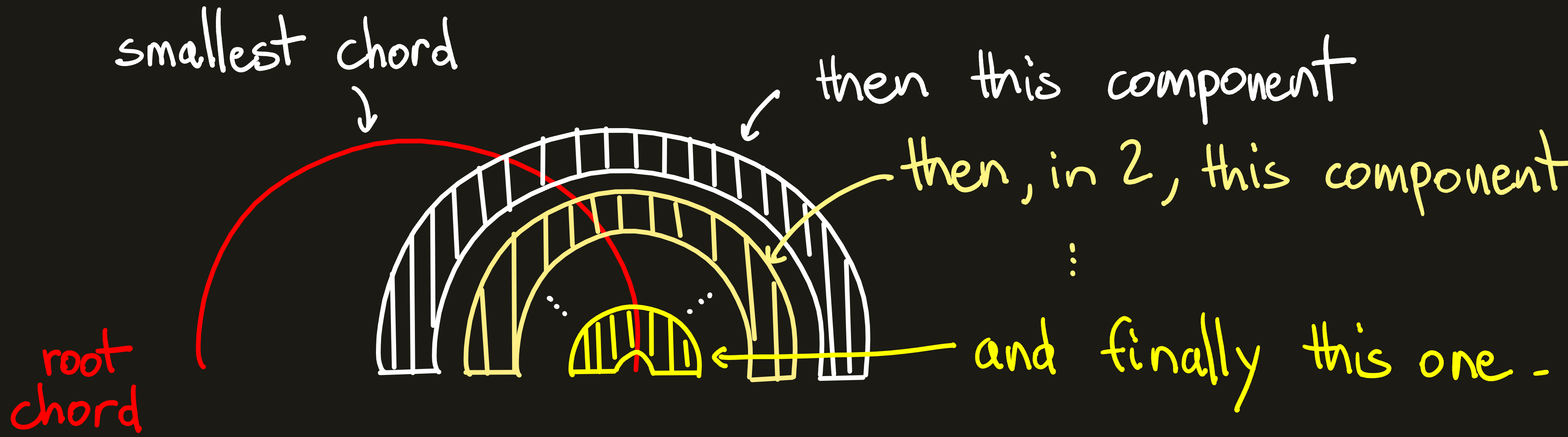

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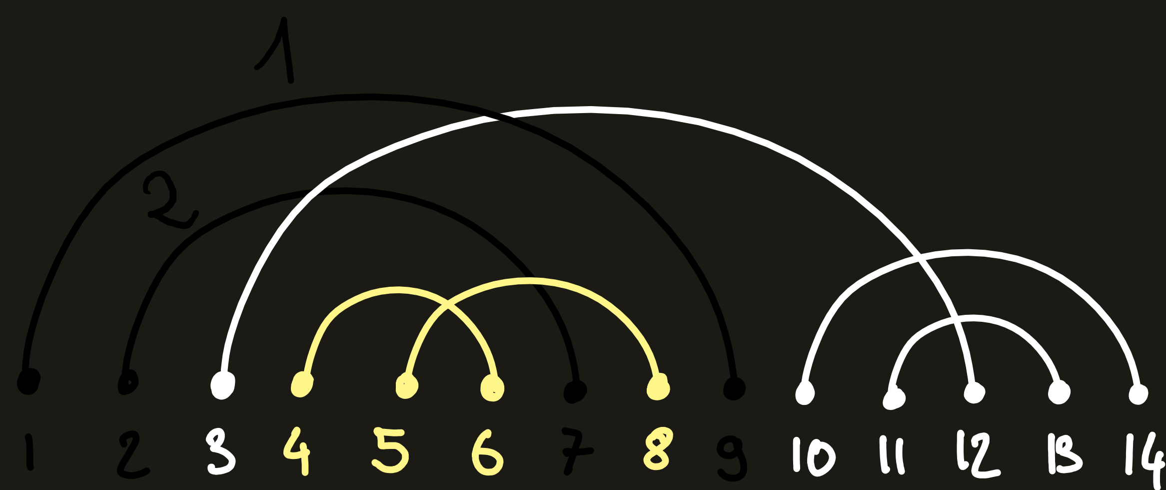

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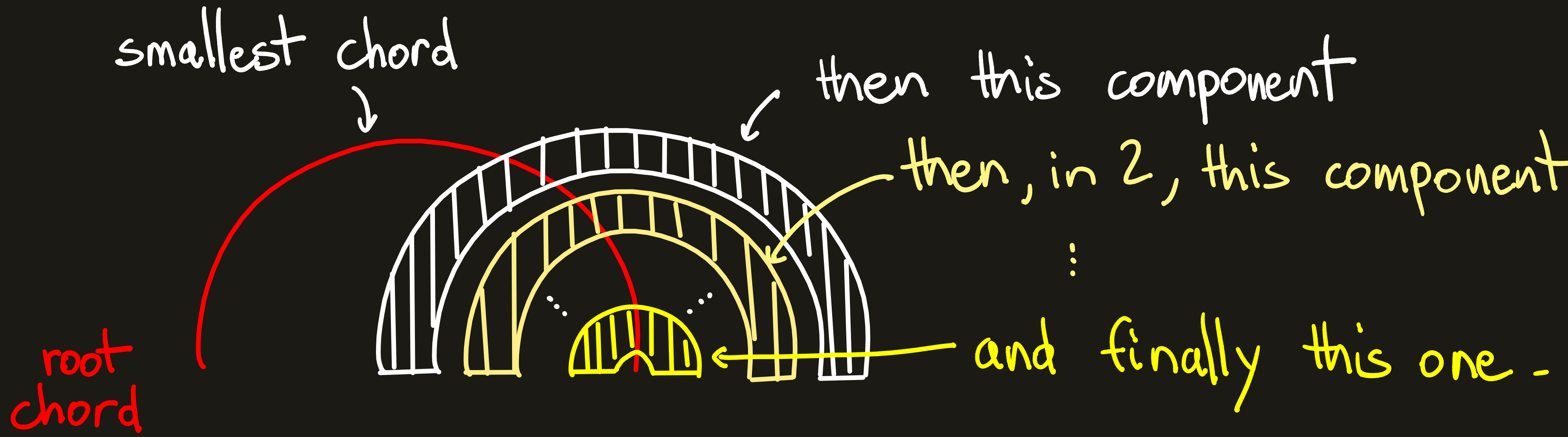

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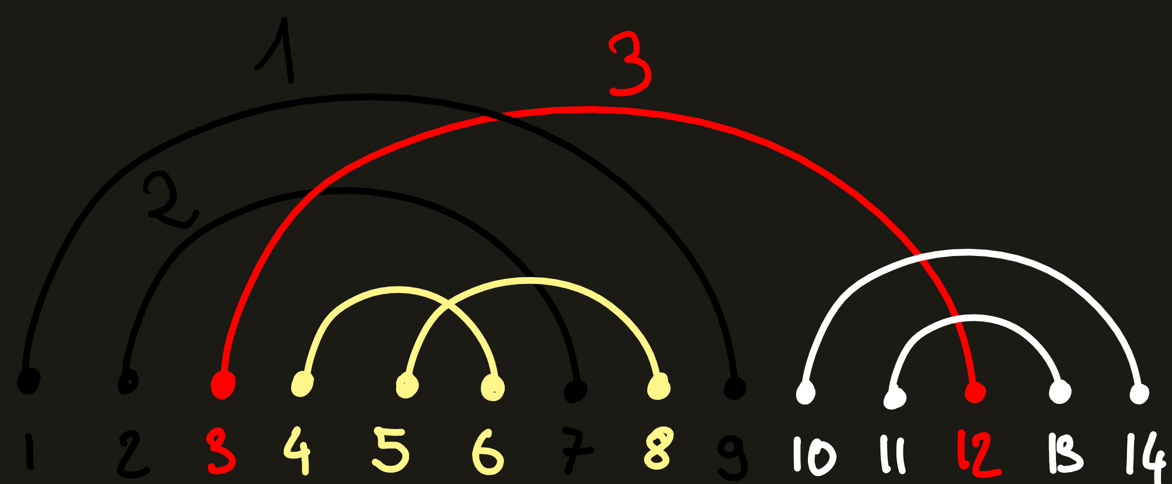

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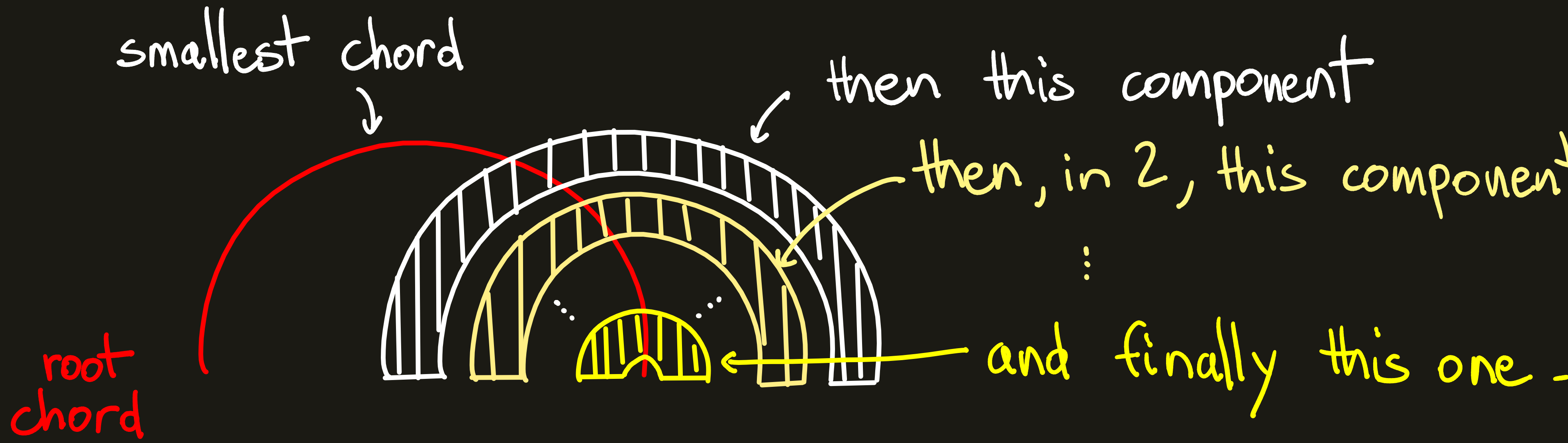

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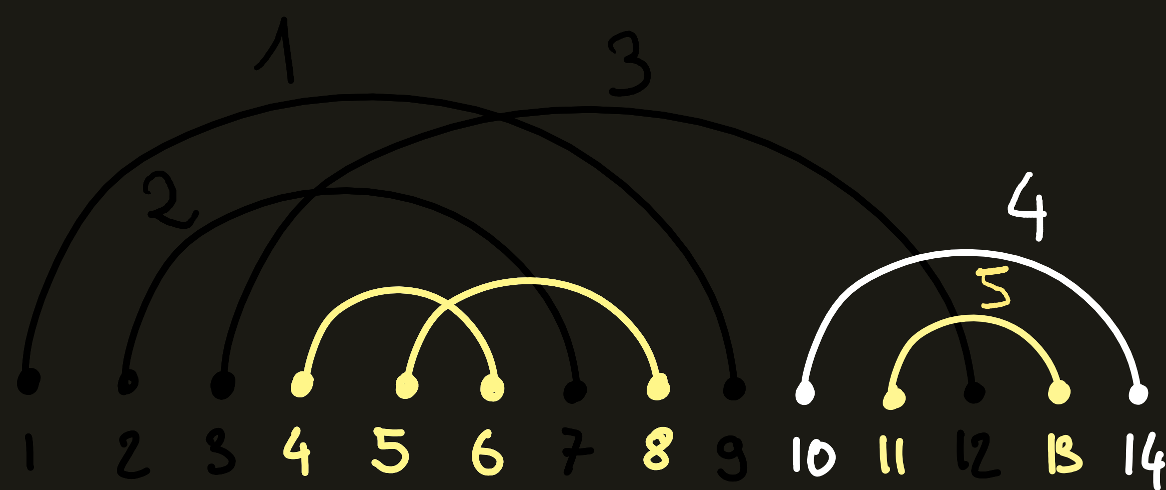

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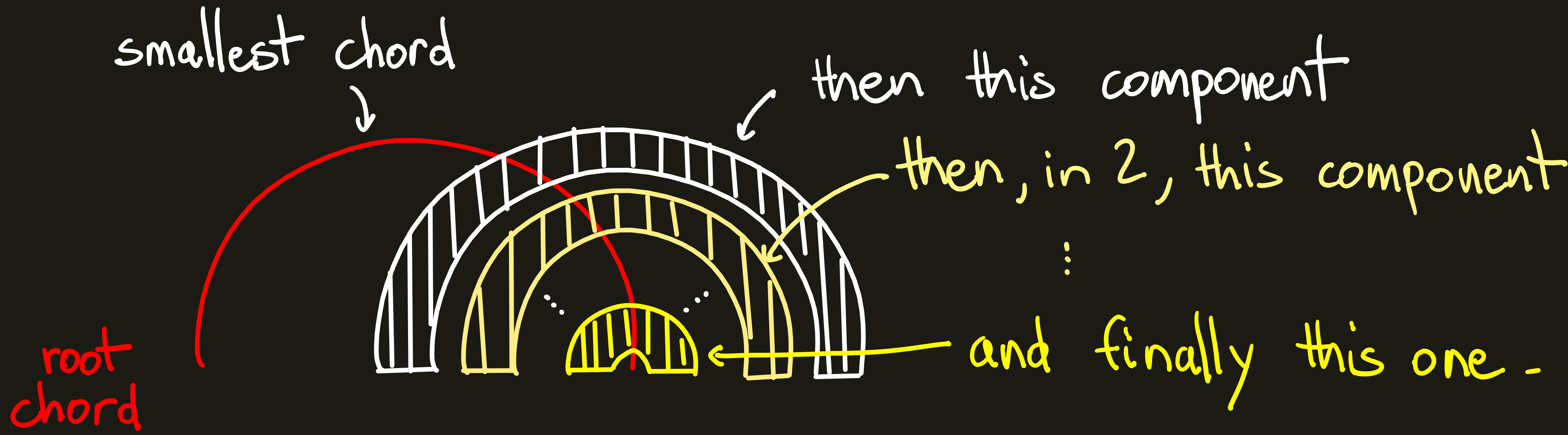

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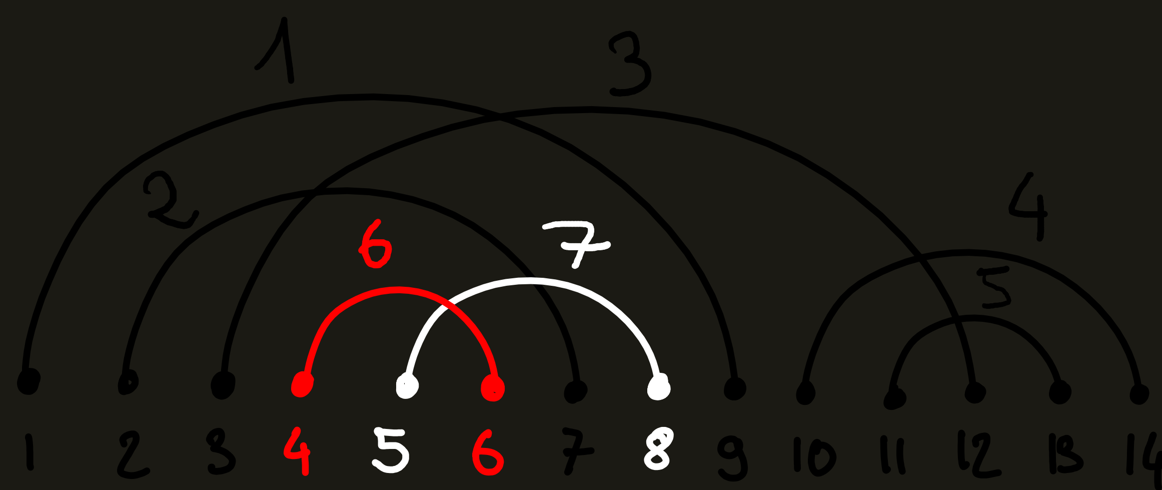

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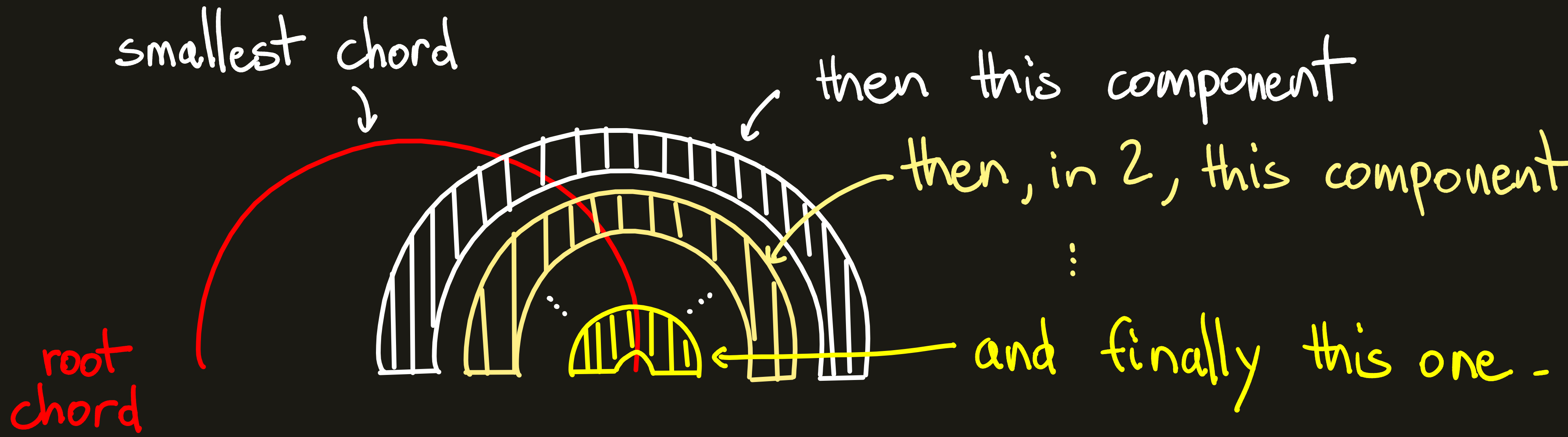

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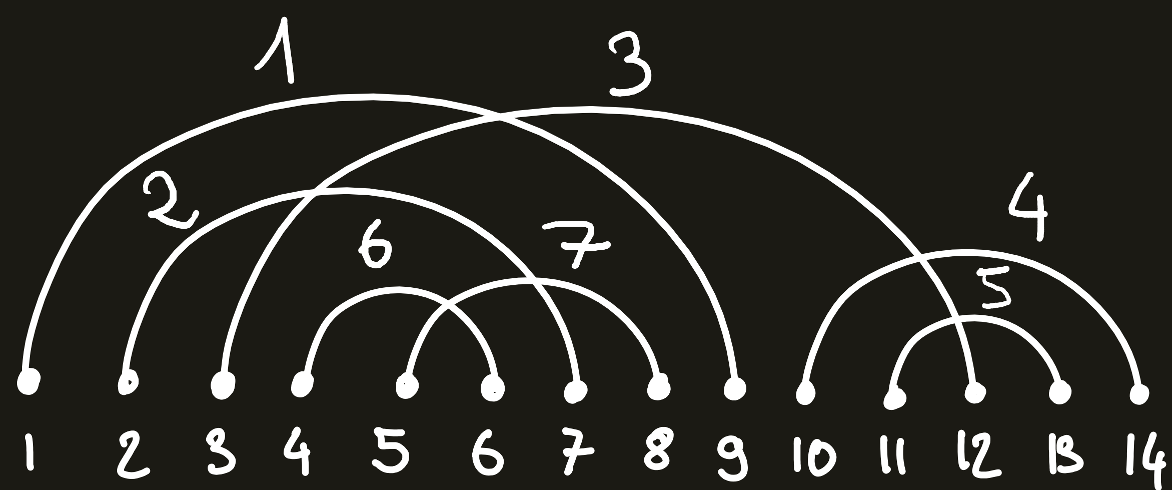

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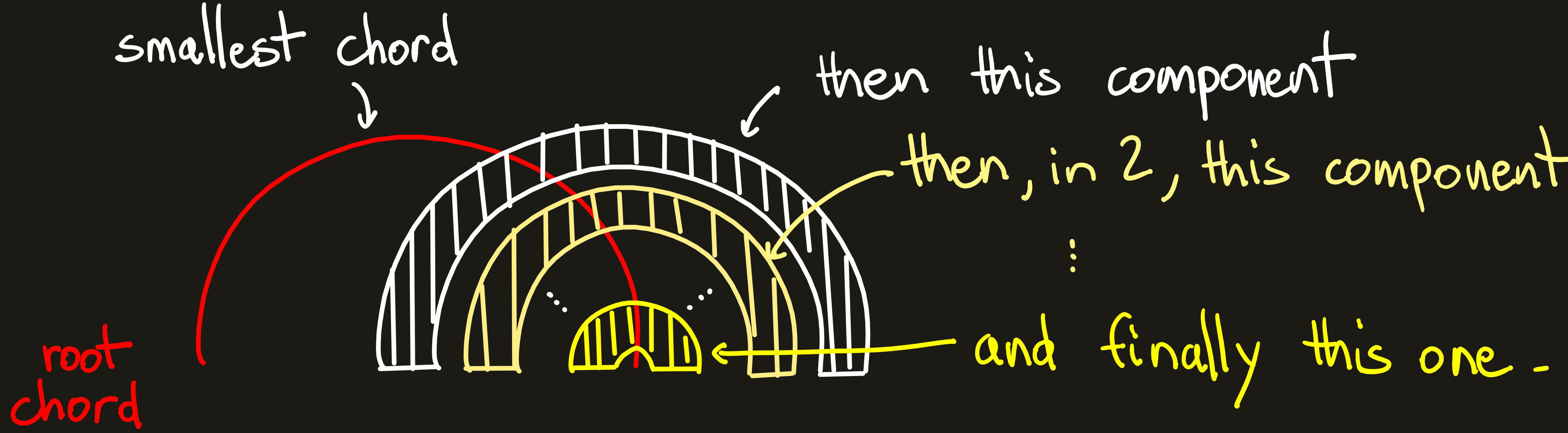

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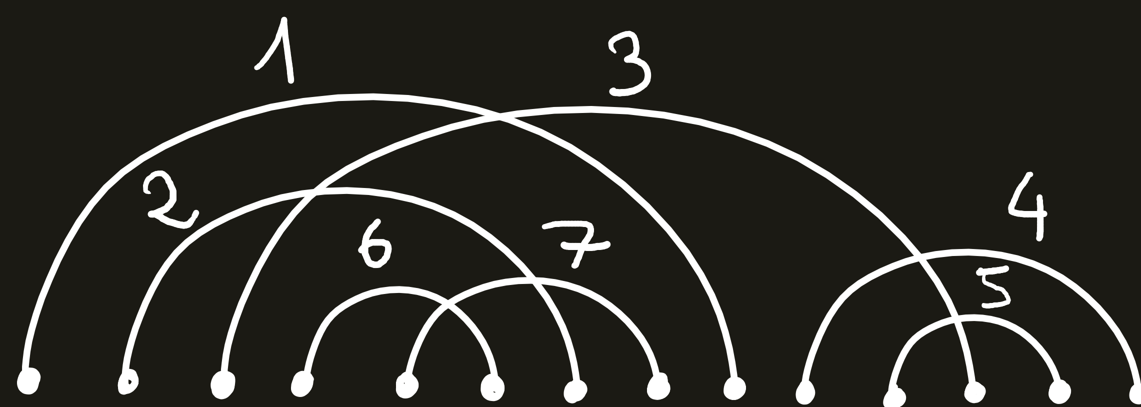

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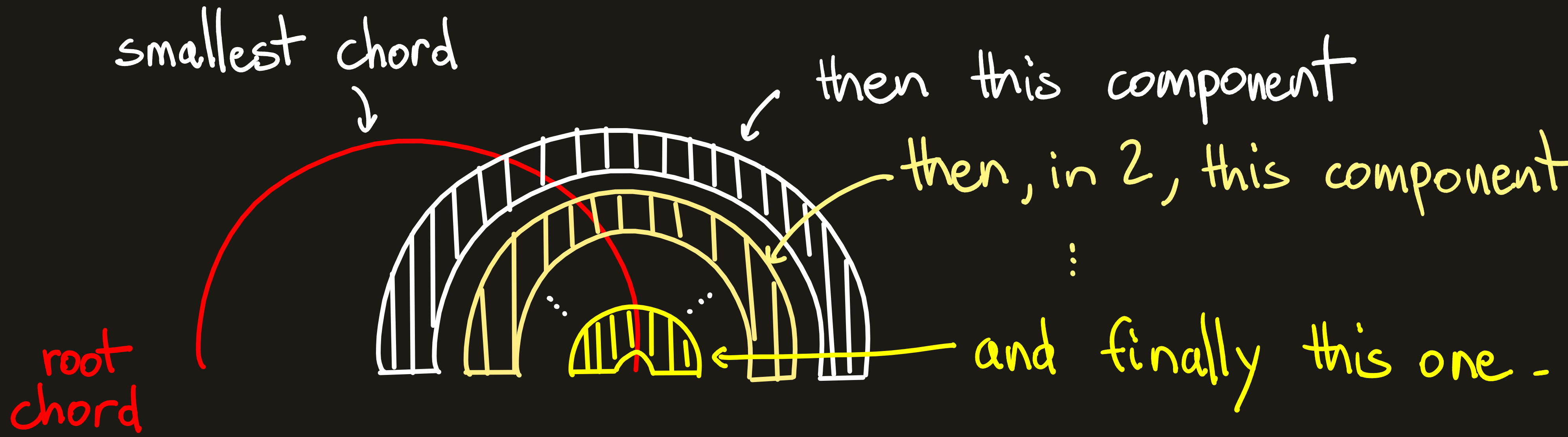

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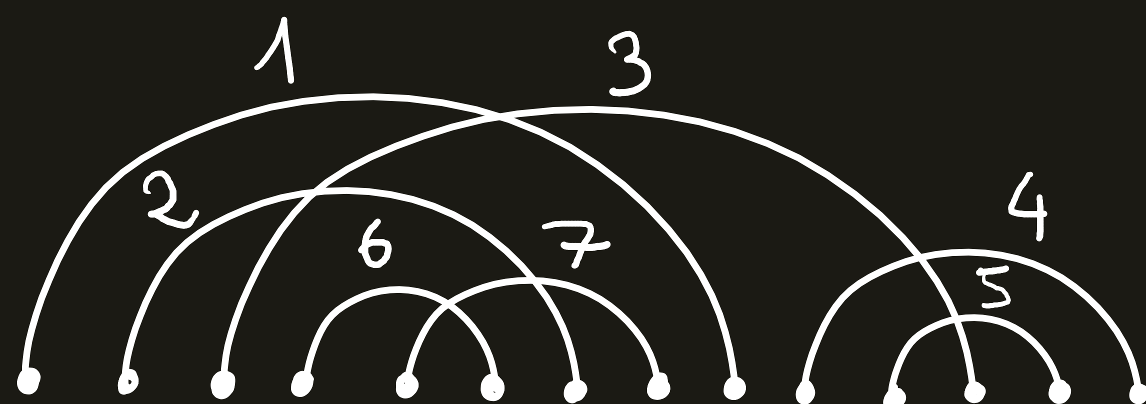

⚠ left-right order ≠ intersection order

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$\checkmark$ such that $t_1 < t_2 < \ldots < t_k$ denote the positions $\checkmark$ of the terminal $\checkmark$ chords of C

où $F(\rho) = \frac{f_0}{\rho} + f_1 + f_2\rho + f_3\rho^2 + \ldots$ = regularized Feynman integral of the one-loop graph

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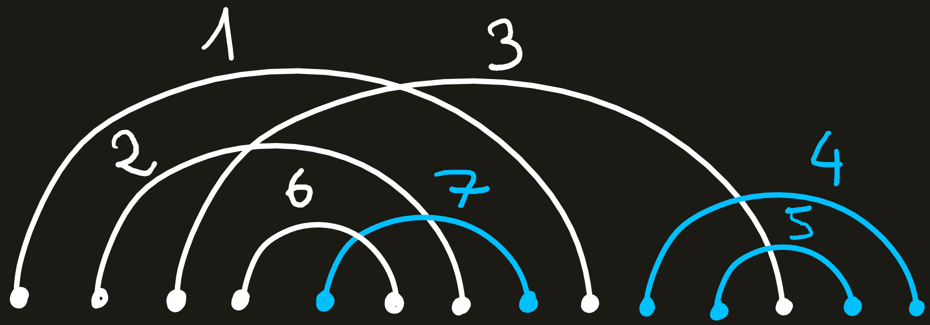

$t_1 = 4 \quad t_2 = 5 \quad t_3 = 7$

$$(*) = \left(-f_3 L + f_2 \frac{L^2}{2} - f_1 \frac{L^3}{3!} + f_0 \frac{L^4}{4!}\right) \times x^7 \times f_0^4 \times f_1 \times f_2$$

$$G(x,L) = 1 - \sum_{C} \underbrace{\left(\sum_{i=1}^{t_1} f_{t_1-i} \frac{(-L)^i}{i!} \right) x^{|C|} f_0^{|C|-k} f_{t_2-t_1} f_{t_3-t_2} \cdots f_{t_k-t_{k-1}}}_{(*)}$$

C connected chord diagram ✓
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denote the positions ✓
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GOAL

1. Computing the leading-log expansions

cf [Krüger Kreimer]

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2. Do some asymptotic

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2. Do some asymptotic

3. Repeat

PART II

SOME TRIVIA ABOUT CHORD DIAGRAMS

ELEMENTARY ENUMERATION

number of diagrams with n chords = ???

For $n = 1$: 

For $n = 2$: 

ELEMENTARY ENUMERATION

number of diagrams with n chords $= (2n-1)!!$

$$= (2n-1) \times (2n-3) \times \dots \times 3 \times 1$$

For $n=1$: 

For $n=2$: 

ELEMENTARY ENUMERATION

number of connected diagrams with n chords = c_n

$c_1 = 1 \quad c_2 = 1 \quad c_3 = 4 \quad c_4 = 27 \quad c_5 = 248$

For $n = 3$:

ELEMENTARY ENUMERATION

number of connected diagrams with n chords $= c_n$

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For $n = 3$:

Induction formula [Stein]

$$c_n = (n-1) \times \sum_{k=1}^{n-1} c_k \times c_{n-k}$$

ELEMENTARY ENUMERATION

Proof of $\quad c_n = (n-1) \times \sum_{k=1}^{n-1} c_k \times c_{n-k}$?

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$k \leftarrow n-k$

$$c_n = \sum_{k=1}^{n-1} (2(n-k)-1) \times c_{n-k} \times c_k$$

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$$2c_n = \sum_{k=1}^{n-1} (2n-2) \times c_k \times c_{n-k}$$

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$\% \; 2$

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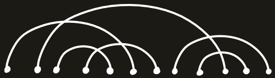

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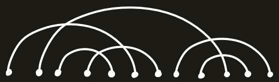

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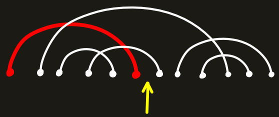

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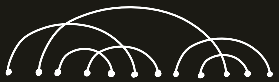

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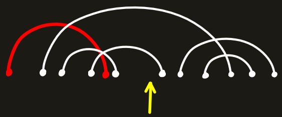
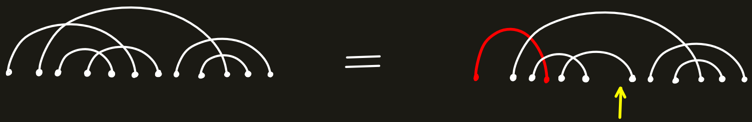


ELEMENTARY ENUMERATION

Proof of $\quad c_n = \sum_{k=1}^{n-1} (2k-1) \times c_k \times c_{n-k}$?



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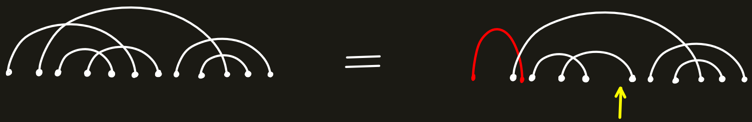

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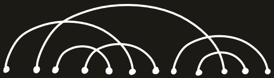

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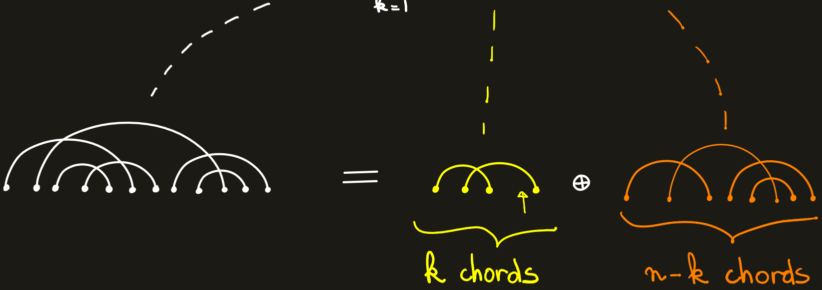

PART III

LEADING-LOG EXPANSIONS

LEADING-LOG EXPANSIONS

$$G(x,L) = 1 - \sum_{\substack{C \text{ connected} \\ \text{chord diagram}}} \left(\sum_{i=1}^{t_1} f_{t_1 - i} \frac{(-L)^i}{i!} \right) x^{|C|} f_0^{|C|-k} f_{t_2 - t_1} f_{t_3 - t_2} \cdots f_{t_k - t_{k-1}}$$

such that $t_1 < t_2 < \ldots < t_k$

denote the positions of the terminal chords of C

Remember

LEADING-LOG EXPANSIONS

$$G(x,L) = 1 - \sum_{\substack{C \text{ connected} \\ \text{chord diagram}}} \left(\sum_{i=1}^{t_1} f_{t_1-i} \frac{(-L)^i}{i!} \right) x^{|C|} f_0^{|C|-k} f_{t_2-t_1} f_{t_3-t_2} \cdots f_{t_k-t_{k-1}}$$

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such that $t_1 < t_2 < \ldots < t_k$ denote the positions of the terminal chords of C

We can write $G(x,L) = 1 + \sum_{k \geq 0} H_k(xL)\, x^k$

<u>Definition:</u>

$H_0(z)$ = leading-log expansion

$H_1(z)$ = next-to leading-log expansion

$H_k(z) = \underbrace{\text{next-to next-to} \ldots \text{next-to}}_{k \text{ times}}$ leading-log expansion.

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---

number of connected diagrams with n chords and only 1 terminal chord

$$\| $$

$$(2n-3)!! = (2n-3) \times (2n-1) \times \ldots \times 3 \times 1$$

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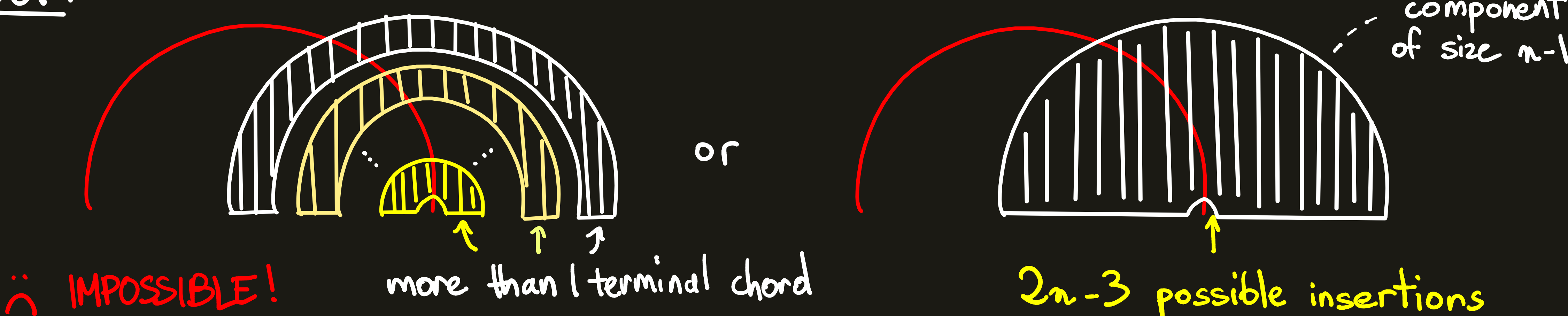

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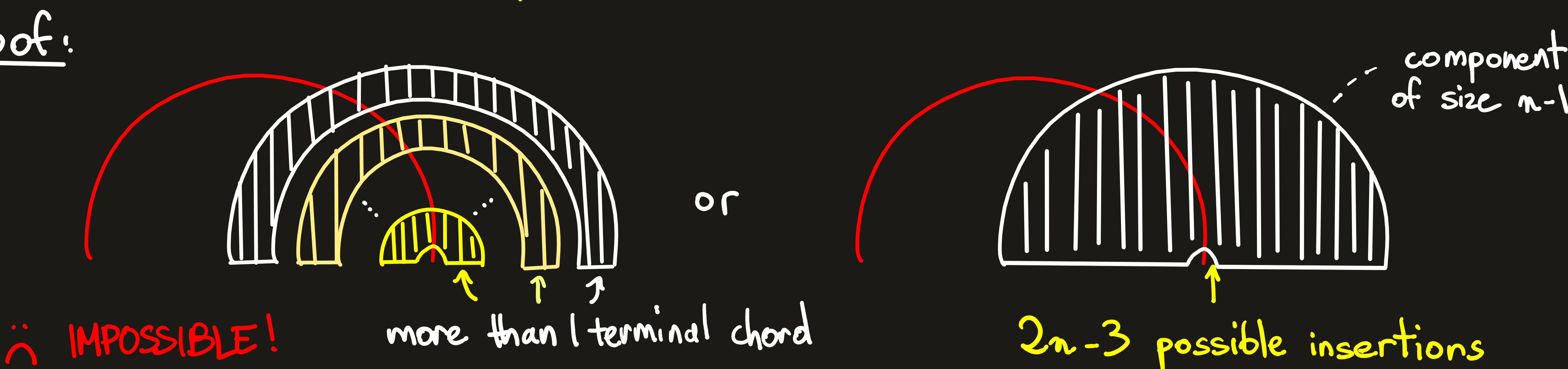

Corollary: n^{th} coefficient of the leading-log expansion $= \dfrac{(2n-3)!!}{n!} f_0^{\,n}$

LEADING-LOG EXPANSIONS

$$G(x,L) = 1 - \sum_{i \geq 1} \sum_{\substack{C \text{ connected} \\ \text{chord diagram}}} f_{t_1 - i} \frac{(Lx)^i}{i!} (-1)^i x^{|C|-i} f_0^{|C|-k} f_{t_2-t_1} f_{t_3-t_2} \cdots f_{t_k-t_{k-1}}$$

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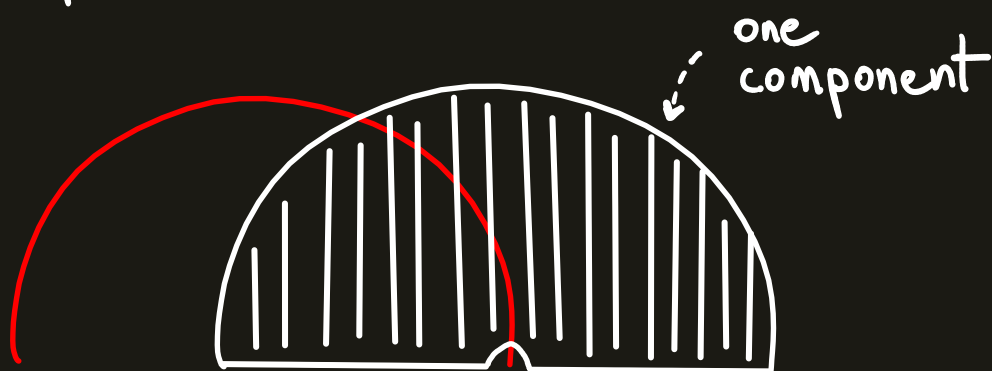
DIAGRAMS SUCH THAT $t_1 \geq |C|-1$

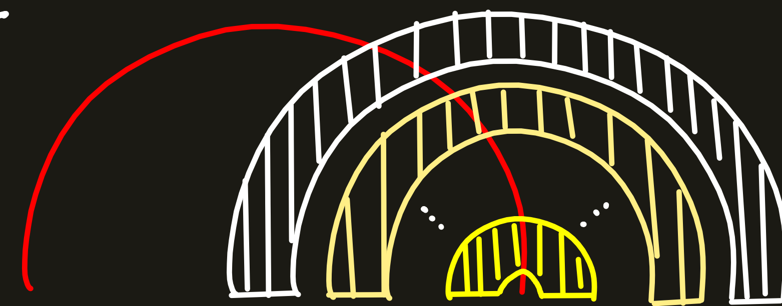
a_n = number of connected diagrams with n chords such that $t_1 \geq |C|-1$

<u>Two possibilities:</u>

1.



2.



insertion of a root chord into a diagram

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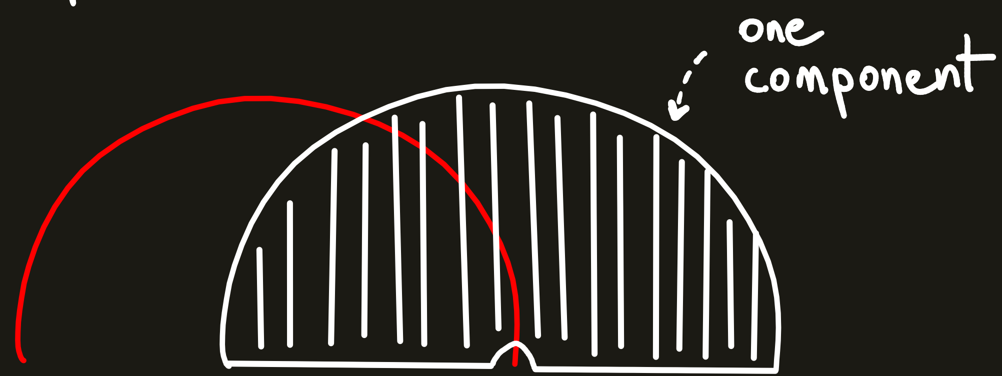

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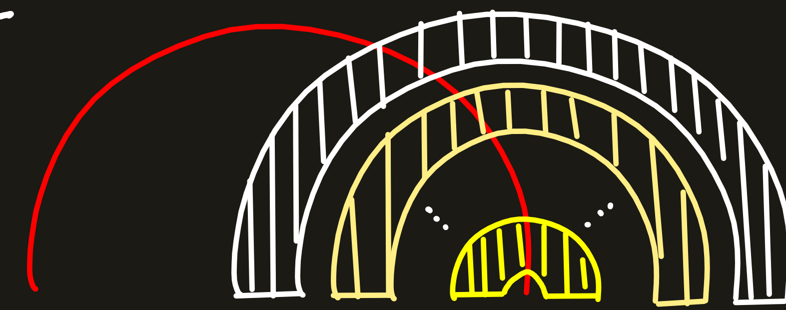


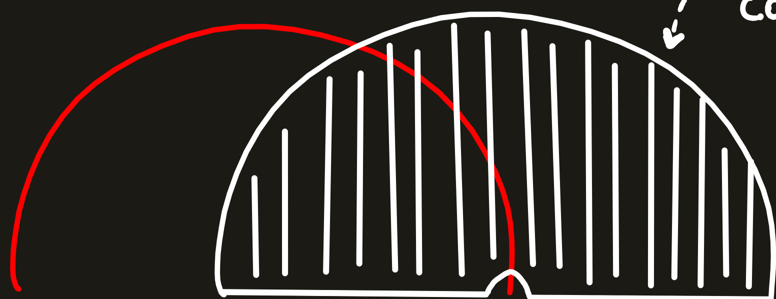
insertion of a root chord into a diagram

DIAGRAMS SUCH THAT $t_1 \geq |C| - 1$

a_n = number of connected diagrams with n chords such that $t_1 \geq |C| - 1$

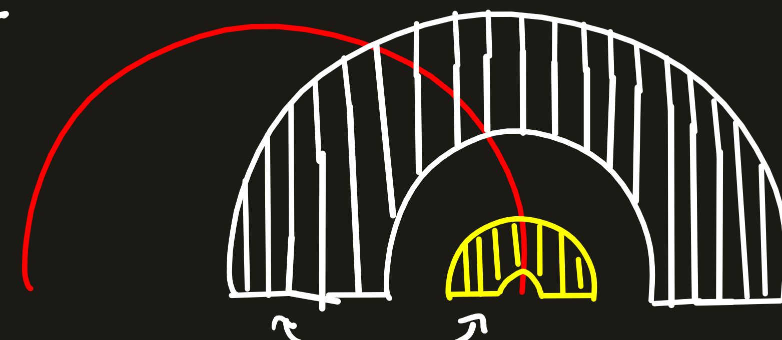
<u>Two possibilities:</u>

1. one component



insertion of a root chord into a diagram

2.



two components because at most 2 terminal chords

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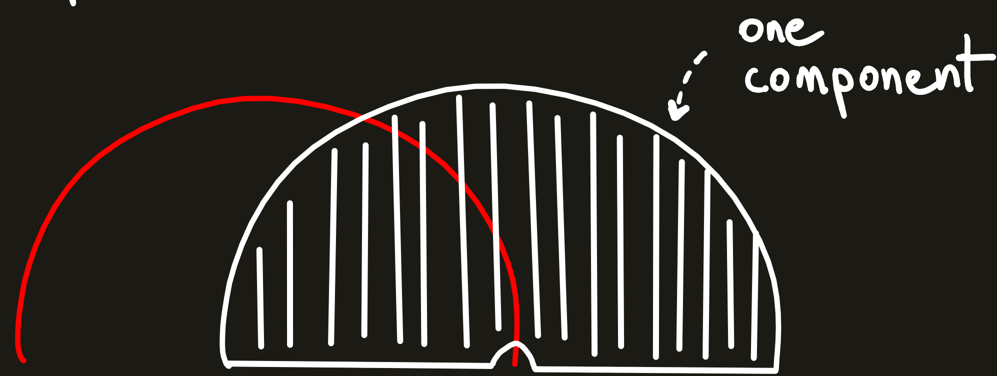

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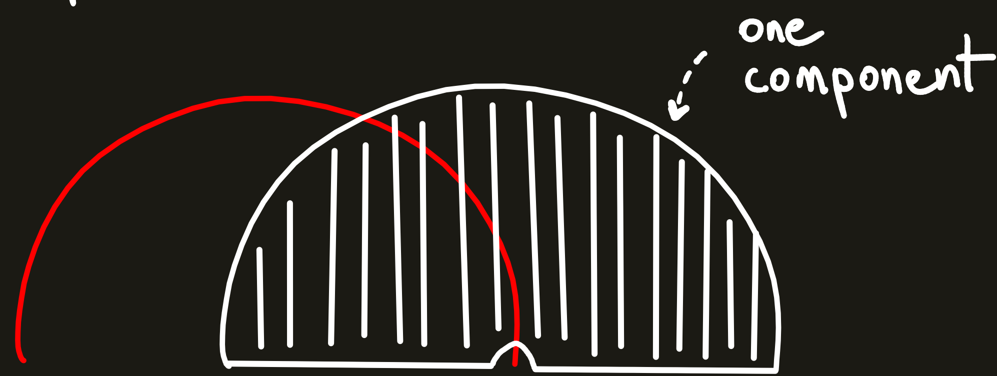

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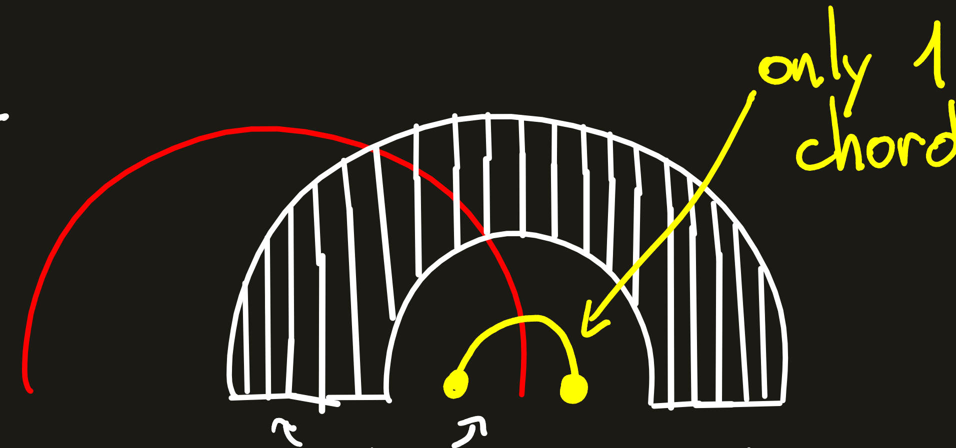

two components because at most 2 terminal chords

Recurrence: $a_n = (2n-3)\, a_{n-1} + (2n-5)!!$

DIAGRAMS SUCH THAT $t_1 \geq |C|-1$

a_n = number of connected diagrams with n chords such that $t_1 \geq |C|-1$

Generating function: $\sum_{n \geq 0} \frac{a_n}{n!} z^n = 1 + z + \sqrt{1-2z}\left(\ln(1-2z)+1\right)$

Recurrence: $a_n = (2n-3)\, a_{n-1} + (2n-5)!!$

DIAGRAMS SUCH THAT $t_1 \geq |C|-1$

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Next-to leading-log expansion: $H_1(z) = \frac{\beta_1}{2} \frac{\ln(1+2\beta_0 z)}{\sqrt{1+2\beta_0 z}}$

Generating function: $\sum_{n\geq 0} \frac{a_n}{n!} z^n = 1 + z + \sqrt{1-2z}\left(\ln(1-2z)+1\right)$

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DIAGRAMS SUCH THAT $t_1 \geq |C|-1$

a_n = number of connected diagrams with n chords such that $t_1 \geq |C|-1$

Asymptotic behaviour: nth coeff of $H_1 \sim f_0^{n-1} f_1 \dfrac{\ln(n)\, n^{-\frac{3}{2}}\, n!}{4\sqrt{\pi}}\, 2^n$

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DIAGRAMS SUCH THAT $t_1 \geq |C| - \ell$

→ Same type of decomposition applies

Theorem: For $\ell \geq 0$,

numbers of connected diagrams with n chords such that $t_1 \geq |C| - \ell$

$$\sim \frac{1}{\sqrt{\pi}\, 2^{\ell+1}\, \ell!}\, 2^n \frac{\ln(n)^{\ell}}{n^{\frac{3}{2}}} \times n!$$

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→ Exact expression for the $\underbrace{\text{next-to} \ldots \text{next-to}}_{\ell}$ leading-log expansion impossible to compute for general ℓ

→ how about an asymptotic estimate?

DIAGRAMS WHERE THE $\ell+1$ TERMINAL CHORDS ARE IN LAST

→ Same type of decomposition applies

Theorem: For $\ell \geq 0$,

numbers of connected diagrams with n chords such that the $\ell+1$ last chords are the only terminal chords

$$\sim \frac{1}{\sqrt{\pi}\, 2^{\ell+1}\, \ell!} 2^n \frac{\ln(n)^\ell}{n^{\frac{3}{2}}} \times n!$$

Here $f_0^{|C|-k} f_{t_1-i} f_{t_2-t_1} f_{t_3-t_2} \cdots f_{t_k-t_{k-1}} = f_0^{n-\ell} f_1^{\ell}$

$(\text{NEXT-TO})^{\ell}$ LEADING LOG EXPANSIONS

Thus the connected diagrams such that the $\ell+1$ last chords are terminal are dominant amongst diagrams such that $t_1 \geq |C| - \ell$

Theorem: For $\ell \geq 0$,

n^{th} coefficient of the $(\text{next-to})^{\ell}$ leading-log expansion:

$$\sim \frac{1}{\sqrt{\pi}\, 2^{\ell+1}\, \ell!} \times 2^n \times \frac{\ln(n)^{\ell}}{n^{\frac{3}{2}}} \times n! \times f_0^{\,n-\ell} \times f_1^{\,\ell}$$

Only f_0 and f_1 matter!

JUST BY CURIOSITY... HOW ABOUT AN UNIFORM RANDOM CONNECTED DIAGRAM?

What we also proved:

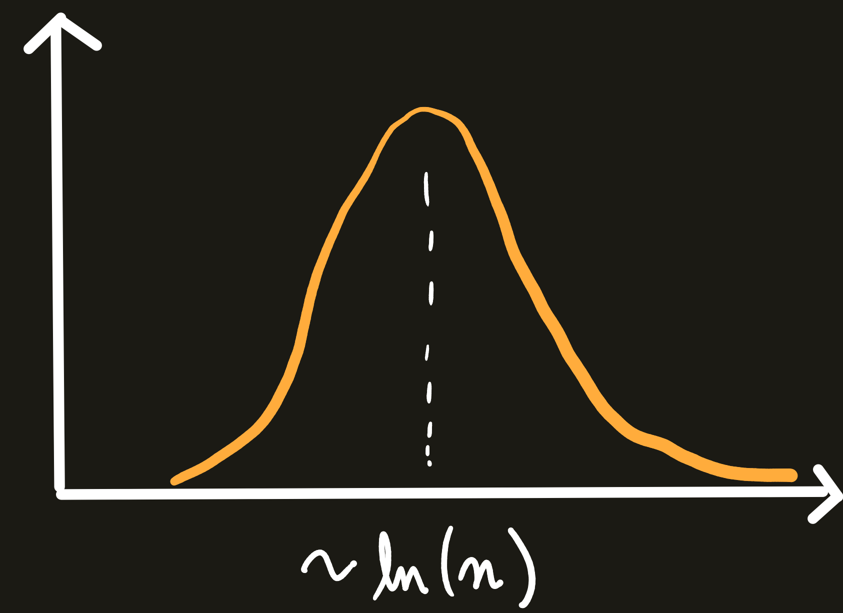
JUST BY CURIOSITY... HOW ABOUT AN UNIFORM RANDOM CONNECTED DIAGRAM?

What we also proved:

number of terminal chords $\xrightarrow[\text{law}]{}$ Gaussian law

mean $\sim \ln(n)$

var $\sim \ln(n)$



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number of i such that $t_i - t_{i-1} = 1$ $\xrightarrow[\text{law}]{}$ Gaussian law
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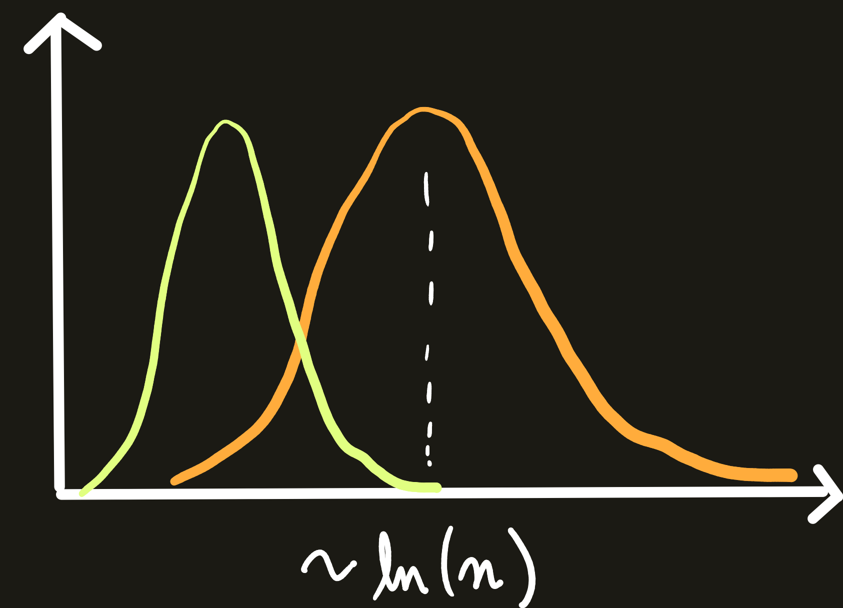

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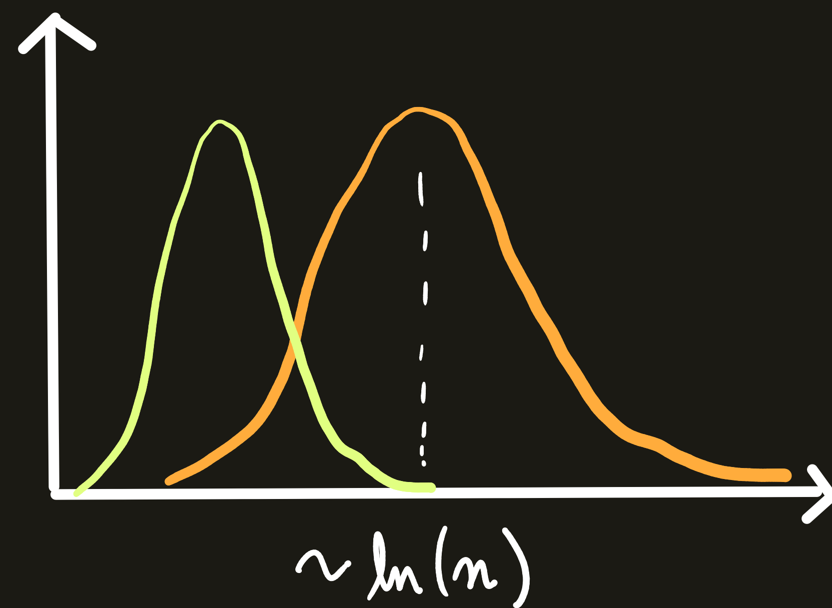


In average, $\beta_0^{|C|\;k}\,\beta_{t_1-i}\,\beta_{t_2-t_1}\beta_{t_3-t_2}\cdots\beta_{t_k-t_{k-1}} \sim \beta_0^{\,n-\ln n}\,\beta_{t_1-i}\,\beta_1^{\frac{\ln n}{2}}\cdots$

$\rightarrow$ confirms the importance of β_0 and β_1

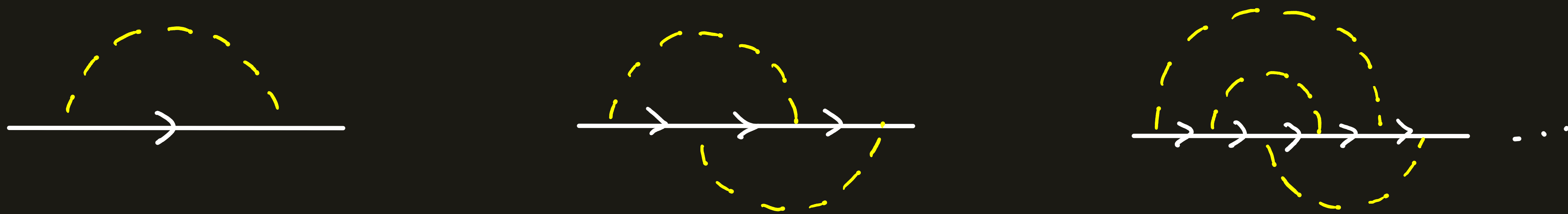
PART IV

WHAT AM I DOING HERE?

THE NEW PHYSICAL CONTEXT

→ paper from [Hihn Yeats]

- Yukawa theory



<u>Every</u> renormalized Feynman integral contributes to a new Green function

THE NEW PHYSICAL CONTEXT

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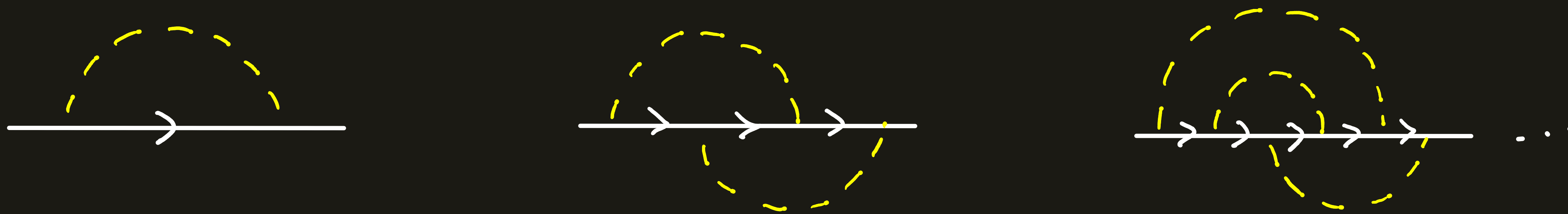
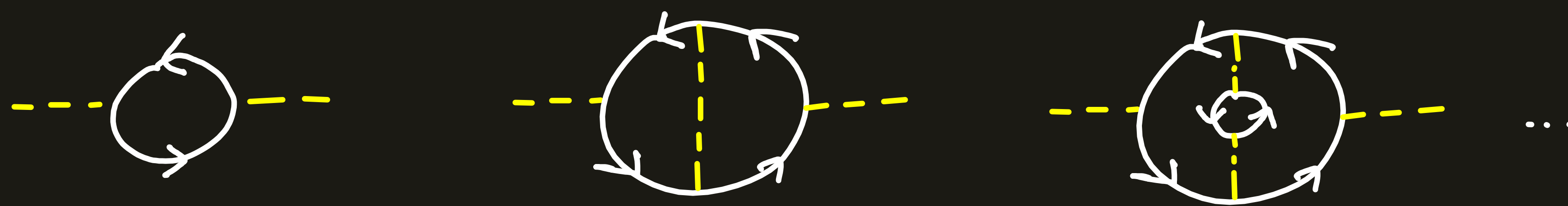


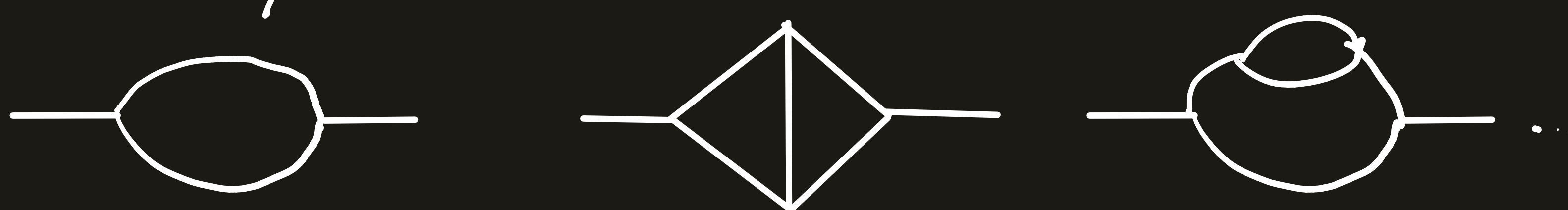
Every renormalized Feynman integral contributes to a new Green function

- QED



- Scalar ϕ^3-theory

⋮



THE NEW PHYSICAL CONTEXT

The corresponding Dyson-Schwinger equation is :

$$G(x, L) = 1 - \sum_{k \geq 1} x^k \, G(x, \partial_{-p})^{1-sk} \left(e^{-Lp} - 1\right) F_k(p)$$

where $F_k(p)$ =

the regularized Feynman integral of the primitive graphs of size k

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What is s? The insertion growth number!

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---

What is s? The insertion growth number!

Ex : Yukawa theory

THE NEW PHYSICAL CONTEXT

The corresponding Dyson-Schwinger equation is :

$$G(x,L) = 1 - \sum_{k\geq 1} x^k \, G(x,\partial_{-p})^{1-\delta k} \left(e^{-Lp}-1\right) F_k(p)$$

where $F_k(p)$ =
the regularized Feynman integral of the primitive graphs of size k

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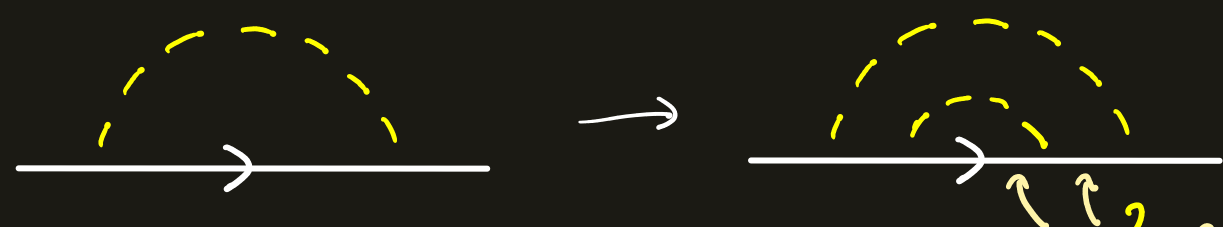
What is δ? The insertion growth number!

<u>Ex</u> : Yukawa theory

$\boxed{\delta = 2}$



2 extra places

THE NEW PHYSICAL CONTEXT

The corresponding Dyson-Schwinger equation is :

$$G(x,L) = 1 - \sum_{k\geq 1} x^k \, G(x, \partial_{-p})^{1-sk} \, (e^{-Lp} - 1) \, F_k(p)$$

where $F_k(p)$ =
the regularized Feynman integral of the primitive graphs of size k

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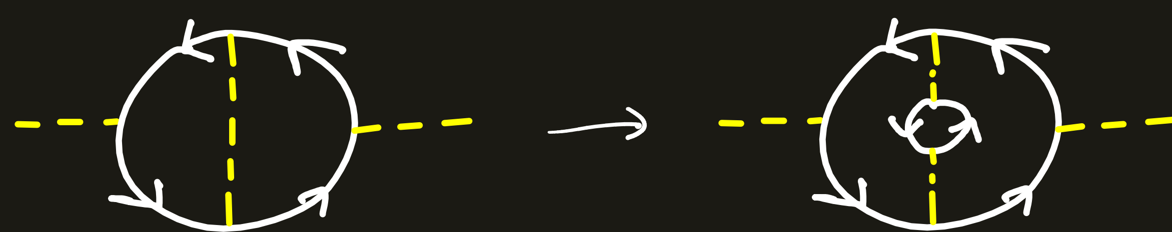
What is s? The insertion growth number!

Ex : QED



THE NEW PHYSICAL CONTEXT

The corresponding Dyson-Schwinger equation is :

$$G(x,L) = 1 - \sum_{k\geq 1} x^k \, G(x, \partial_{-\rho})^{1-sk} \, (e^{-L\rho} - 1) \, F_k(\rho)$$

where $F_k(\rho)$ =

the regularized Feynman integral of the primitive graphs of size k

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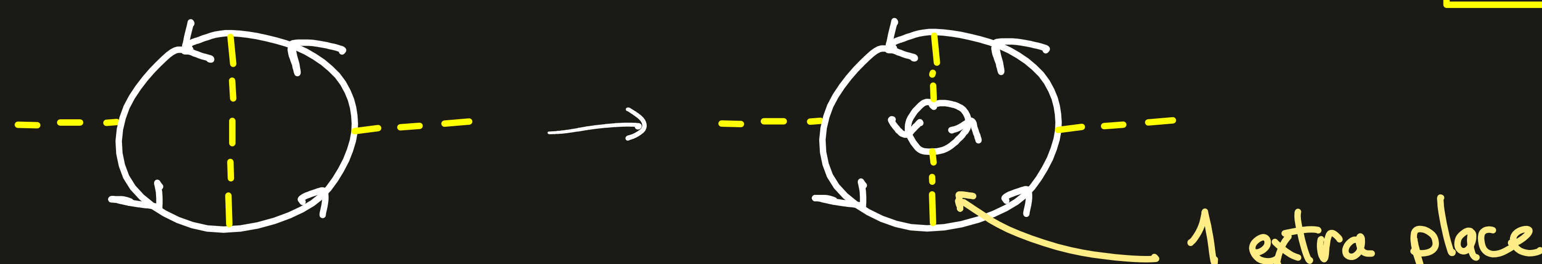
What is s ? The insertion growth number!

Ex : QED

$$\boxed{s = 1}$$



THE NEW PHYSICAL CONTEXT

The corresponding Dyson-Schwinger equation is :

$$G(x,L) = 1 - \sum_{k\geq 1} x^k \, G(x,\partial_{-p})^{1-sk} \left(e^{-Lp}-1\right) F_k(p)$$

where $F_k(p)$ =
the regularized Feynman integral of the primitive graphs of size k

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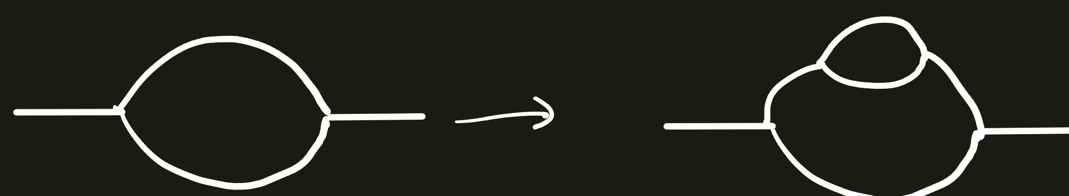
What is s? The insertion growth number!

Ex: Scalar ϕ^3-theory

$$\boxed{s = 3}$$



THE NEW PHYSICAL CONTEXT

The corresponding Dyson-Schwinger equation is:

$$G(x,L) = 1 - \sum_{k\geq 1} x^k \, G(x, \partial_{-p})^{1-\Delta k} \, (e^{-Lp} - 1) \, F_k(p)$$

where $F_k(p)$ =
the regularized Feynman integral of the primitive graphs of size k

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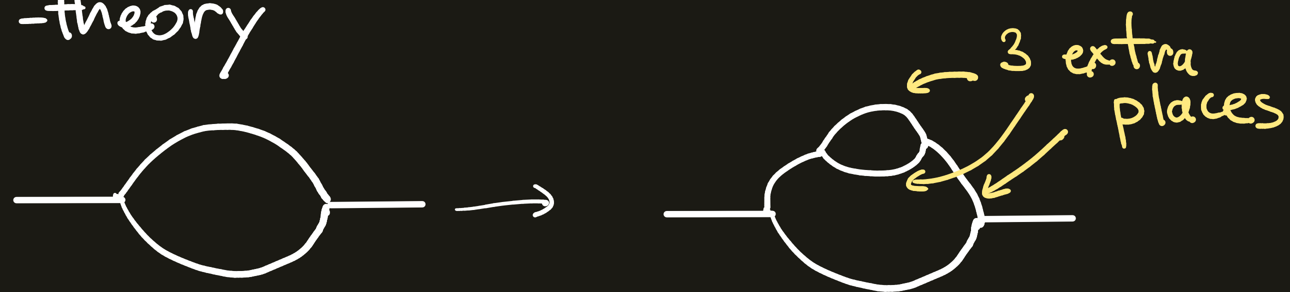
What is Δ? The insertion growth number!

<u>Ex</u>: Scalar ϕ^3-theory

$\boxed{\Delta = 3}$



NEW CHORD DIAGRAM EXPANSION

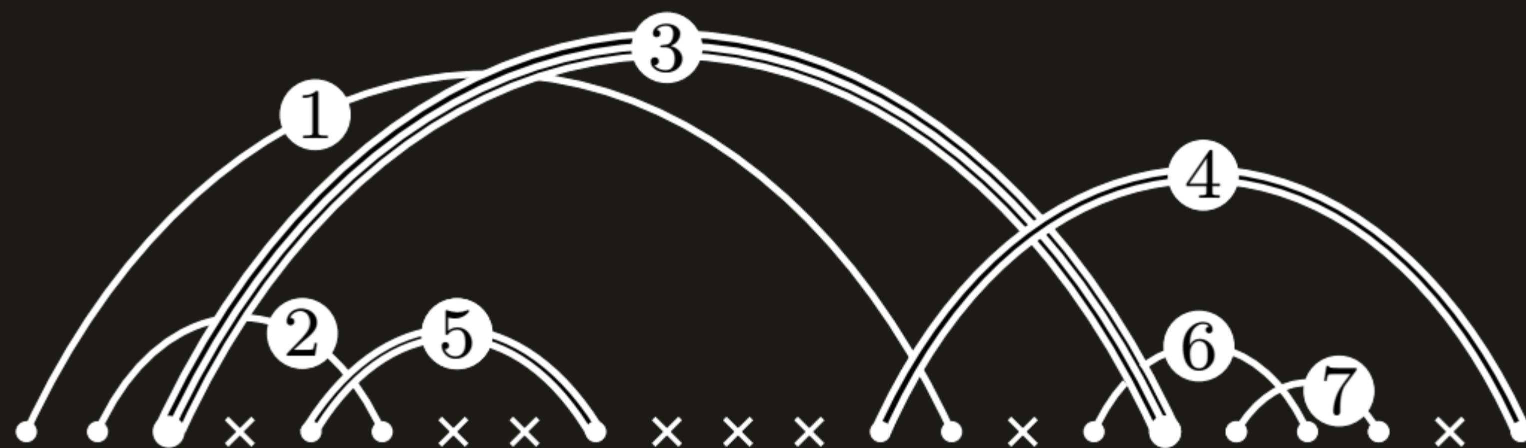
<u>Theorem</u> [Hihn Yeats] [Courtiel Yeats]

The previous Dyson - Schwinger equation has for solution:

$$G(x,L) = 1 - \sum_{\substack{C \text{ co-marked} \\ \text{diagram such} \\ \text{that } t_1 < t_2 < \dots < t_k \\ \text{are the positions of the terminal chords}}} \left(\sum_{i=1}^{t_1} f_{d(t_1),\, t_1 - i} \frac{(-L)^i}{i!} \right) \times \prod_{\substack{c \text{ non} \\ \text{terminal}}} f_{d(c),0} \times \prod_{i=1}^{k-1} f_{d(t_i),\, t_i - t_{i-1}} \, x^{\|C\|}$$

where $\frac{f_{k,0}}{\rho} + f_{k,1} + f_{k,2}\,\rho + f_{k,3}\,\rho^2 + \dots = F_k(\rho) =$ regularized Feynman integral of the primitive graphs of size k

AN EXAMPLE OF AN ω-MARKED DIAGRAM



(s = 2)

OUR NEW RESULTS

→ automatic computation of the $(\text{next-to})^{\ell}$ leading-log expansions
→ asymptotic results: dichotomy with respect to s!

OUR NEW RESULTS

→ automatic computation of the $(\text{next-to})^{\ell}$ leading-log expansions
→ asymptotic results: dichotomy with respect to s!

ℓ^{th} leading-log expansion

$s = 1$	$s \geq 2$
Domination of diagrams with ℓ chords of decoration 2 $n^{\underline{th}}$ coefficient $\sim$ $C \times \frac{\ln(n)^{\ell-2}}{n^{\frac{3}{2}}} \times n! \times \mathfrak{f}_{1,0}^{\,n-\ell} \times \mathfrak{f}_{2,0}^{\,\ell}$	Domination of diagrams with ℓ_1 chords of decoration 2 and ℓ_2 terminal chords in last positions where $\ell_1 + \ell_2 = \ell$ $n^{\underline{th}}$ coefficient $\sim$ $\sum_{\ell_1,\ell_2} C_{\ell_1,\ell_2} \frac{\ln(n)^{\ell}}{n^{\frac{3}{2}}} s^n n! \times \mathfrak{f}_{0,1}^{\,n-\ell} \times \mathfrak{f}_{1,1}^{\,\ell_1} \times \mathfrak{f}_{2,0}^{\,\ell_2}$

THANK YOU!