A GENERAL NOTION OF ACTIVITY FOR THE TUTTE POLYNOMIAL

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WHY THE TUTTE POLYNOMIAL?

→ Graph invariant

→ Numerous interesting specializations
  e.g. the chromatic polynomial

→ Closely related to the Potts model
  (statistical physics)

→ A lot of nice properties ...
THE TUTTE POLYNOMIAL

The Tutte polynomial of a connected graph $G = (V, E)$ is:

$$T_G(x, y) = \sum_{S \text{ spanning subgraph of } G} \frac{cc(S)-1}{(x-1)^{cc(S)}} \frac{\text{cycl}(S)}{(y-1)^{\text{cycl}(S)}}$$

$cc(S) = \text{number of connected components of } S$

$\text{cycl}(S) = \text{cyclomatic number of } S$

$= \text{minimal number of edges we need to remove from } S \text{ to obtain an acyclic graph}$

Prop:

$$T_G(x, y) \in \mathbb{N}[x,y]$$
THE TUTTE POLYNOMIAL

The Tutte polynomial of a connected graph \( G = (V, E) \) is defined by

\[
T_G(x, y) = \sum_{S \text{ spanning subgraph of } G} \frac{cc(S) - 1}{(x-1)^{cc(S)} (y-1)^{cycl(S)}}
\]

where

- \( cc(S) \) is the number of connected components of \( S \),
- \( cycl(S) \) is the cyclomatic number of \( S \),
- \( \text{minimal number of edges we need to remove from } S \text{ to obtain an acyclic graph} \).

Prop: \( T_G(x, y) \in \mathbb{N}[x,y] \)
\[(x-1)^2\]

\[4(x-1)\]

\[(x-1)(y-1)\]

\[5\]

\[4(y-1)\]

\[(y-1)^2\]
\[(x - 1)^2 \quad \text{# conn. comp} - 1 \quad \times \quad (y - 1) \quad \text{# cycles}\]
\[ T_G(x,y) = (x-1)^2 + 4(x-1) + (x-1)(y-1) + 5 + 4(y-1) + (y-1)^2 \]
\[ T_6(x,y) = (x-1)^2 + 4(x-1) + (x-1)(y-1) + 5 + 4(y-1) + (y-1)^2 \]

\[ = x^2 + x + xy + y + y^2. \]
**How to Interpret the Coefficients**

**Principle**: Map each spanning tree onto a set of edges called “active” edges such that

\[ T_G(x, y) = \sum_{T \text{ spanning tree of } G} x^{i(T)} y^{e(T)} \]

where \( i(T) = \) number of active edges inside \( T \)

\( e(T) = \) number of active edges outside \( T \).
How to Interpret the Coefficients

Principle: Map each spanning tree onto a set of edges called "active" edges such that

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\( e(T) \) = number of active edges outside \( T \)
Let $T$ be a spanning tree and $e$ an edge.
Let $T$ be a spanning tree and $e$ an edge outside $T$.

Fundamental cycle = unique cycle in $T + e$.
**FUNDAMENTAL CYCLE / COCYCLE**

Let $T$ be a spanning tree and $e$ an edge outside $T$,

fundamental cycle = unique cycle in $T \cup \{e\}$
FUNDAMENTAL CYCLE / COCYCLE

Let $T$ be a spanning tree and $e$ an edge inside $T$.

Fundamental cocycle = unique cocycle in $T \cup \{e\}$
Let $T$ be a spanning tree and $e$ an edge inside $T_j$

fundamental cocycle $= \text{unique cocycle in } T_j \cup \{e\}$
We label and order the edges: \( a < b < c < d \)
Tutte's Activity

We label and order the edges:
\[ a < b < c < d \]

Active edge = minimal edge inside its fundamental cycle/cocycle
**Tutte's Activity**

\[ a < b < c < d \]

\[
\begin{align*}
x & \quad x^2 & \quad xy & \quad y^2 & \quad y \\
\ast & \quad \ast & \quad \ast & \quad \ast & \quad \ast
\end{align*}
\]

\[
T_6(x, y) = x^2 + x + xy + y + y^2
\]
BERNARDI’S ACTIVITY: TOUR OF THE TREE

We embed and root the graph:
Bernardi's Activity: Tour of the Tree

We embed and root the graph:

Rules:
- Inside the tree
- Outside the tree
Bernardi's Activity: Tour of the Tree

We embed and root the graph:

Rules:

- Inside the tree
- Outside the tree

We walk along.
Bernardi's Activity: Tour of the Tree

We embed and root the graph:

Rules:

- inside the tree: We walk along.
- outside the tree: We cross.
Bernardi's Activity: Tour of the Tree

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Rules:

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Bernardi's Activity: Tour of the Tree

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Rules:

- Inside the tree: We walk along.
- Outside the tree: We cross.
Bernardi’s Activity: Definition

We embed and root the graph:

![Graph Diagram]

Active edge = minimal edge inside its fundamental cycle/cocycle (for the first visit order)
Bernardi's Activity: Definition

\[ T_G(x,y) = x^2 + x + xy + y + y^2 \]
QUESTION

Can we define a "meta-activity" that gathers the two previous notions of activity?
Can we define a “meta-activity” that gathers the two previous notions of activity?

→ Yes, we can! Its name: Δ-activity.
DECISION TREE

Let $G$ be a graph.

Decision tree = plane binary tree $\Delta$ with a labelling $\text{Vertices}(\Delta) \rightarrow \text{Edges}(G)$ such that along every path starting from the root and ending at a leaf, the sequence of the labels forms a permutation of $\text{Edges}(G)$.

Ex: $\quad \begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c} \\
\text{d}
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\text{d} \\
\text{a} \\
\text{c} \\
\text{c}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{c} \\
\text{c} \\
\text{a} \\
\text{a} \\
\text{a} \\
\text{b} \\
\text{b} \\
\text{b} \\
\text{b} \\
\text{b}
\end{array}
\end{array}$
Given a spanning tree, we define an order on the edges under the rule:

\[ e_1 = \text{label of the root of } \Delta \]

- if \( e_i \) is outside the tree, \( e_{i+1} \)
- if \( e_i \) is inside the tree, \( e_{i+1} \)
\( \Delta \text{-activity} \)

Given a spanning tree, we define an order on the edges under the rule:

\[ e_1 = \text{label of the root of } \Delta \]

If \( e_i \) is outside the tree, then \( e_{i+1} \) is inside the tree.

\[ a \quad b \quad c \quad d \]
\[ d \quad d \quad a \quad b \quad b \quad d \]

\( c < d \)
Δ-ACTIVITY

Given a spanning tree, we define an order on the edges under the rule:

\[ e_1 = \text{label of the root of } \Delta \]

- If \( e_i \) is outside the tree, \( e_{i+1} \)
- If \( e_i \) is inside the tree

\[ c \prec d \prec \Delta \prec b \]
Given a spanning tree, we define an order on the edges under the rule:

\[ e_1 = \text{label of the root of } \Delta \]

- If \( e_i \) is outside the tree, \( e_{i+1} \) is chosen.
- If \( e_i \) is inside the tree, \( e_{i+1} \) is chosen.

\[ c < d < b < a \]
**Δ-Activity**

**Δ-active edge** = maximal edge inside its fundamental cycle/cocycle

**Theorem**

For every graph $G$ and decision tree $Δ$,

$$T_G(x,y) = \sum_{T \text{ spanning tree}} x^{i(T)} y^{e(T)}$$

$i(T) = \#Δ-active \text{ edges inside } T$, $e(T) = \#Δ-active \text{ edges outside } T$

Diagram:

```
  c
 / \
 b   d
 / \
 a   a
 / \
 d   d
```

```
  a
 / \  d
 b   c
 / \  /
 a   *
```

$c < d < b < a$
$T_G(x, y) = x^2 + x + xy + y + y^2$
We recover the first activities:

Tutte

\[ \Delta = \begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array} \]

\[ \begin{array}{cccc}
\text{a} & \text{a} & \text{a} & \text{a} \\
\text{a} & \text{a} & \text{a} & \text{a} \\
\text{a} & \text{a} & \text{a} & \text{a} \\
\end{array} \]

\[ \begin{array}{cccc}
x & x^2 & xy & y^2 \\
\end{array} \]
\[ \Delta \text{-ACTIVITY} \]

We recover the first activities:

Bernardi.

\[ \Delta = \]

\[ \]

\[ x^2 \]

\[ x \]

\[ xy \]

\[ y^2 \]

\[ y \]
OTHER PROPERTIES

→ several descriptions

→ Crapo's property:

\[ \text{Subgraphs } (G) = \bigcup_{T \text{ spanning tree free of } G} [T \setminus \text{Act}(T), T \cup \text{Act}(T)] \]

→ induces other "natural" activities.

**Conjecture**

Every activity that describes the Tutte polynomial and that preserves Crapo's property is a Δ-activity.
THANK YOU!

HONK HONK

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TUTTE TUTTE

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BIP BIP