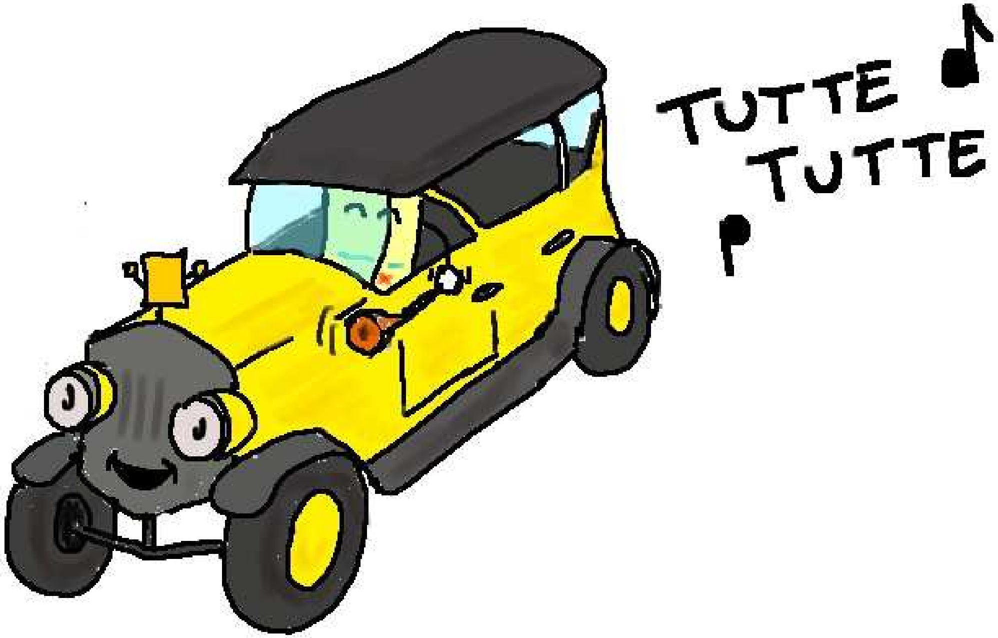
A GENERAL NOTION OF ACTIVITY FOR THE TUTTE POLYNOMIAL

COURTIEL Julien (PIMS/Simon Fraser University)

46th Southeastern International Conference on Combinatorics, Graph Theory & Computing



WHY THE TUTTE POLYNOMIAL?

→ Graph invariant

→ Numerous interesting specializations
e.g. the chromatic polynomial.

→ Closely related to the Potts model.
(statistical physics)

→ A lot of nice properties ...

THE TUTTE POLYNOMIAL

The Tutte polynomial of a connected graph G =

$$T_G(x,y) = \sum_{\substack{S \text{ spanning} \\ \text{subgraph of } G}} (x-1)^{cc(S)-1} (y-1)^{cycl(S)}$$

$cc(S)$ = number of connected components of S.

$cycl(S)$ = cyclomatic number of S
= minimal number of edges we need to remove from S to obtain an acyclic graph.

Prop:

$$T_G(x,y) \in \mathbb{N}[x,y]$$

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BLABLABLA

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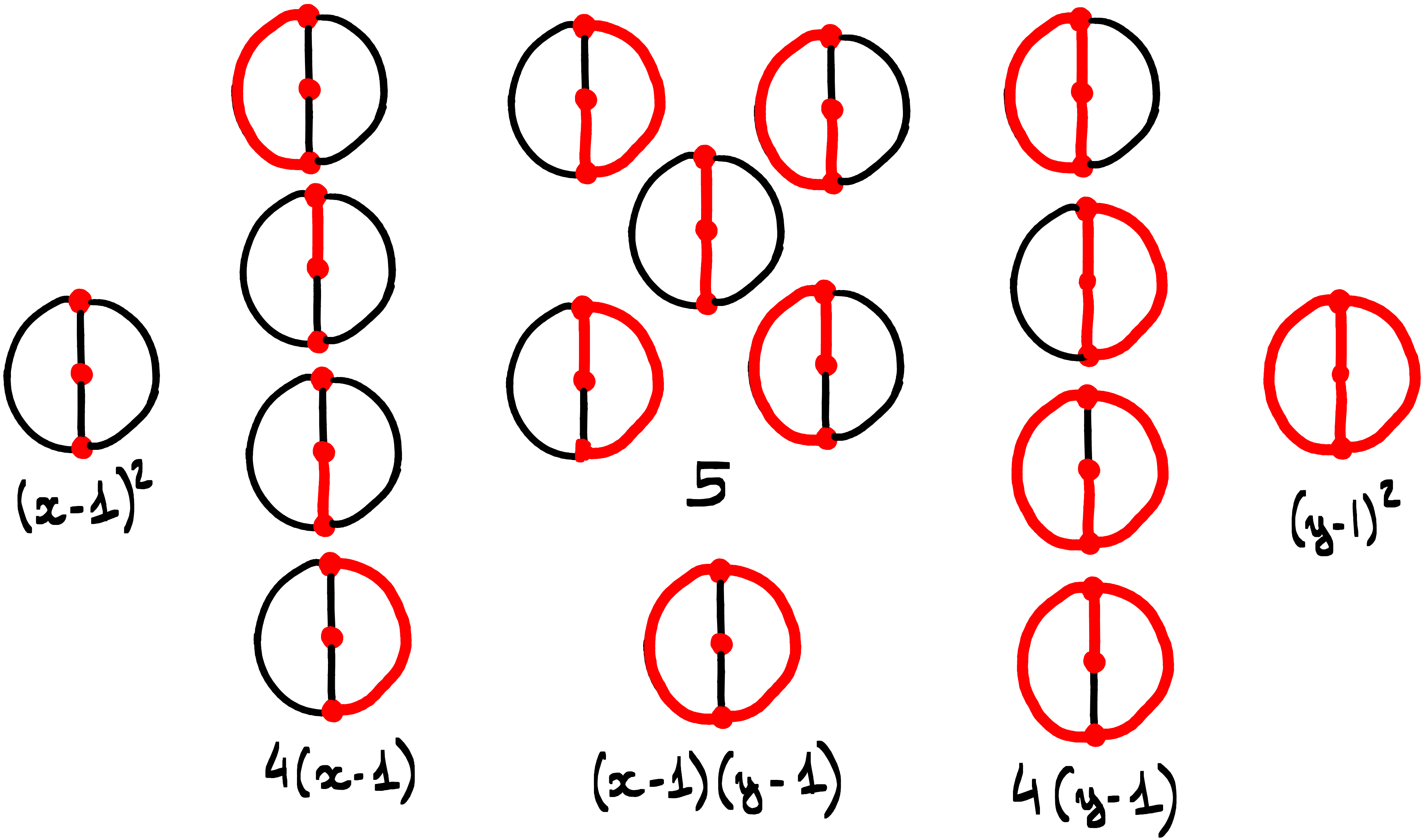
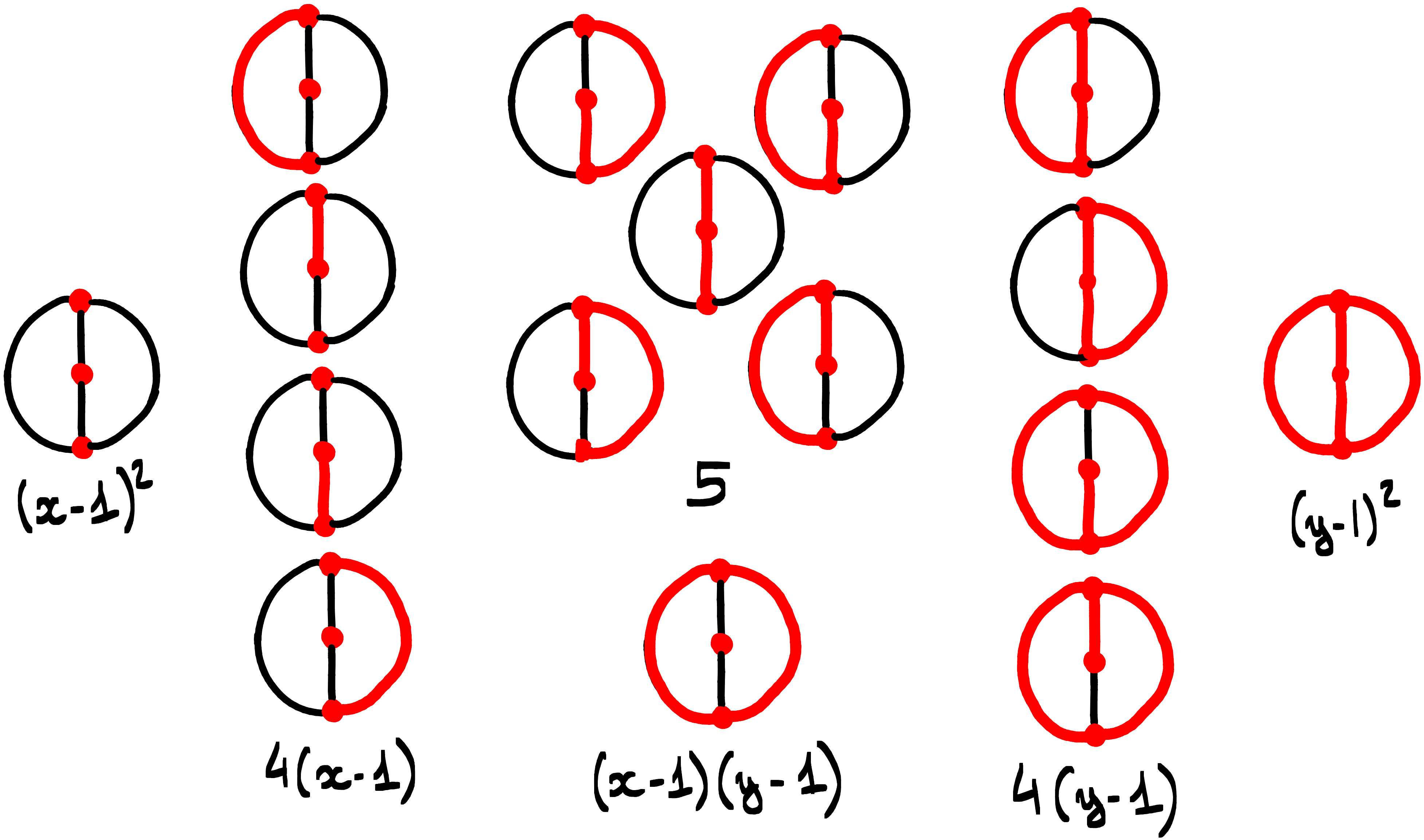
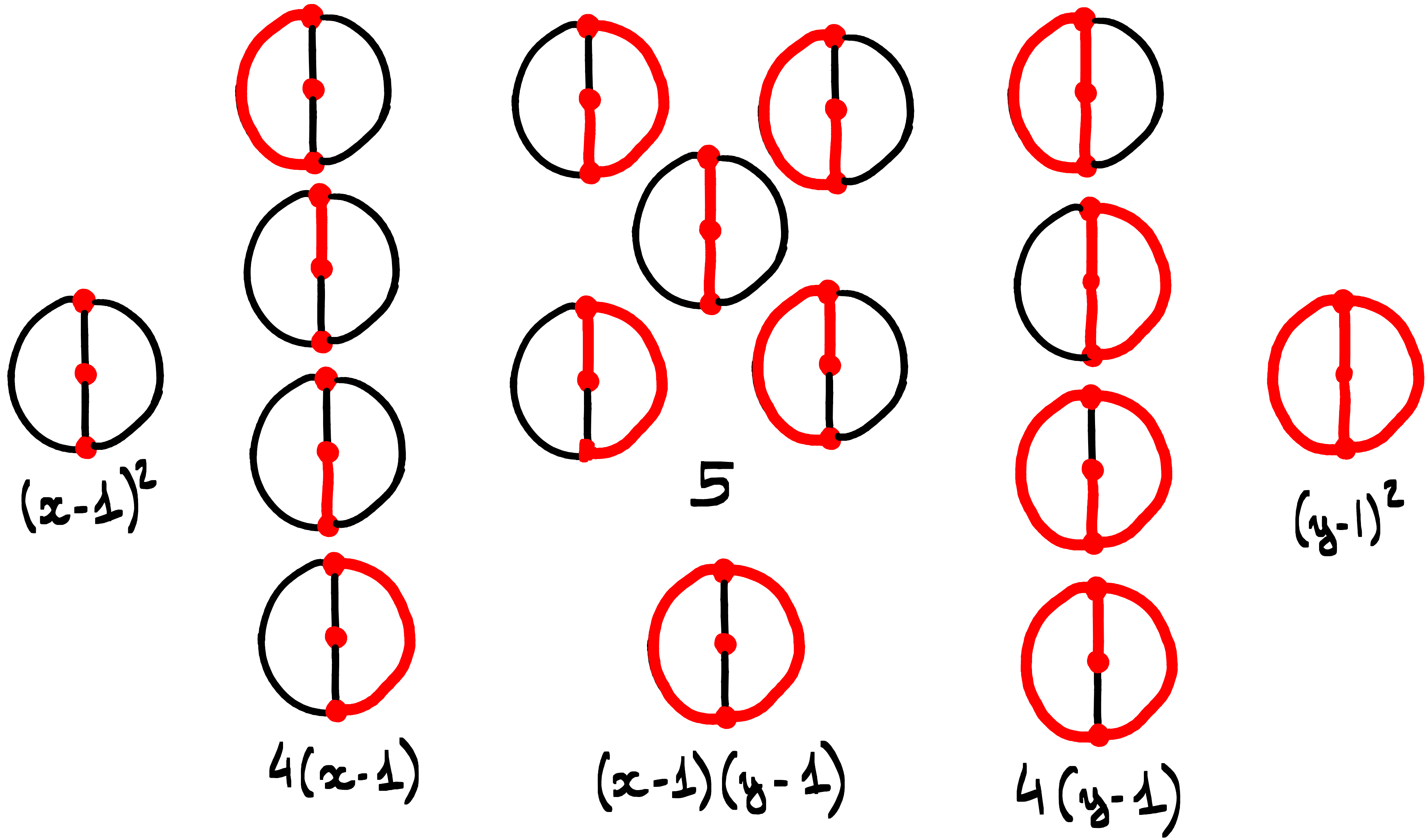
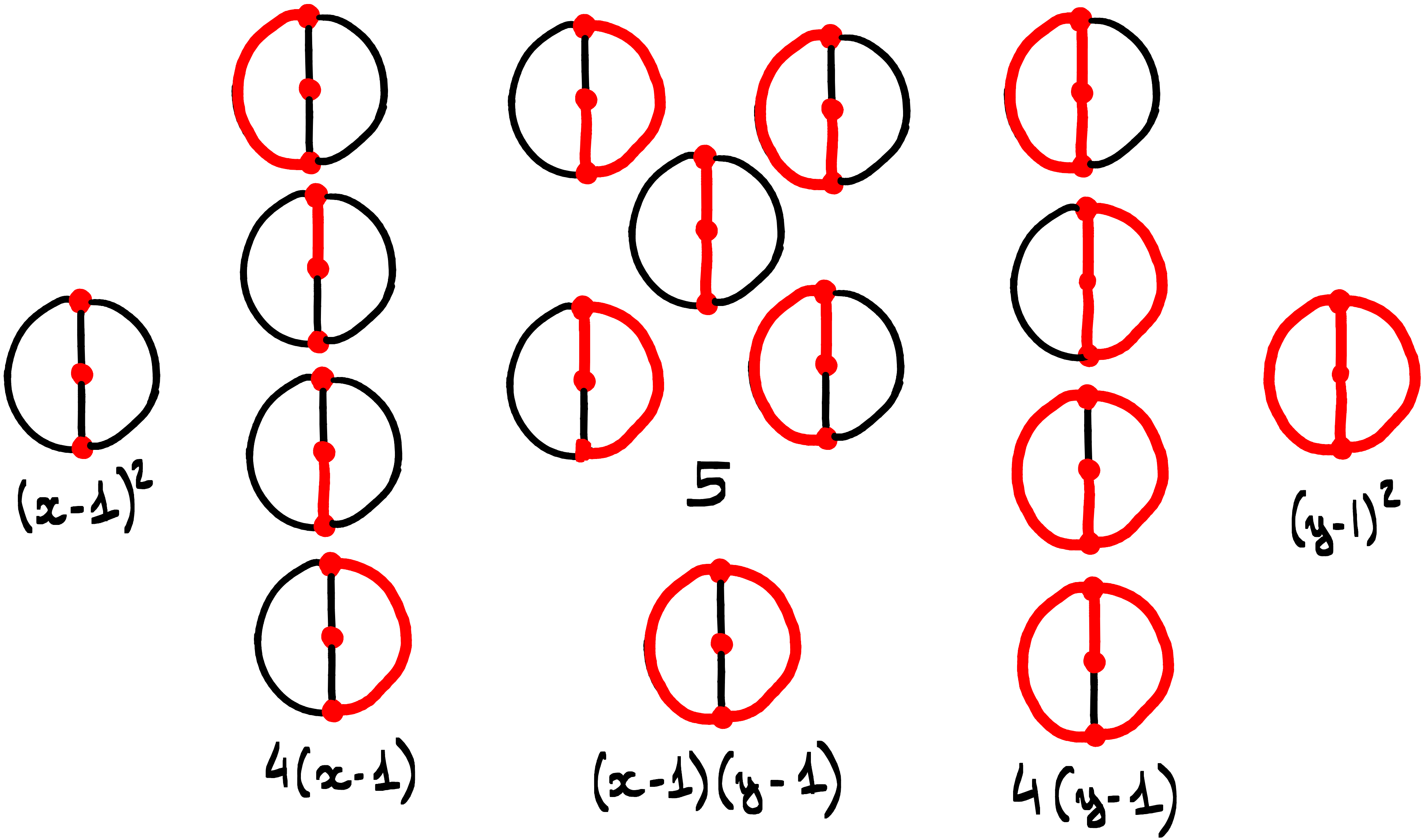

$(x-1)^2$
$4(x-1)$
5
$(x-1)(y-1)$
$4(y-1)$
$(y-1)^2$



$$(x-1)^{\#\text{conn. comp}-1} \times (y-1)^{\#\text{cycles}}$$



$$T_G(x,y) = (x-1)^2 + 4(x-1) + (x-1)(y-1) + 5 + 4(y-1) + (y-1)^2$$



$$T_G(x,y) = (x-1)^2 + 4(x-1) + (x-1)(y-1) + 5 + 4(y-1) + (y-1)^2$$

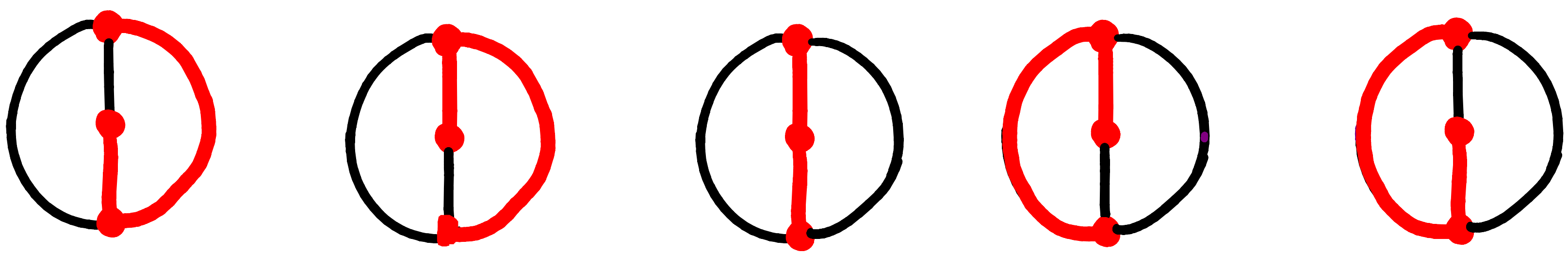
$$= x^2 + x + xy + y + y^2.$$

HOW TO INTERPRET THE COEFFICIENTS

Principle : Map each spanning tree onto a set of edges called "active" edges such that

$$T_G(x,y) = \sum_{T \text{ spanning tree of } G} x^{i(T)} \, y^{e(T)}$$

where $i(T)$ = number of active edges inside T
$e(T)$ = number of active edges outside T.



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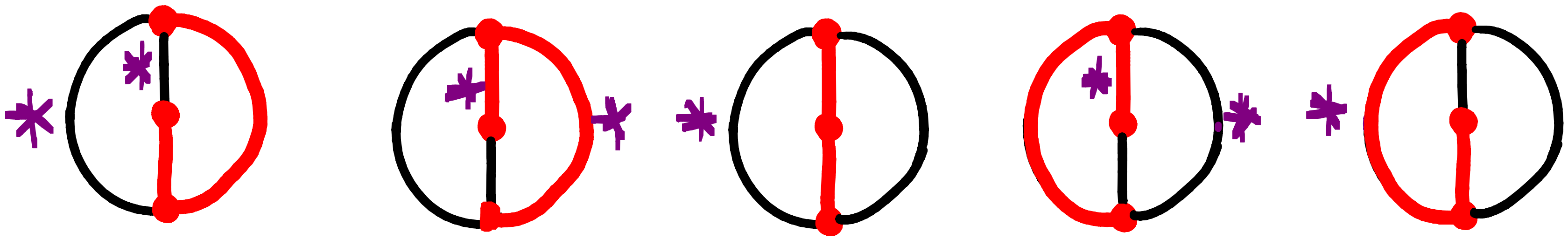

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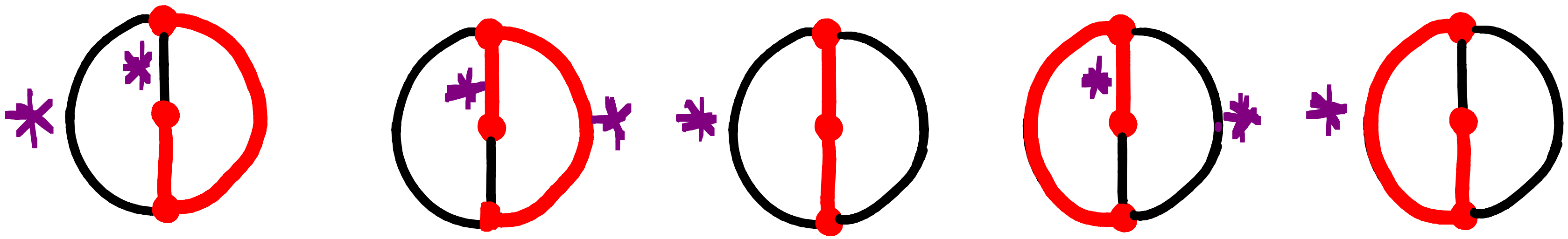
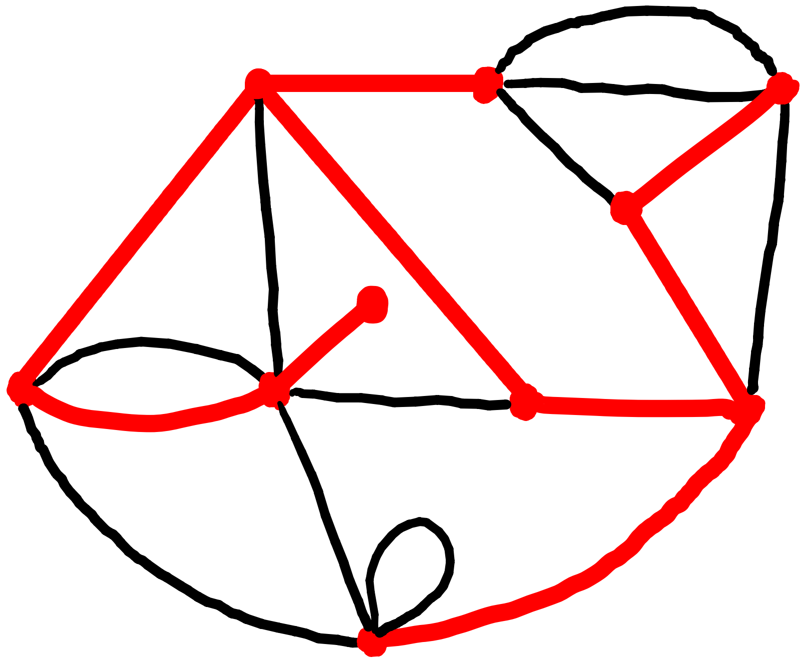


FUNDAMENTAL CYCLE / COCYCLE

Let T be a spanning tree and e an edge,



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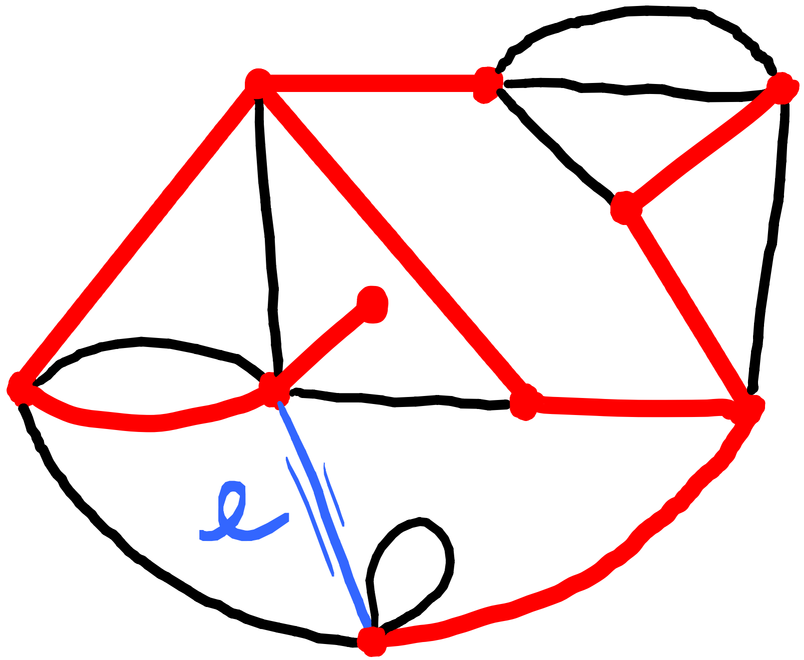

fundamental cycle = unique cycle in $T \cup \{e\}$

Fundamental Cycle / Cocycle

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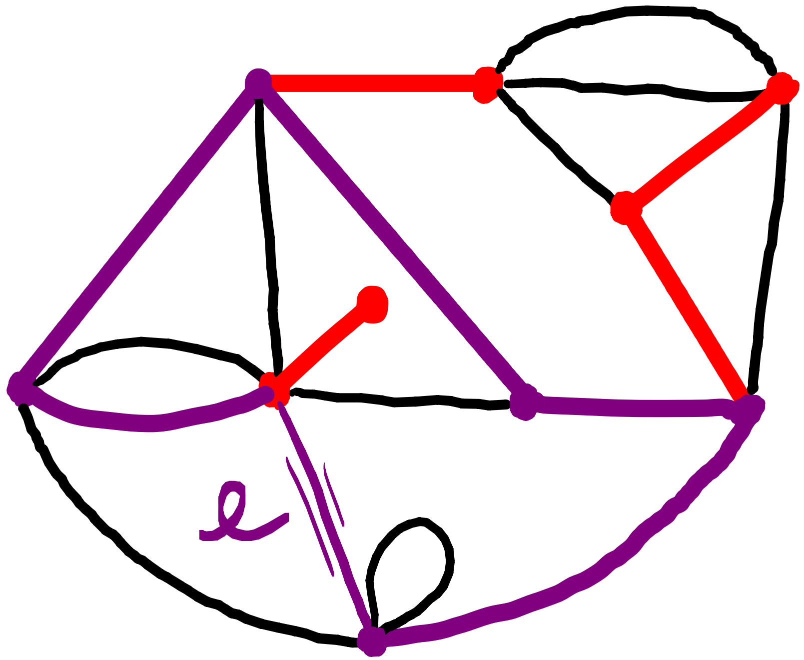

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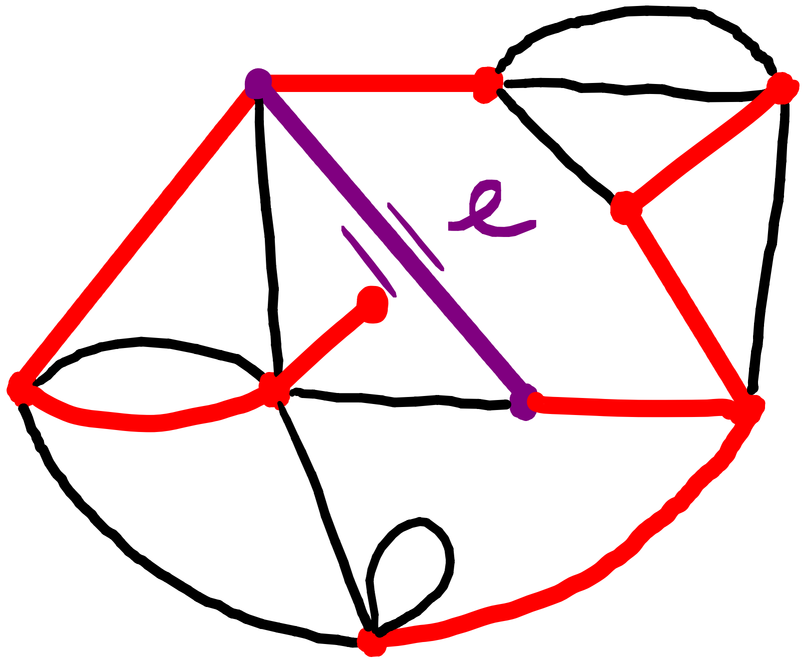

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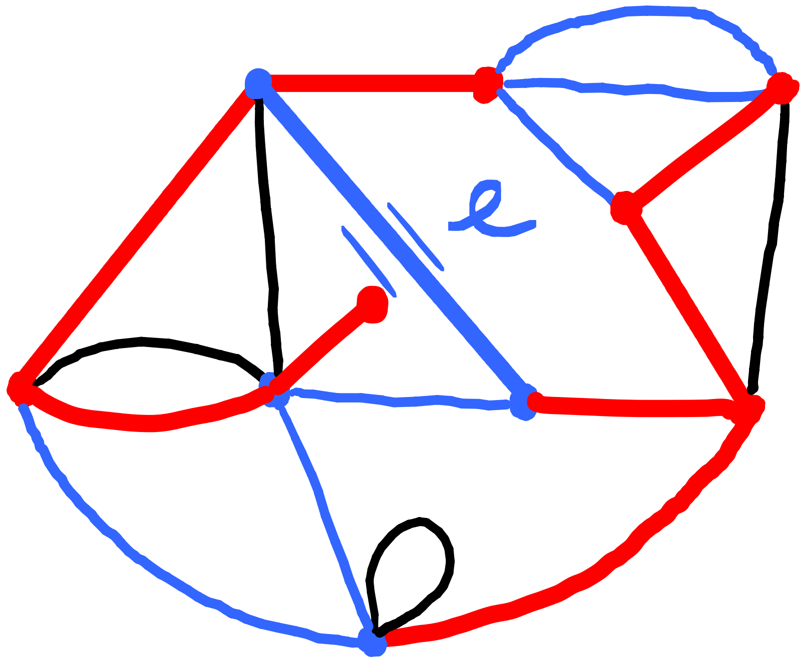

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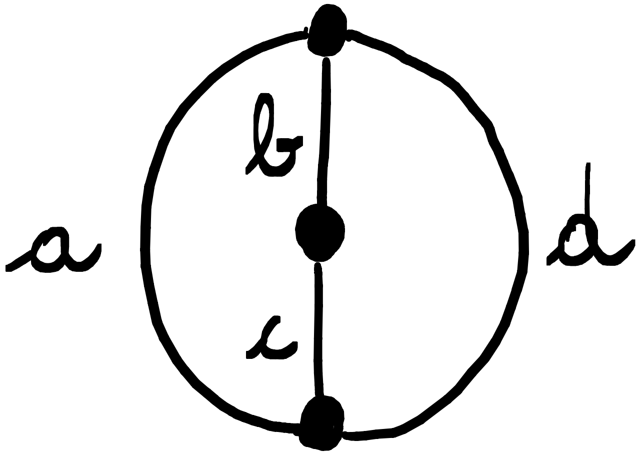
TUTTE'S ACTIVITY



We label and order the edges:

$$a < b < c < d$$

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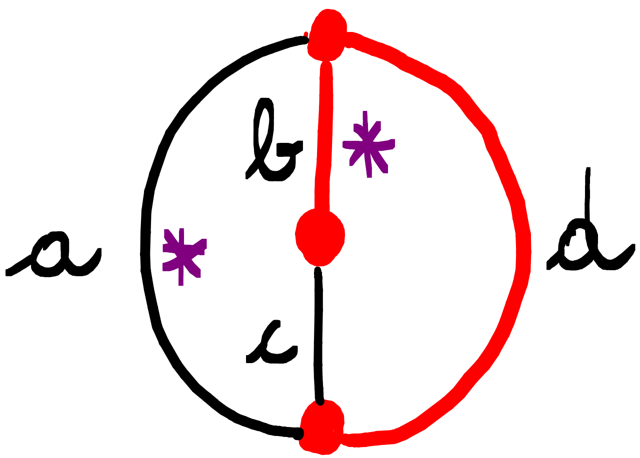

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Active edge = minimal edge inside its fundamental cycle / cocycle

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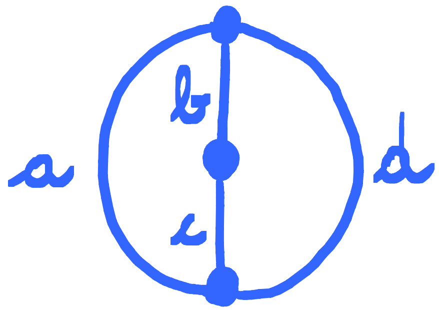

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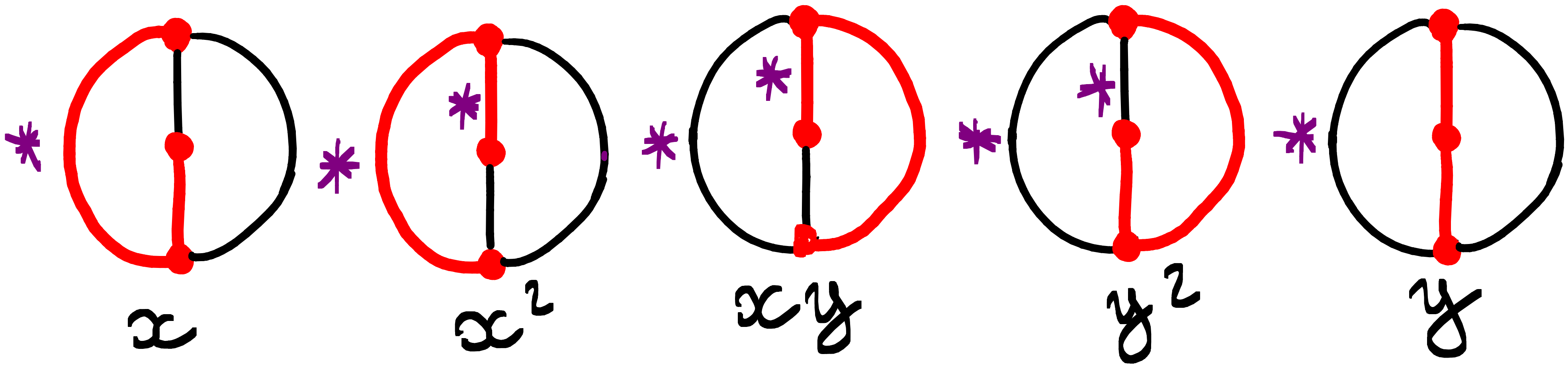
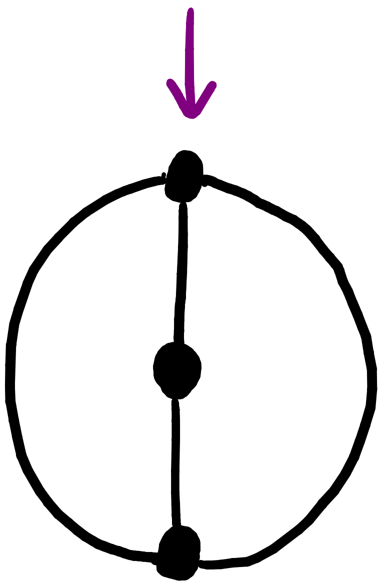


$$T_G(x,y) = x^2 + x + xy + y + y^2$$

BERNARDI'S ACTIVITY: TOUR OF THE TREE

We embed and root the graph:



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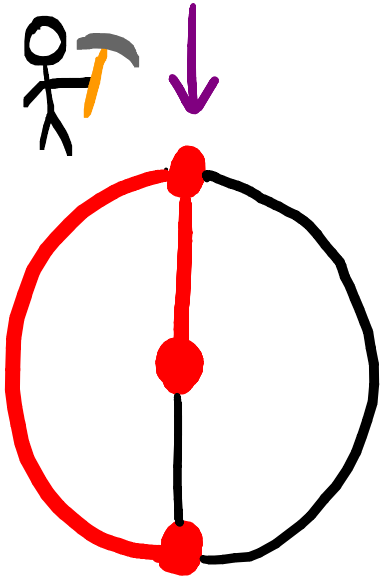
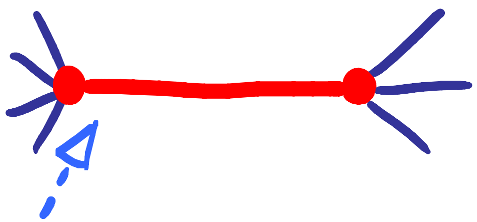
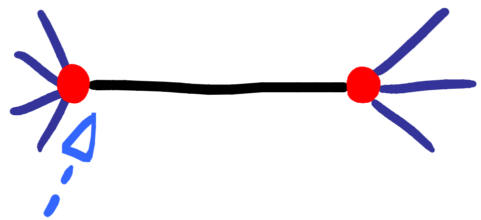


Rules:

inside the tree



outside the tree



BERNARDI'S ACTIVITY: TOUR OF THE TREE

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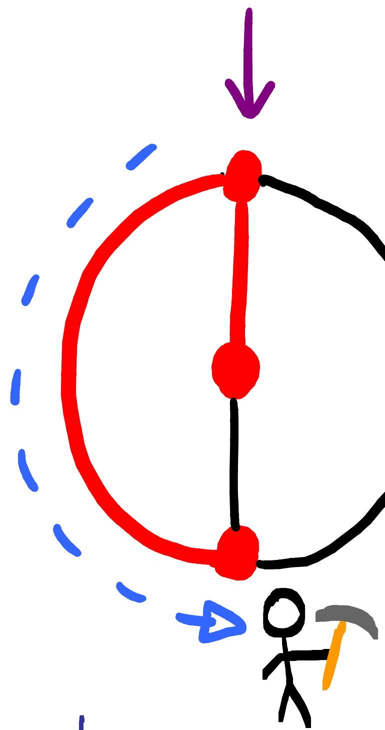

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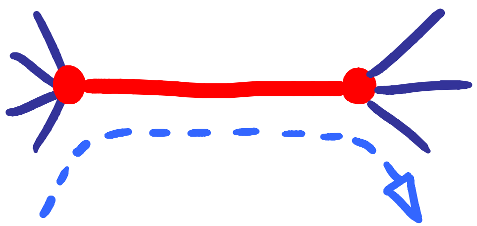
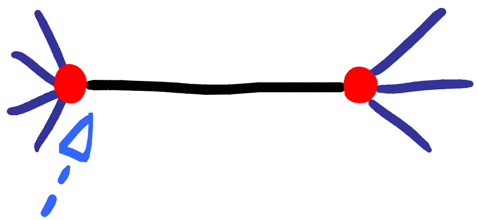


We walk along.

outside the tree



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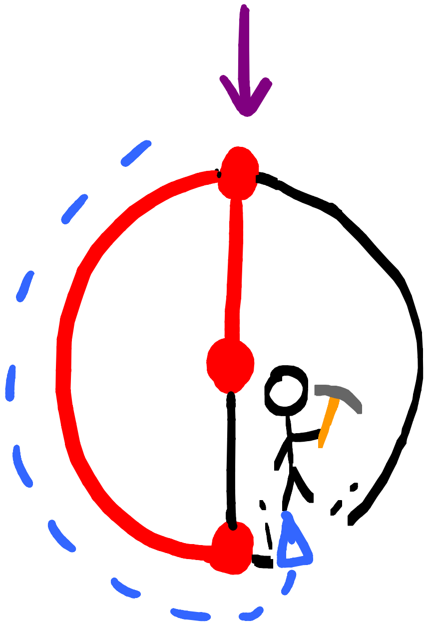

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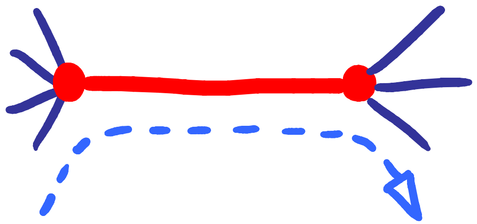
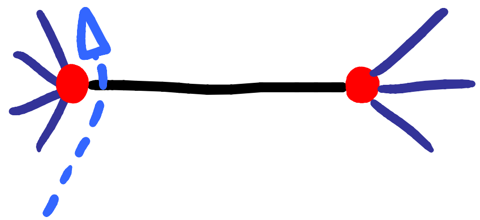


We walk along.

outside the tree



We cross.

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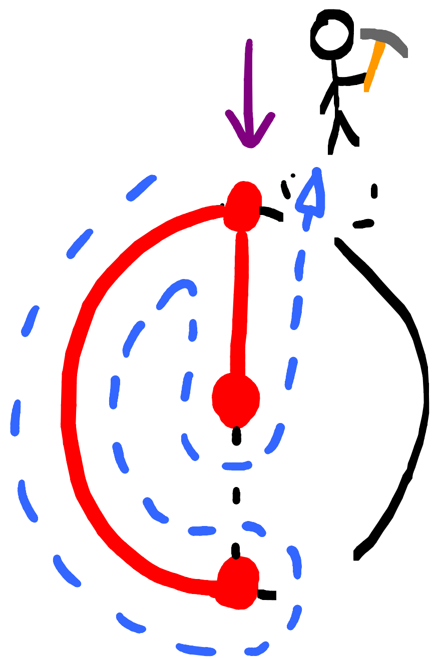

Rules:

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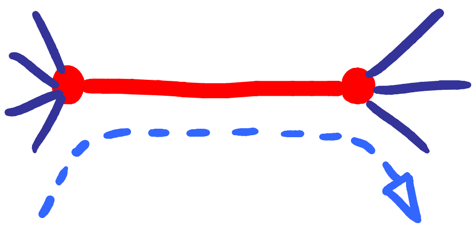
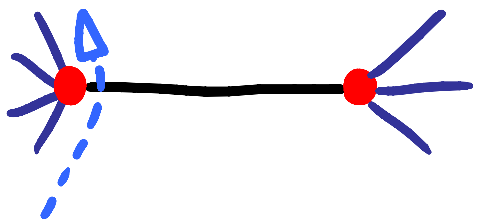


We walk along.

outside the tree



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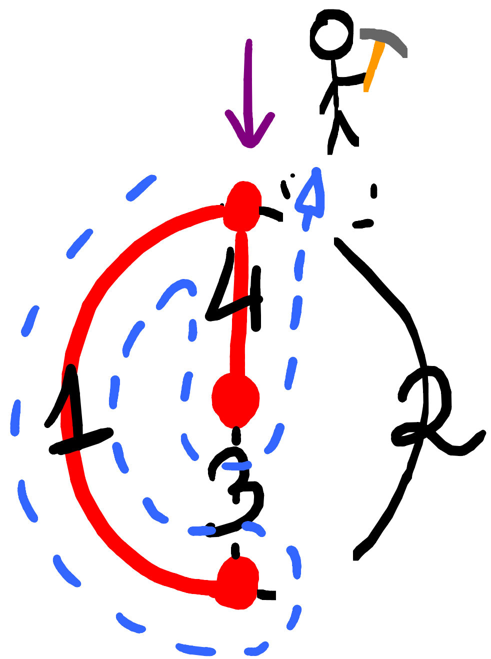

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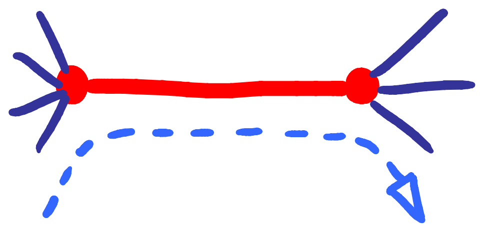
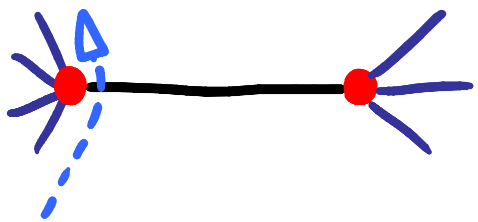


We walk along.

outside the tree



We cross.

BERNARDI'S ACTIVITY: DEFINITION

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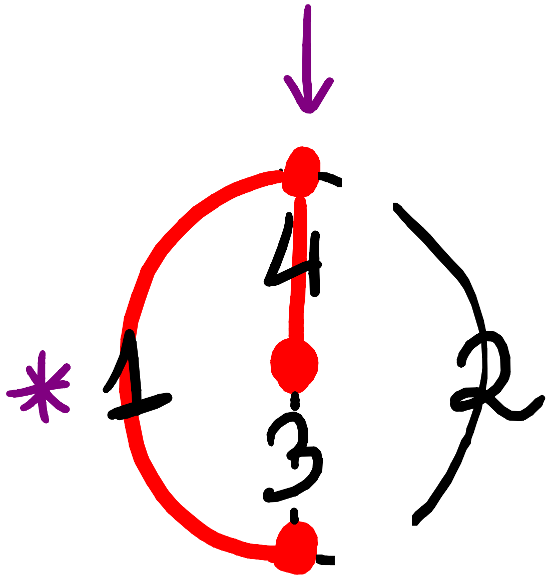
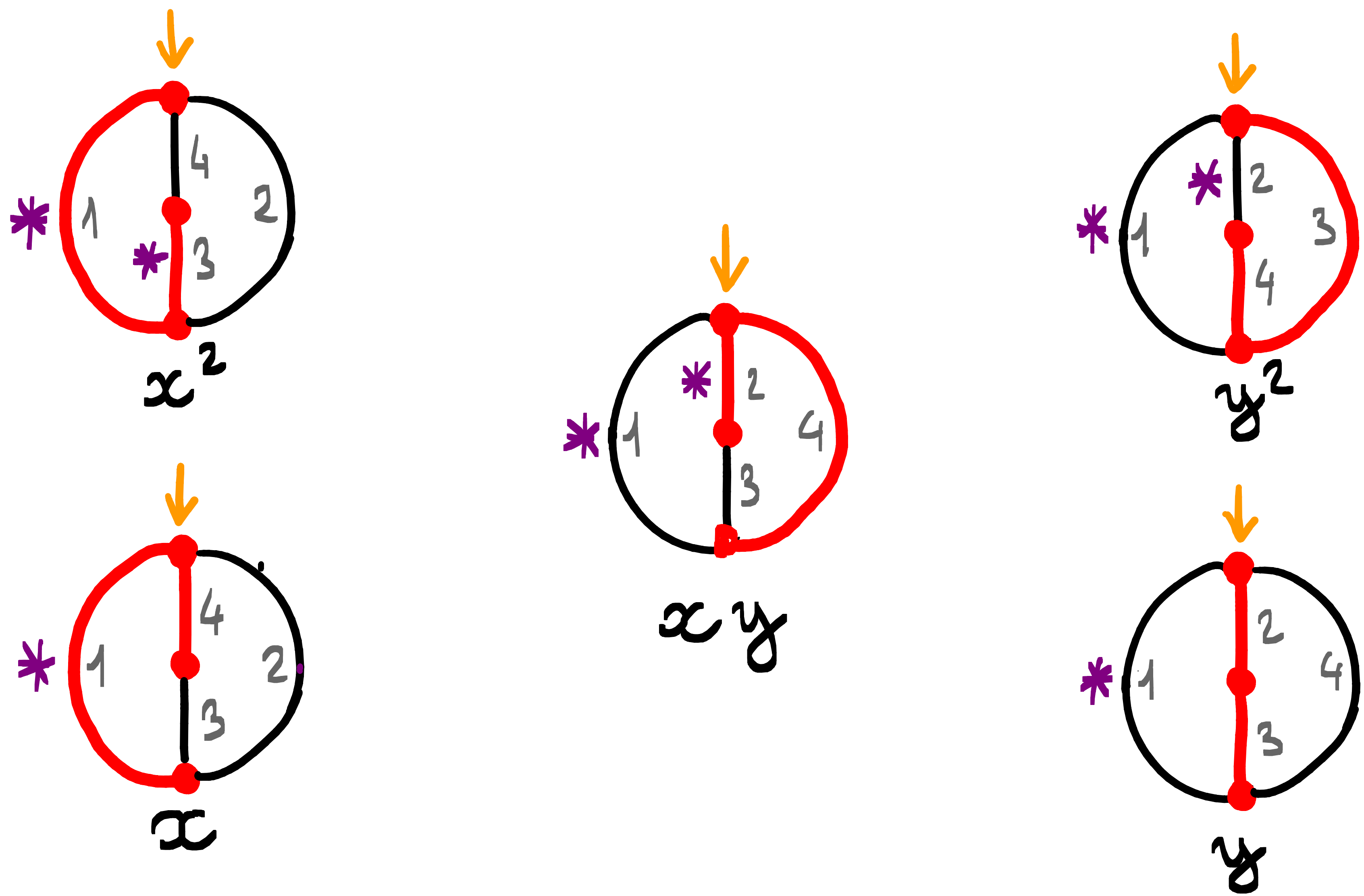


Active edge = minimal edge inside its fundamental cycle / cocycle (for the first visit order)

Bernardi's Activity: Definition



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QUESTION

Can we define a "meta-activity" that gathers the two previous notions of activity?

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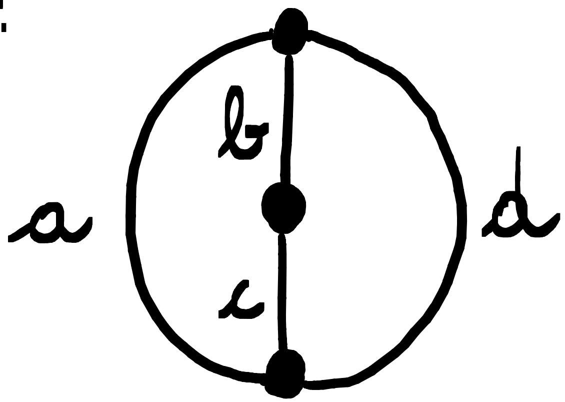
→ Yes, we can! Its name : Δ-activity.

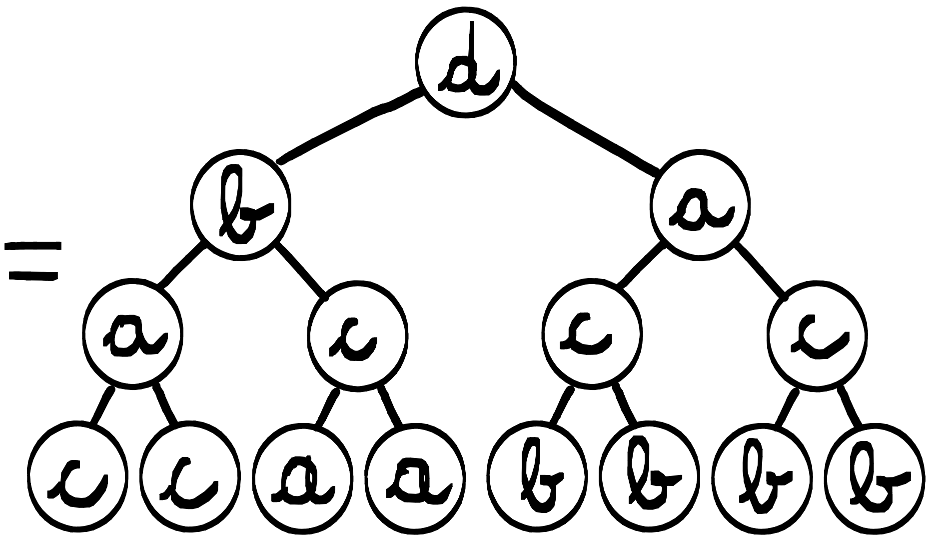
DECISION TREE

Let G be a graph.

Decision tree = plane binary tree Δ with a labelling $\mathrm{Vertices}(\Delta) \to \mathrm{Edges}(G)$ such that along every path starting from the root and ending at a leaf, the sequence of the labels forms a permutation of $\mathrm{Edges}(G)$.

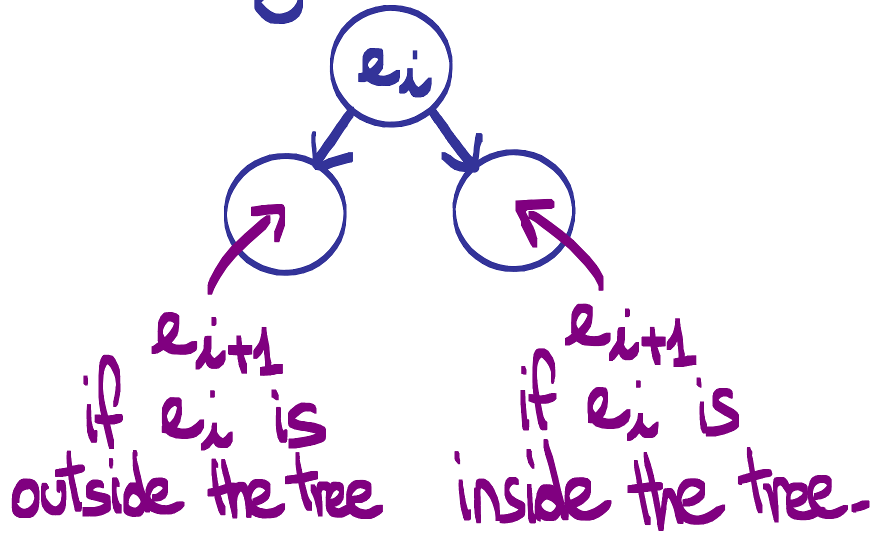
Ex:



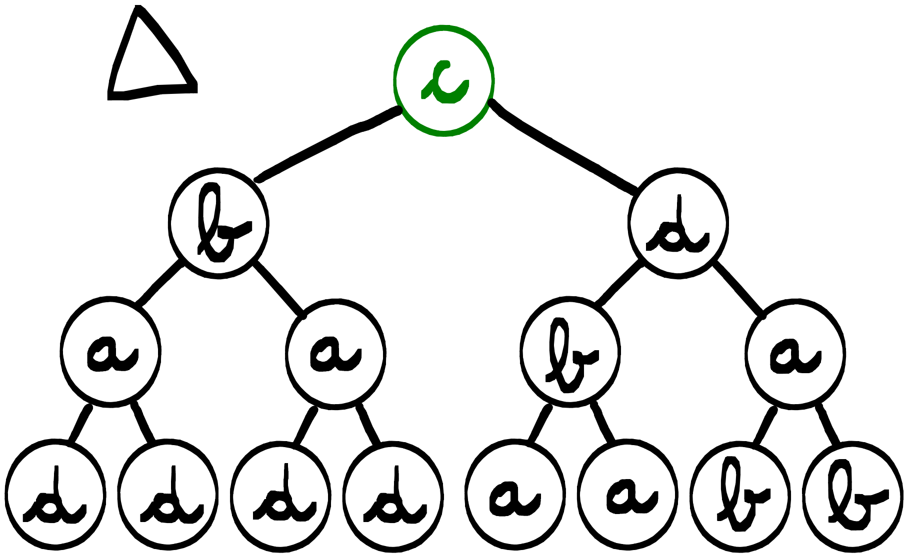
$\Delta =$ 

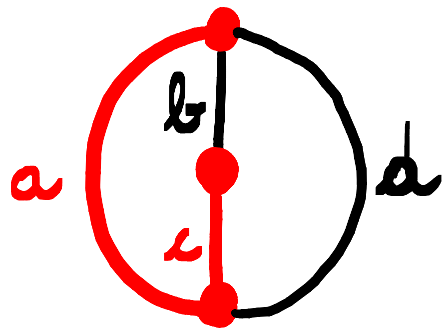
Δ-ACTIVITY

Given a spanning tree,
we define an order on the edges under the rule:



$e_1 =$ label of the root of Δ





c

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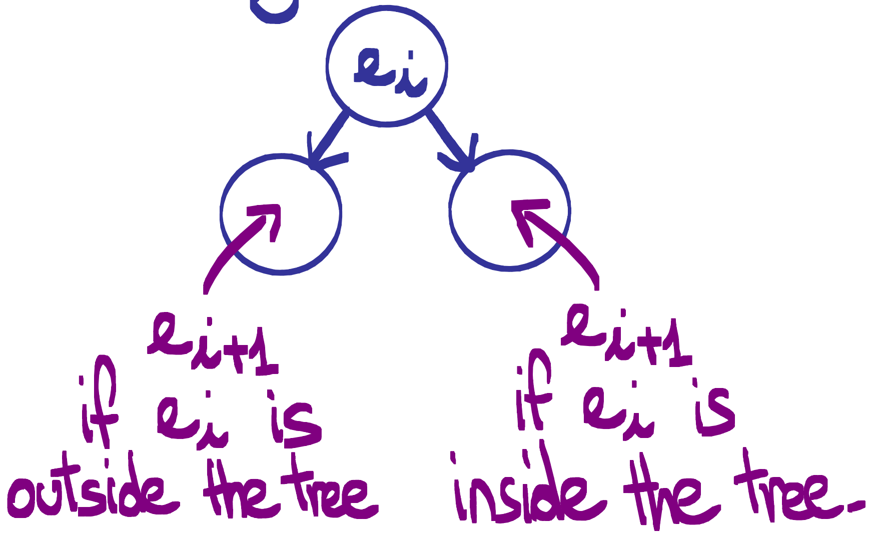

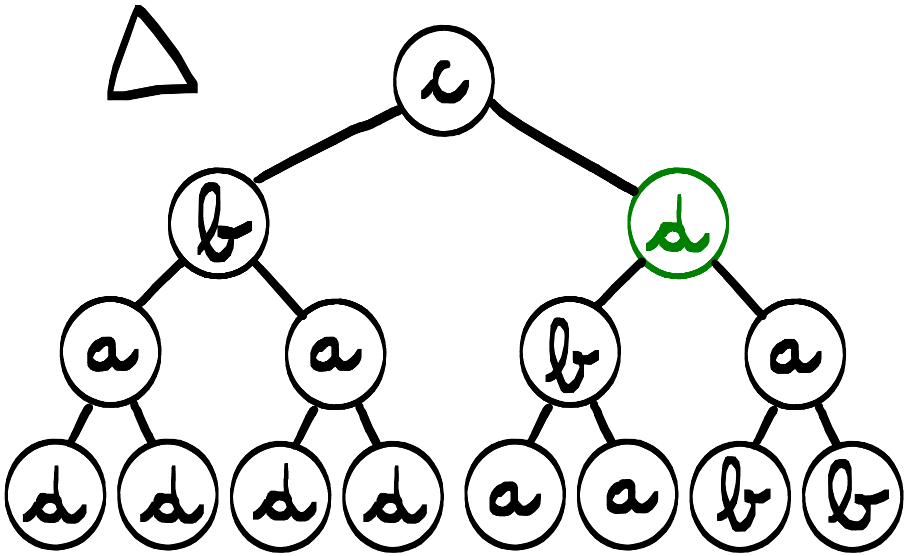

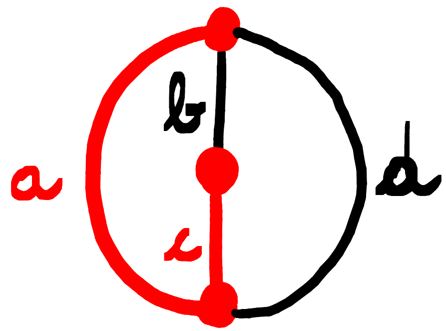

$c < d$

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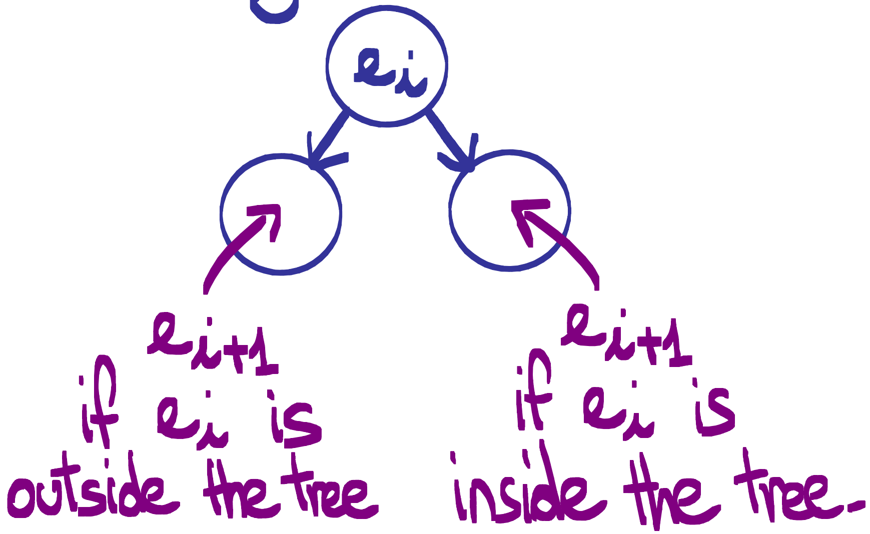
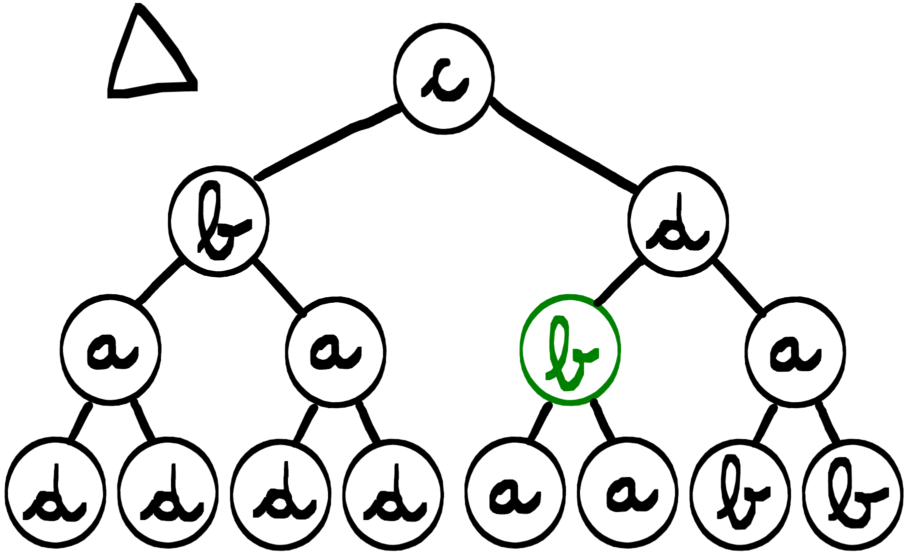
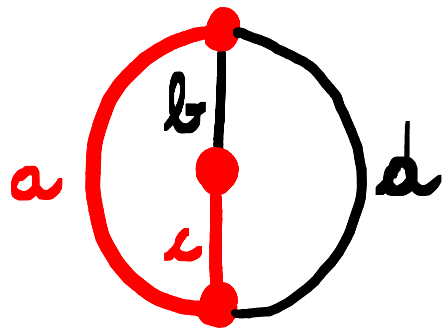


Δ





$c < d < b$

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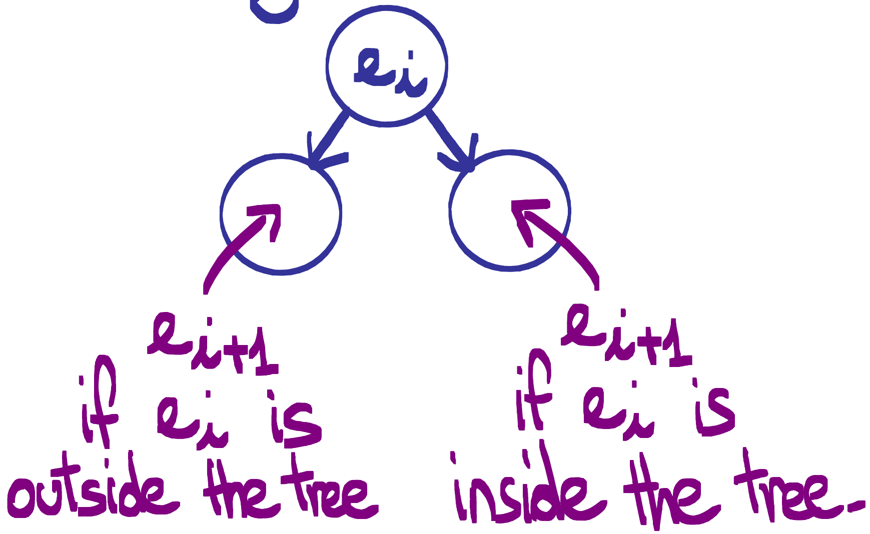
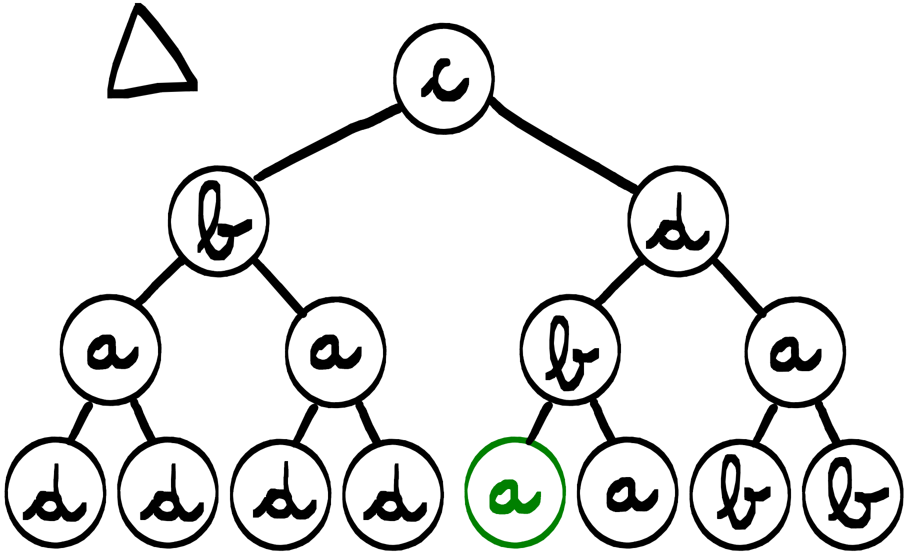
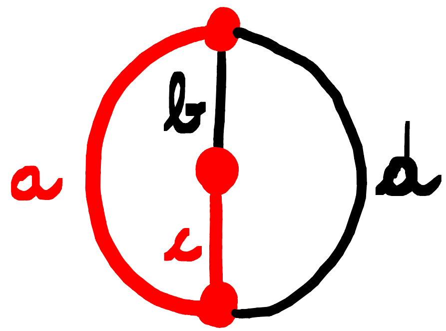


Δ





$c < d < b < a$

Δ-ACTIVITY

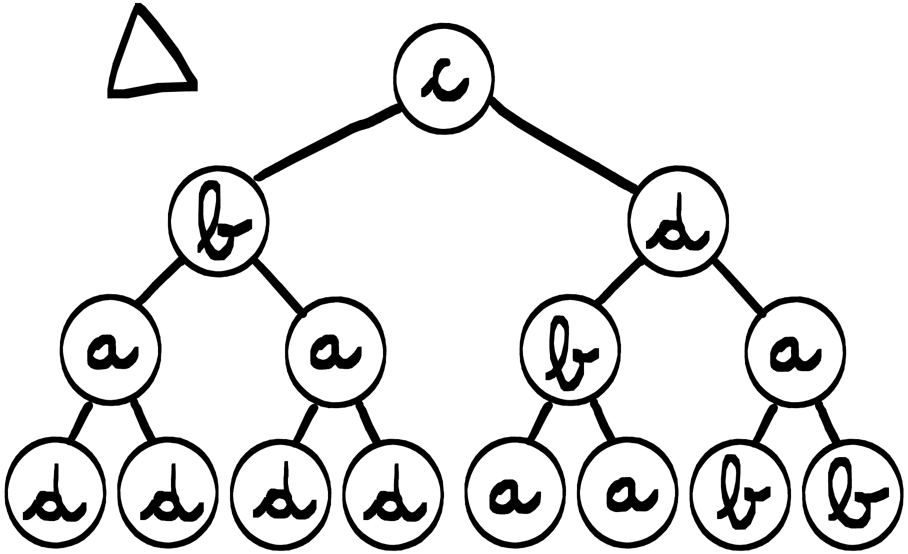
Δ-active edge = maximal edge inside its fundamental cycle/cocycle

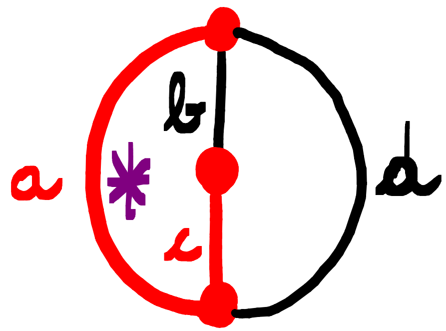
Theorem For every graph G and decision tree Δ,

$$T_G(x,y) = \sum_{T \text{ spanning tree}} x^{i(T)} \, y^{e(T)}$$

$i(T) = \#\,\Delta$-active edges inside T, $e(T) = \#\,\Delta$-active edges outside T

Δ





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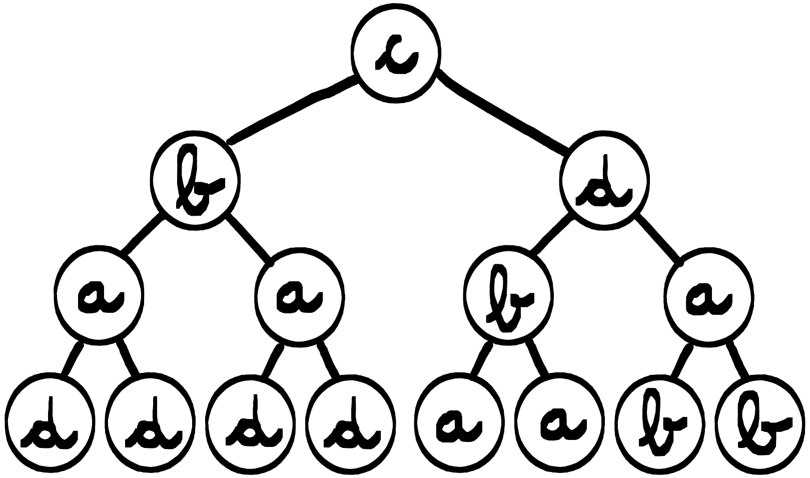
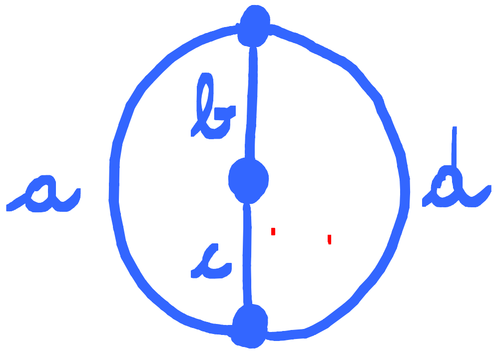
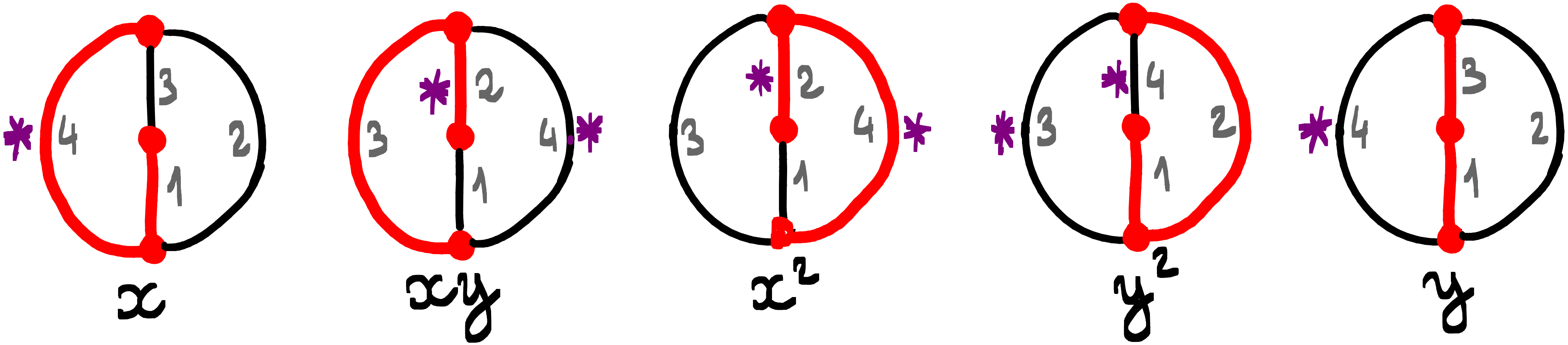
Δ-ACTIVITY



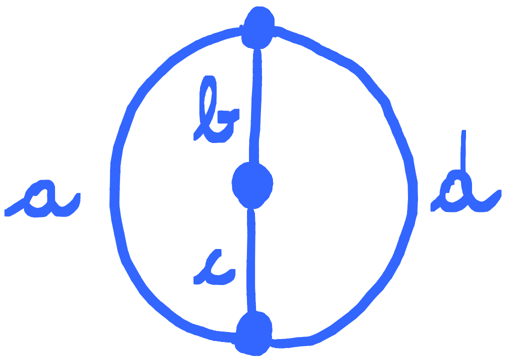




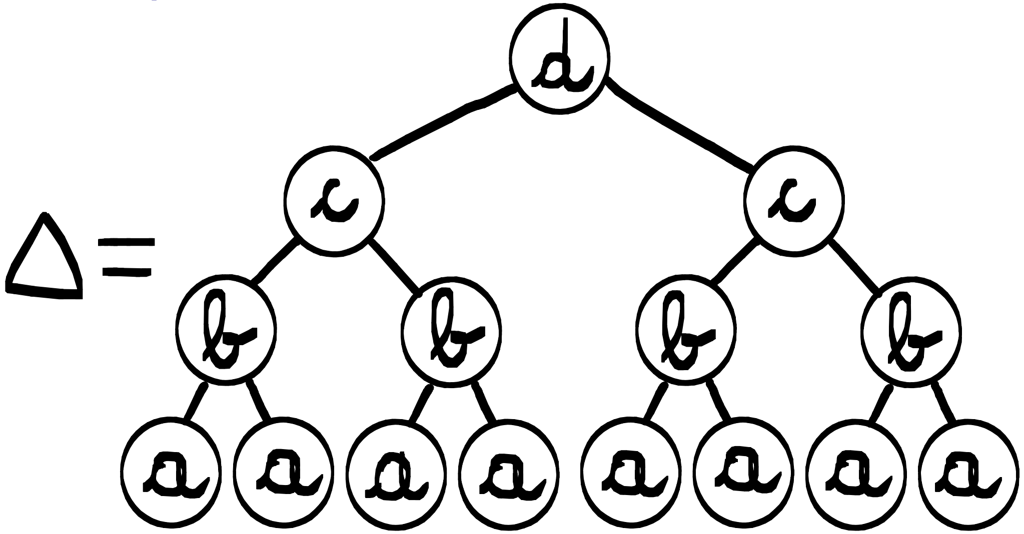
$$T_G(x,y) = x^2 + x + xy + y + y^2$$

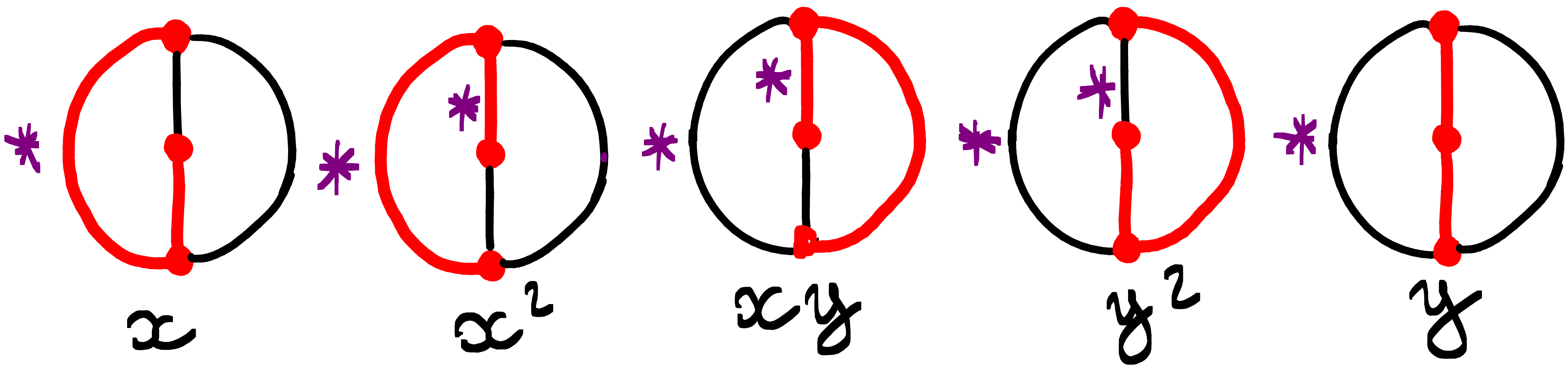
Δ-ACTIVITY

We recover the first activities : Tutte



$a < b < c < d$

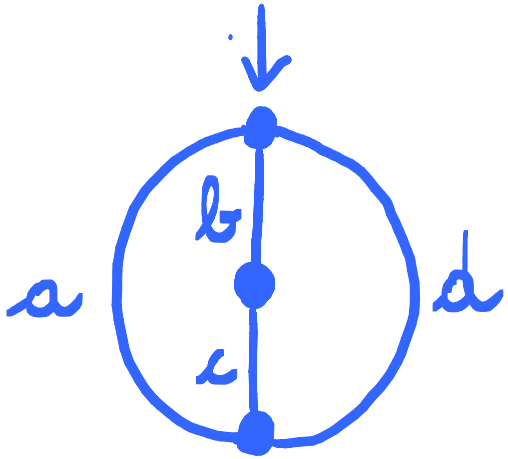




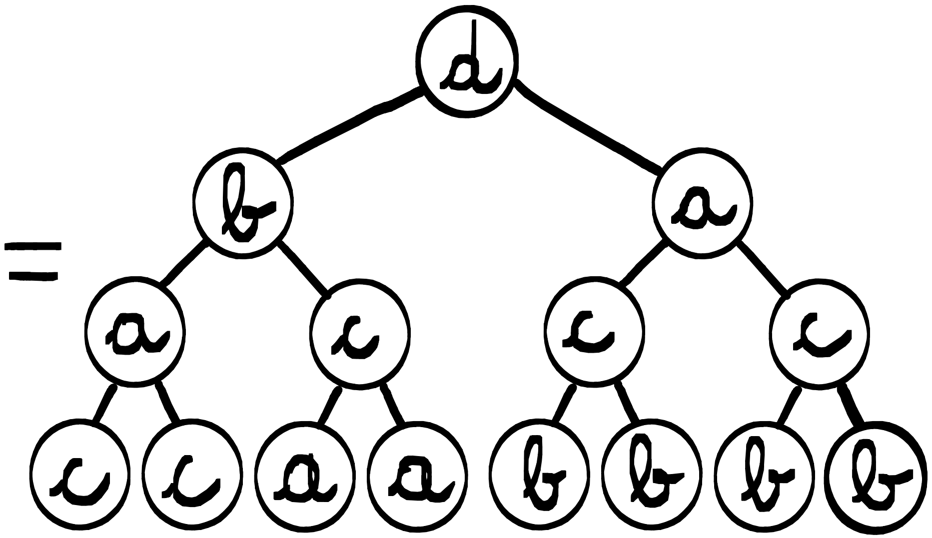
Δ-ACTIVITY

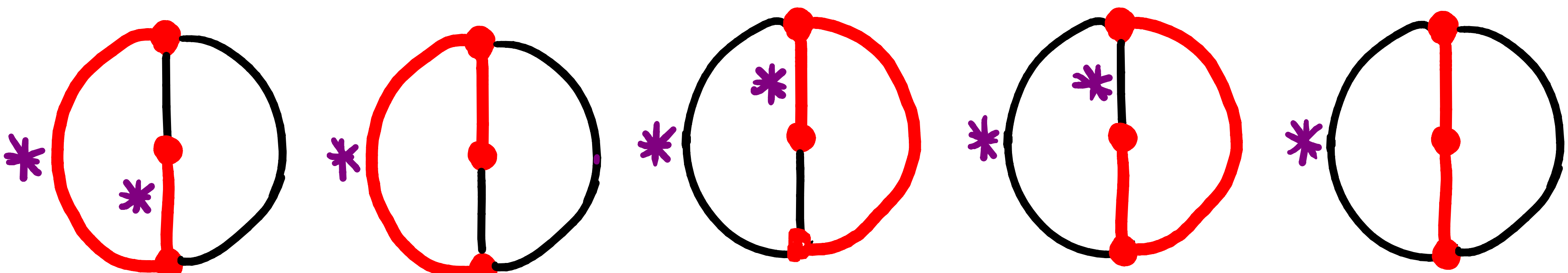
We recover the first activities :

Bernardi.



$\Delta =$





x^2 x xy y^2 y

OTHER PROPERTIES

→ several descriptions

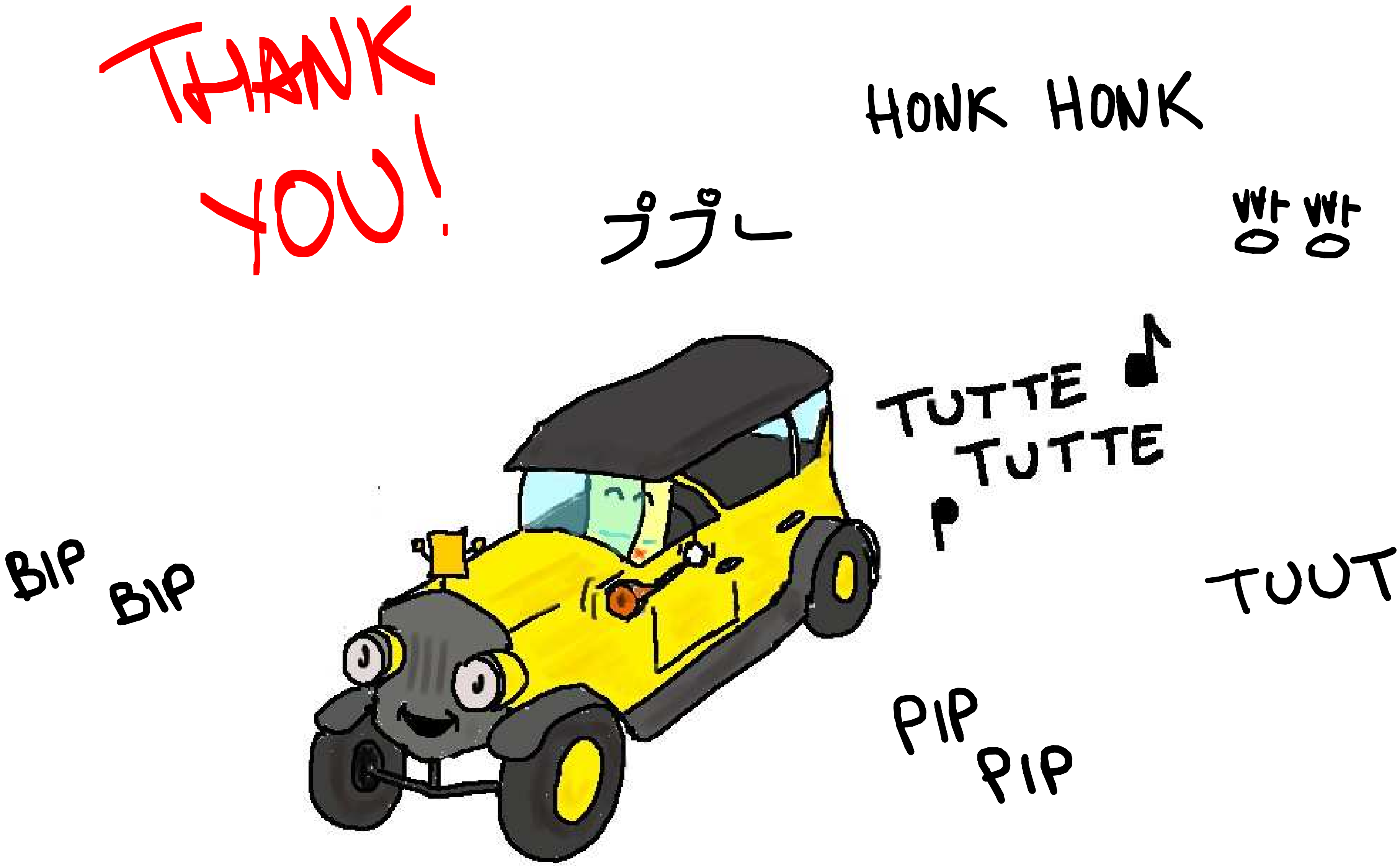
→ Crapo's property :

$$\text{Subgraphs}(G) = \biguplus_{T \text{ spanning tree of } G} [T \setminus Act(T), T \cup Act(T)]$$

→ induces other "natural" activities.

Conjecture

Every activity that describes the Tutte polynomial and that preserves Crapo's property is a Δ-activity.


THANK YOU!
HONK HONK
ププー
빵빵
TUTTE
TUTTE
BIP
BIP
TUUT
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