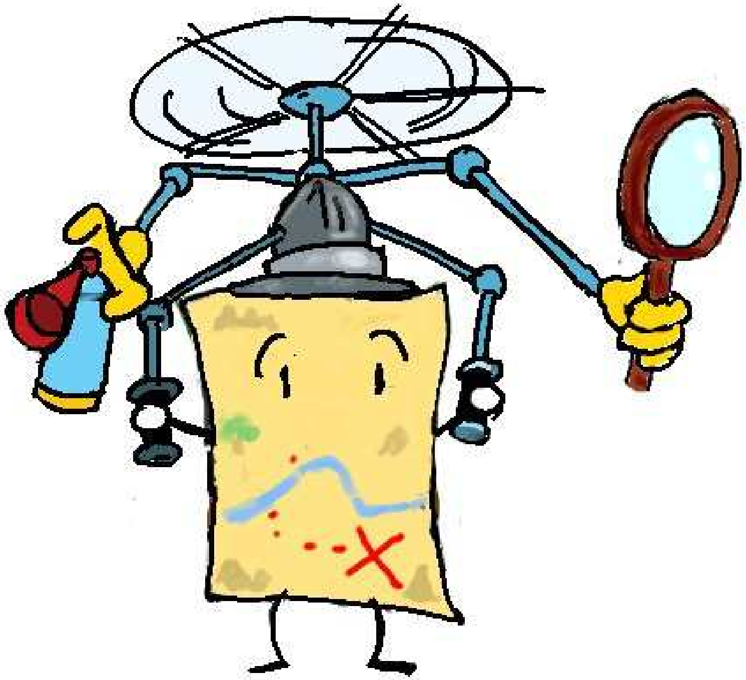
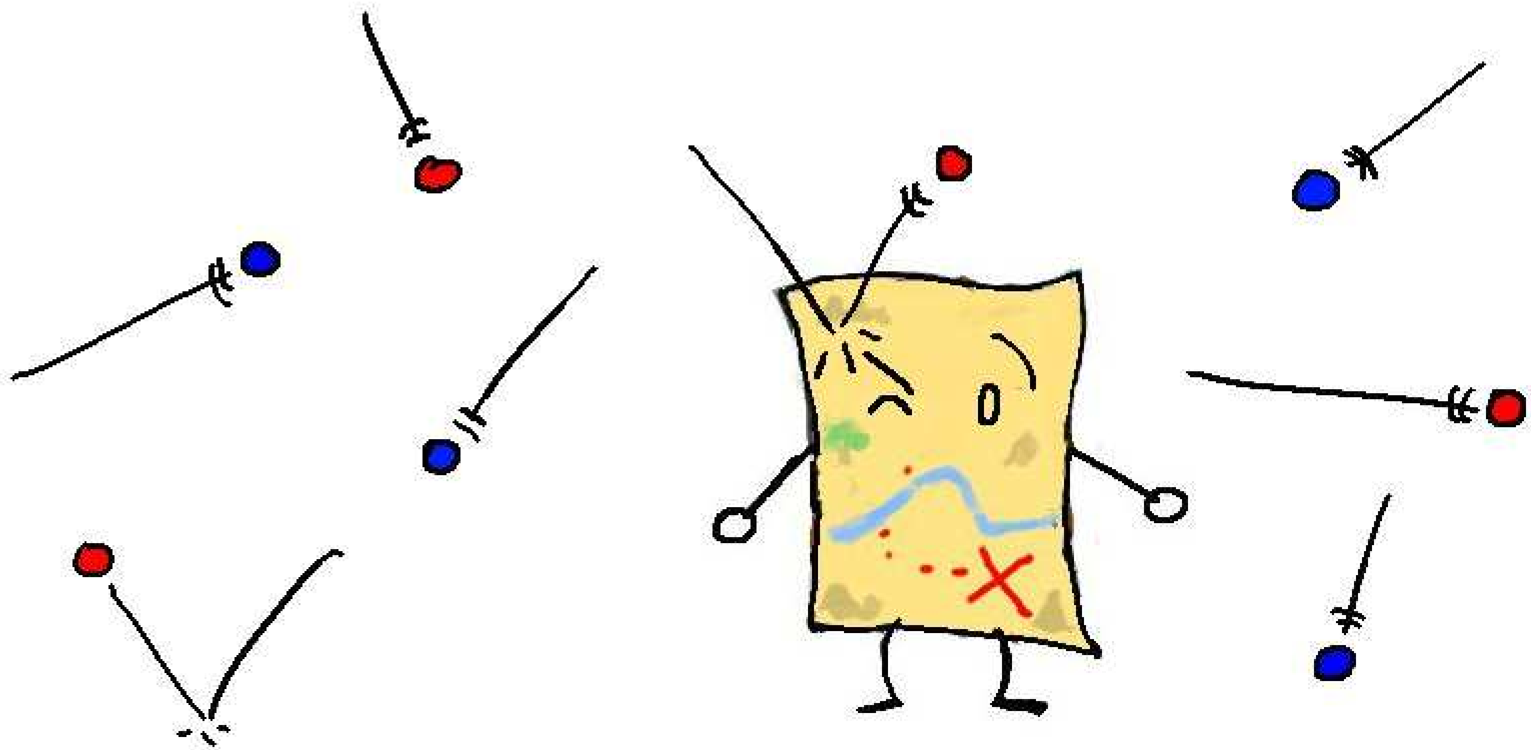
ENUMERATION OF PLANAR MAPS WITH ADDITIONAL STRUCTURES

Julien COURTIEL (Simon Fraser University/PIMS)

CanaDA 2015



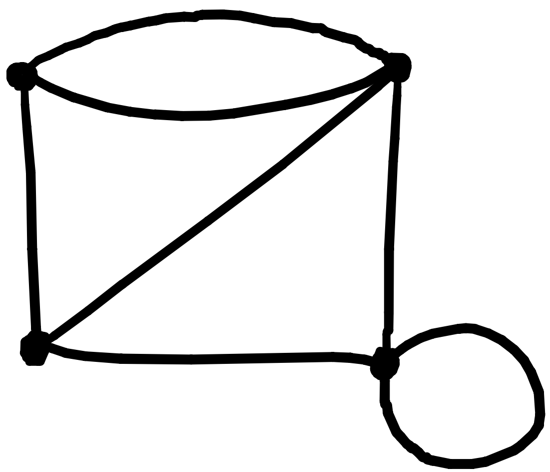
Planar Maps in Statistical Physics



PLANAR MAPS : DEFINITION

Planar map = connected graph

+ embedding of this graph in the plane, considered up to continuous deformation.



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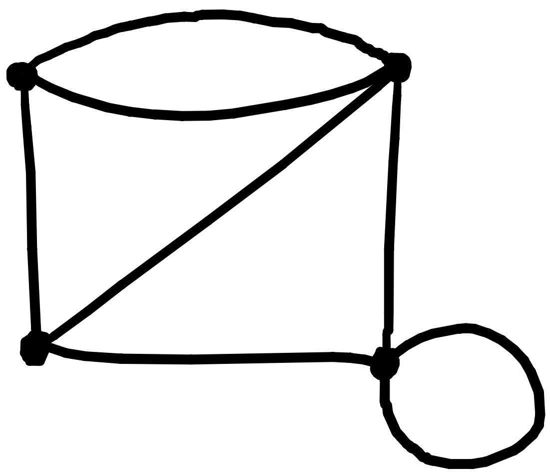

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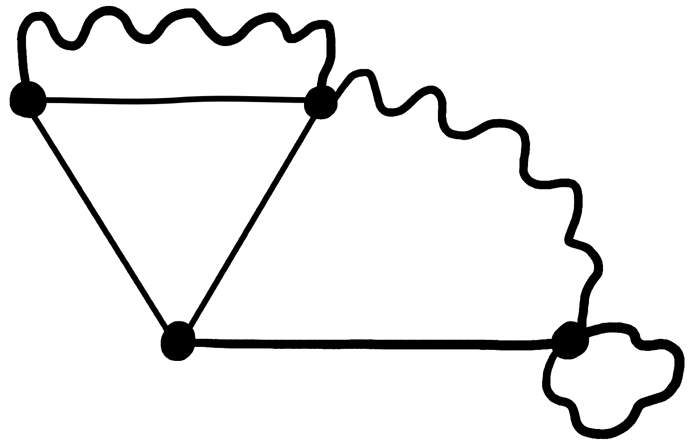

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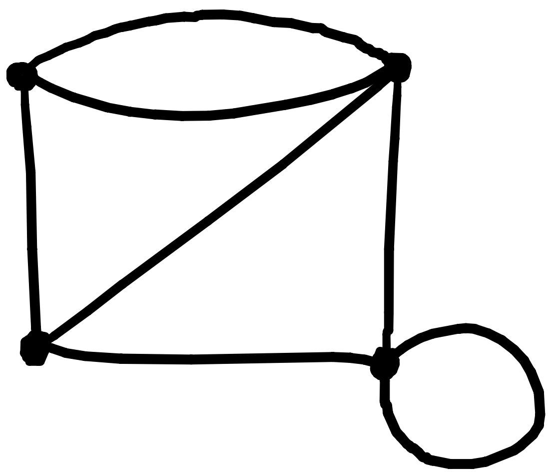
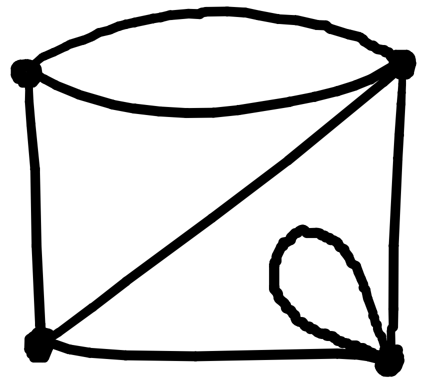


$\neq$



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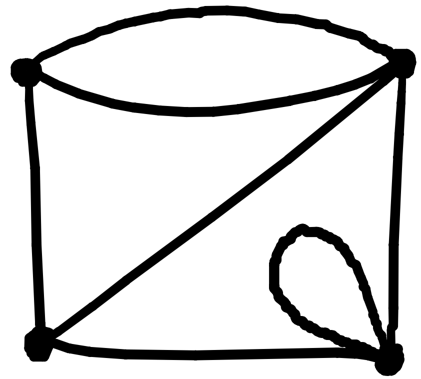

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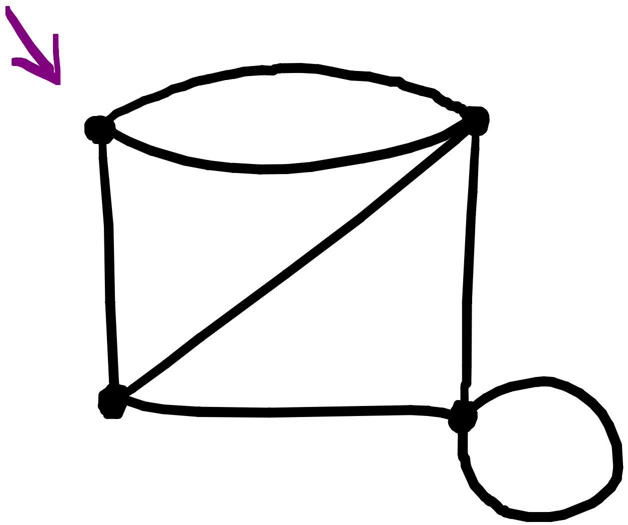

We root every planar map at an outer corner.

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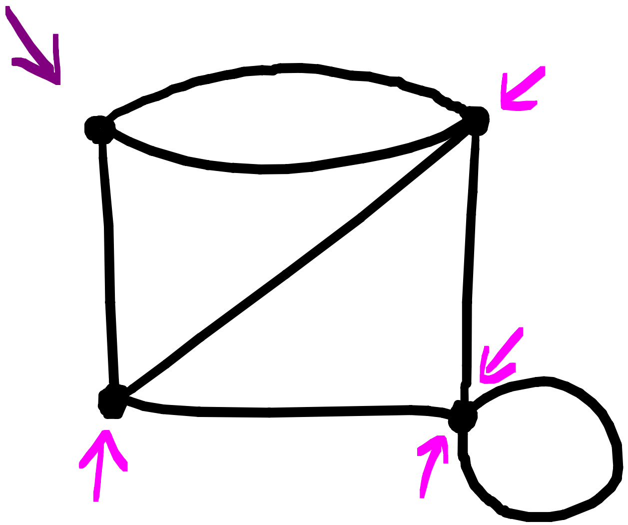

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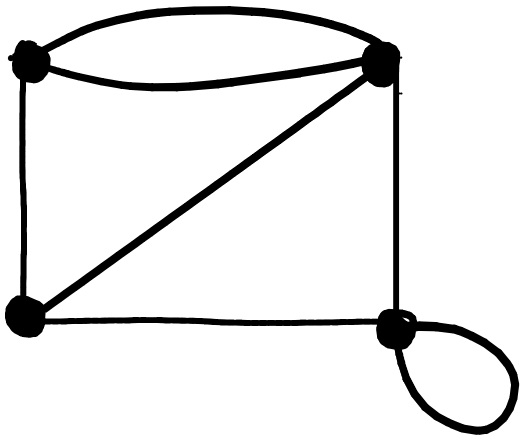
LARGE MAPS



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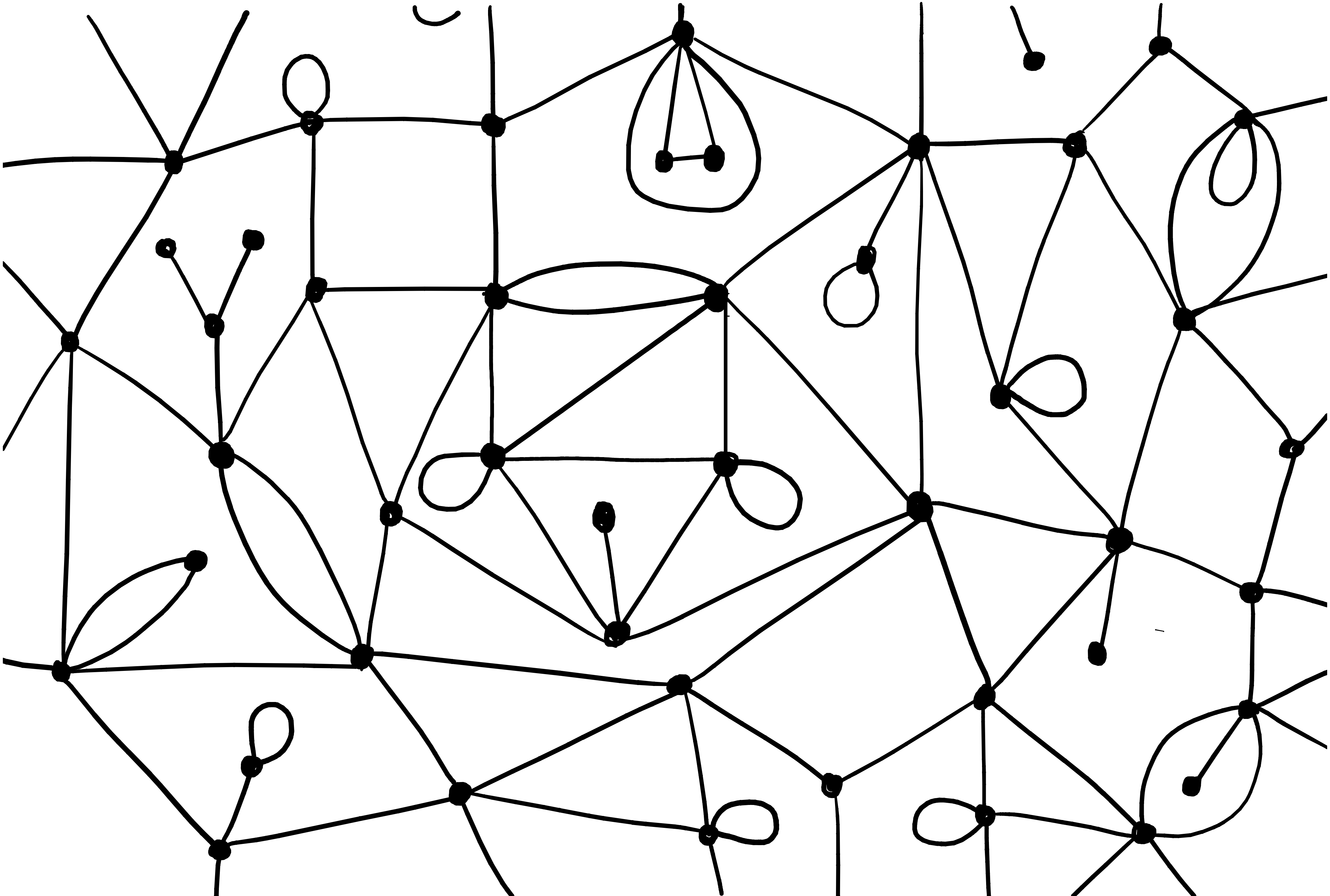


UNIVERSALITY CLASS

Standard asymptotic behaviour of the number of planar maps.

$$\boxed{\sim c\, \rho^{-n}\, n^{-\frac{5}{2}}}$$

Examples: General planar maps : $\frac{2}{\sqrt{\pi}}\, 12^{n}\, n^{-5/2}$

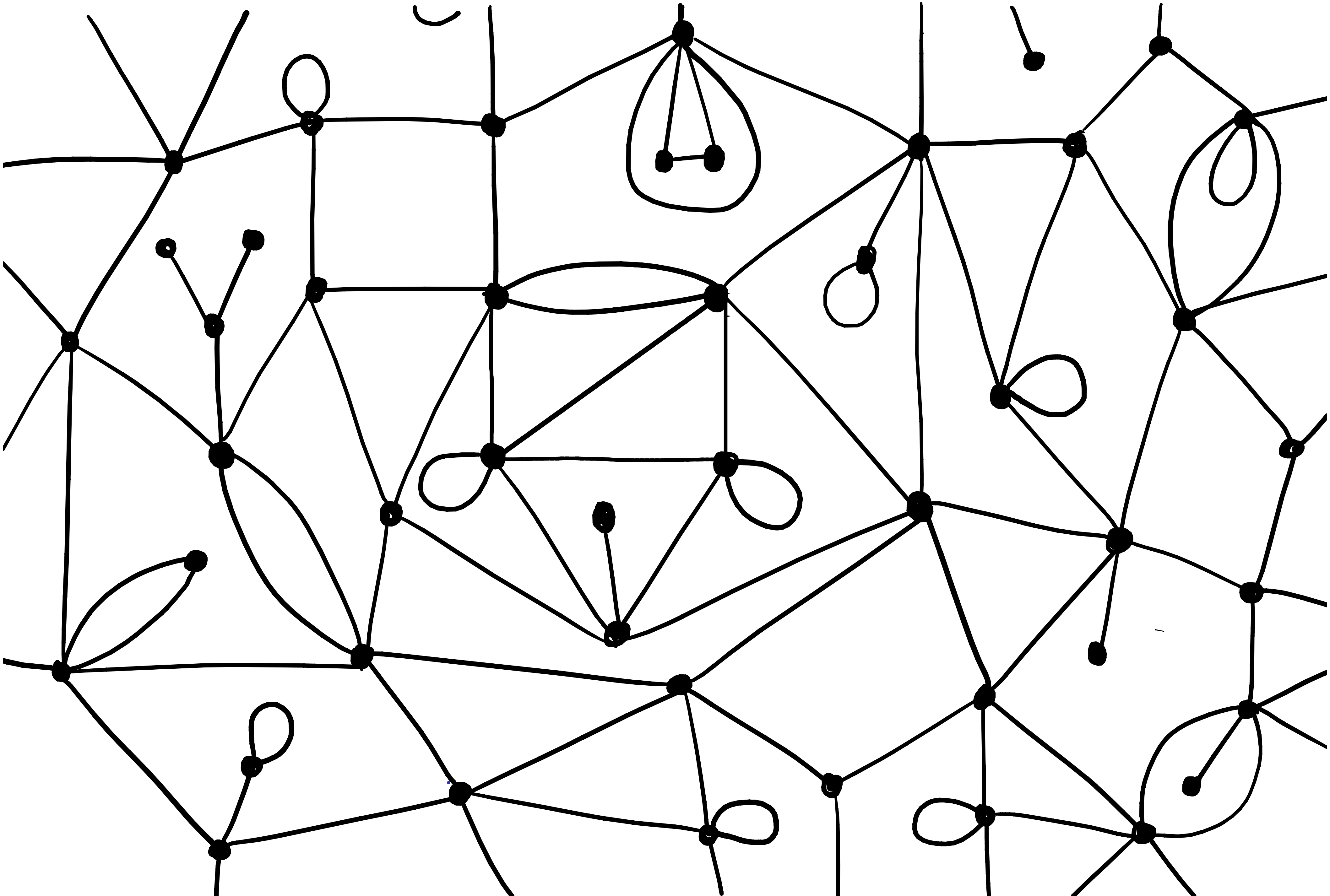
2-connected planar maps : $\frac{\sqrt{3}}{2\sqrt{\pi}} \left(\frac{27}{4}\right)^{n} n^{-5/2}$.

Planar triangulations : $\frac{\sqrt{6}}{32\sqrt{\pi}} \left(\frac{256}{27}\right)^{n} n^{-5/2}$

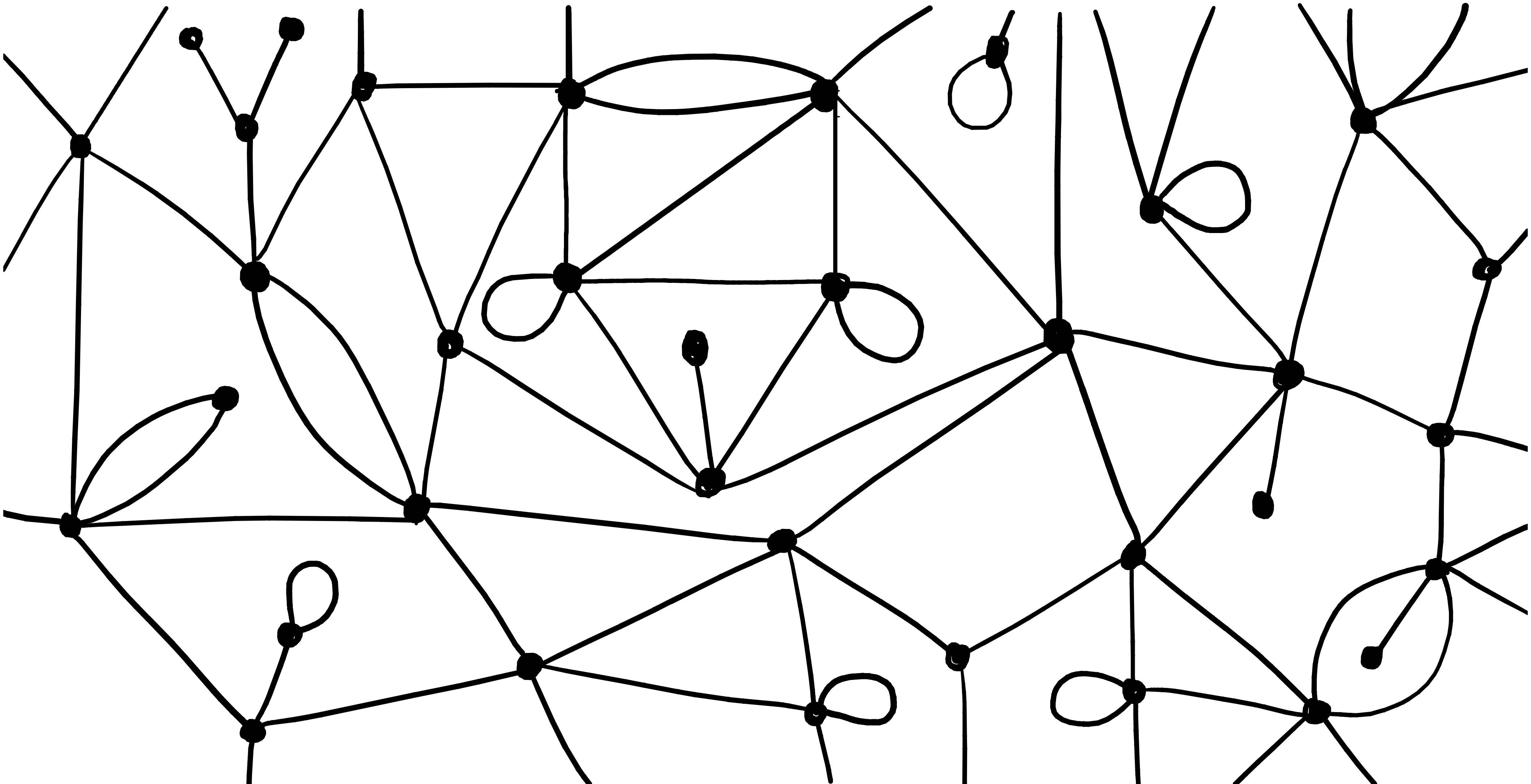
To be compared with the standard asymptotic behaviour of the number of plane trees :

$$\boxed{\sim c\, \rho^{-n}\, n^{-\frac{3}{2}}}$$

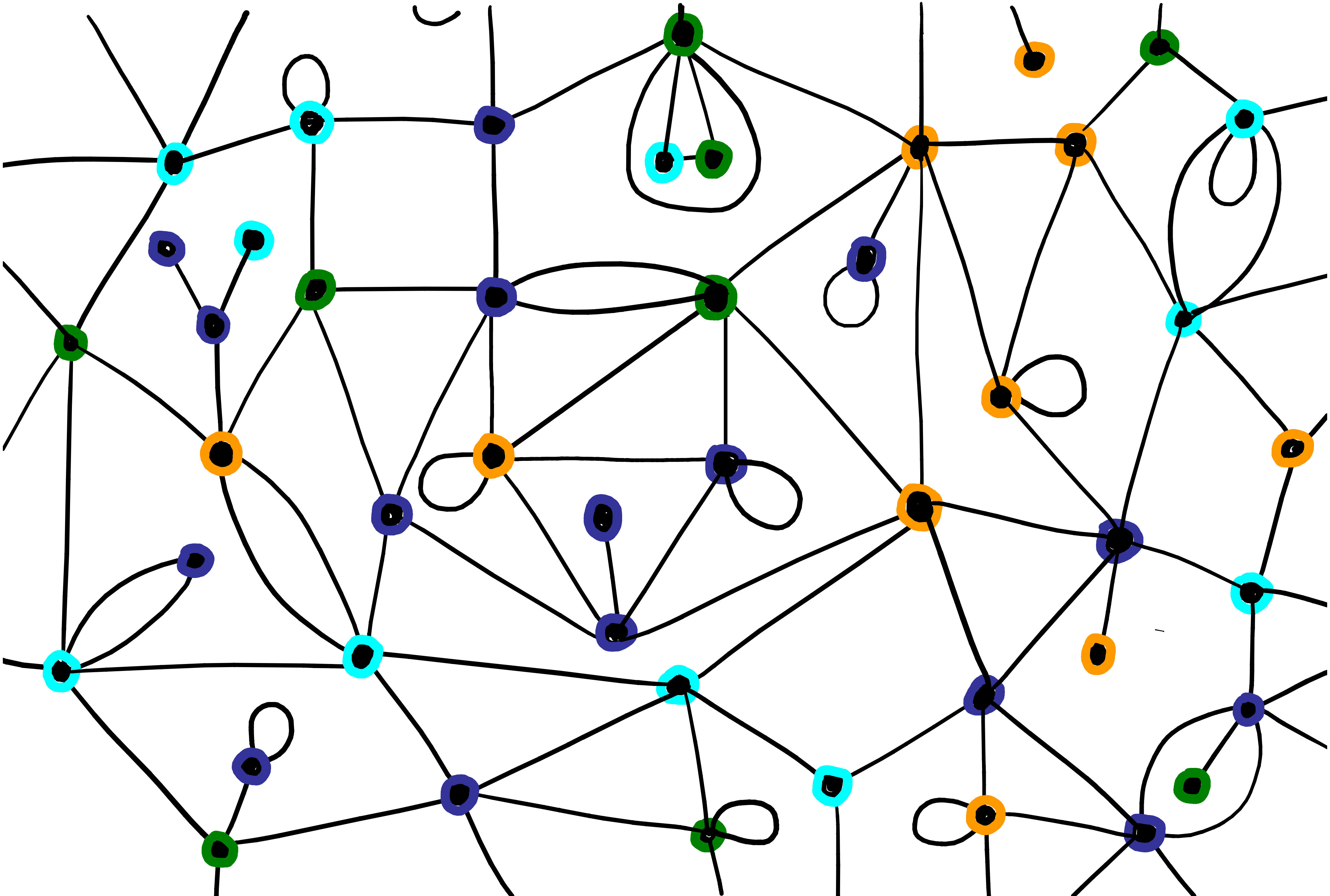
ADDITIONAL STRUCTURES



Additional structures: Spanning trees, colourings, percolation, Ising/Potts model, self-avoiding walks... [Tutte, Mullin, Kazakov, Borot, Bouttier, Guitter, Sportiello, Eynard, Duplantier, Bousquet-Mélou, Schaeffer, Bernardi, Angel ...]



THE POTTS MODEL



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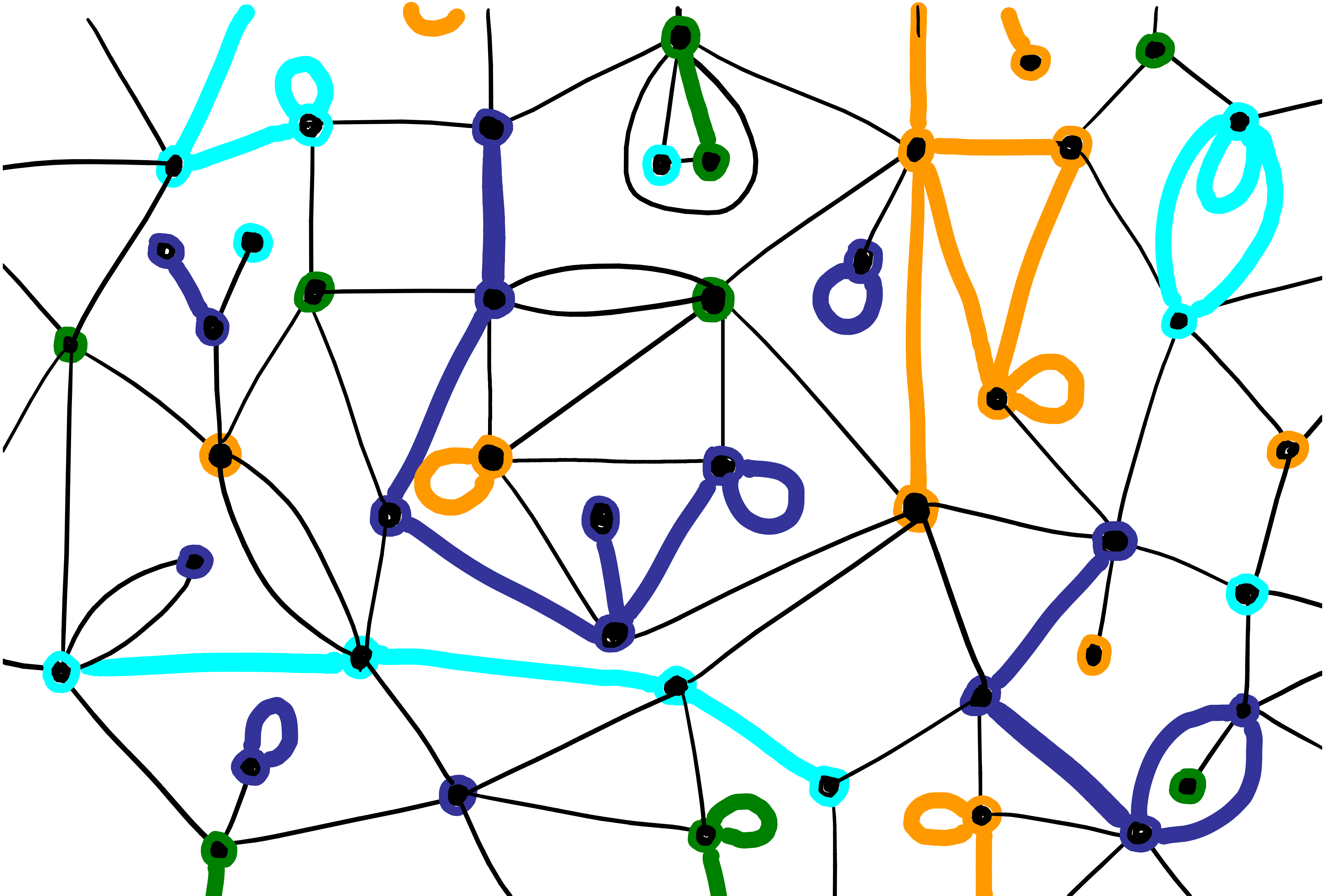
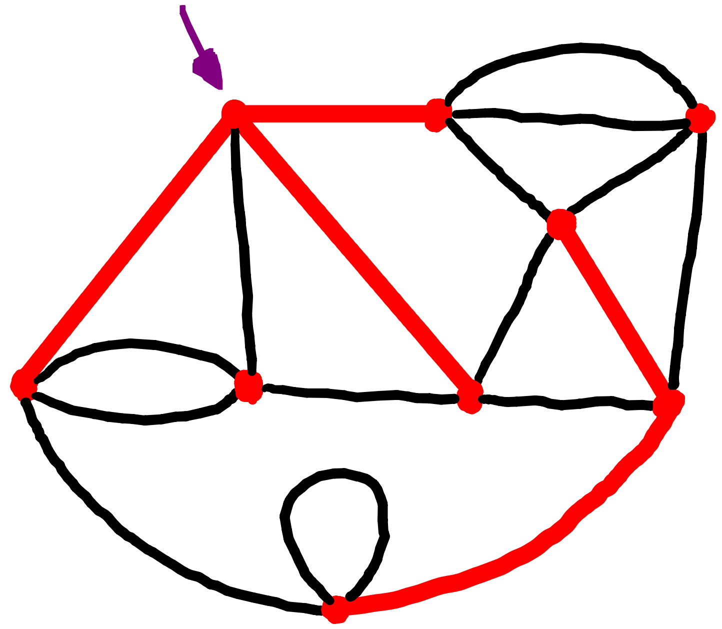


FORESTED MAPS

with Mireille BOUSQUET-MÉLOU (Bordeaux)

Forested Maps : Definition

Spanning forest of M =
graph F such that:
- $V(F) = V(M)$
- $E(F) \subseteq E(M)$ has no cycle.



Forested map (M,F) = Rooted map M with a spanning forest F.

$$F(z,u) = \sum_{\substack{(M,F)\text{ 4-valent} \\ \text{forested map}}} z^{\#\text{faces}} \, u^{\#\text{components} - 1}$$

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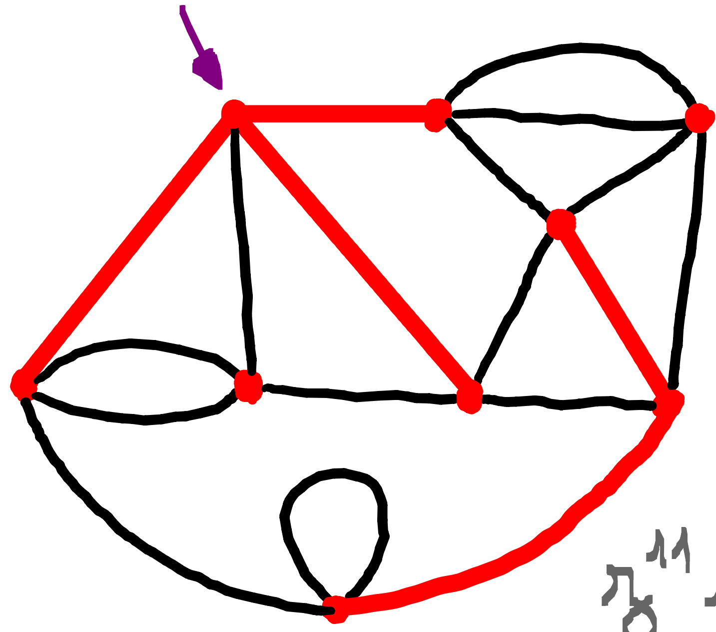

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WHAT DOES IT MODEL?

1) Limit $q \to 0$ of the Potts model.

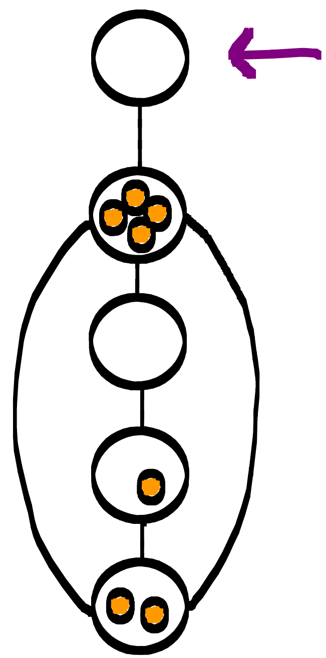
2) Tutte polynomial $T_M(u+1, 1)$

3) Sandpile model

---

3)

$$F(z,u) = \sum_{\substack{\text{quadrangulation} \\ \text{with recurrent} \\ \text{configuration } C}} z^{\#\text{vertices}} (u+1)^{\text{level}(C)}$$



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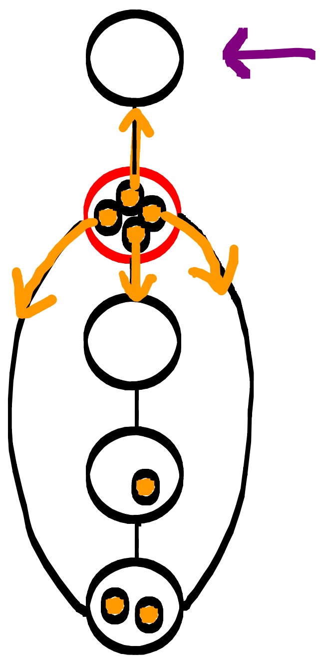

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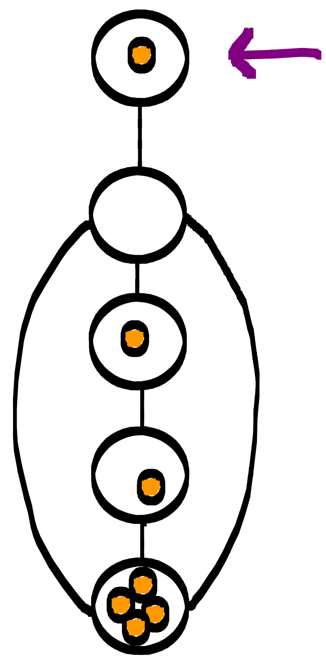

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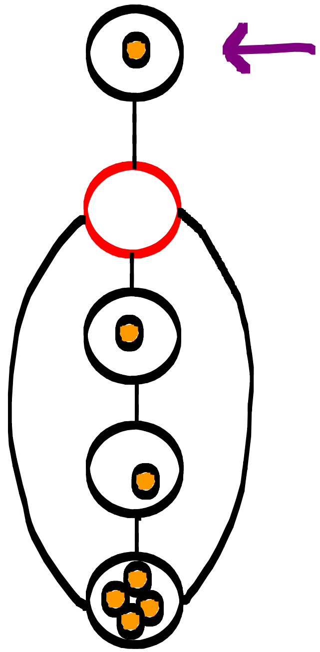

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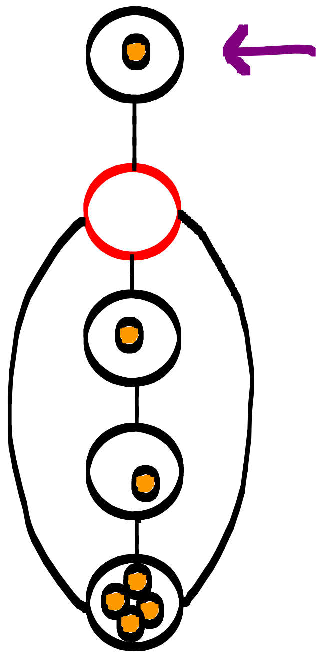

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Natural domain $u \in [-1, +\infty)$

---

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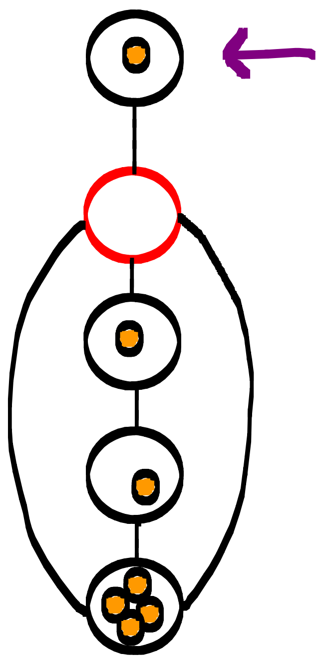


SPECIAL VALUES OF u

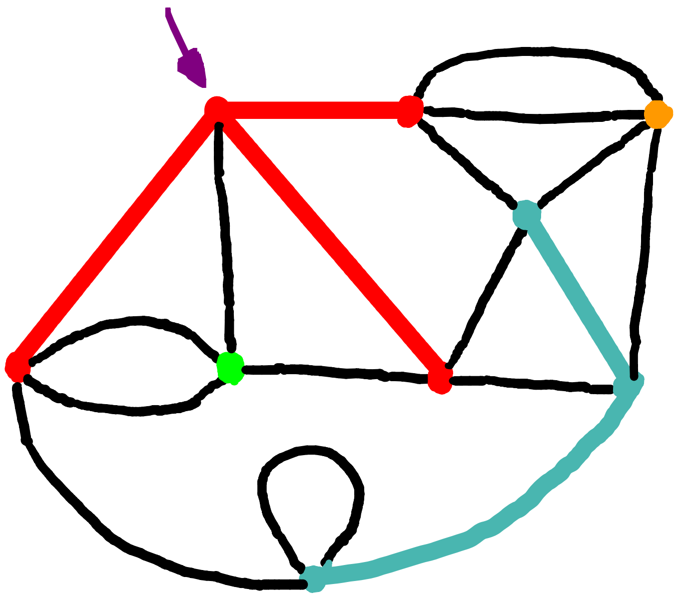
$$F(z,u) = \sum_{\substack{(M,F)\ \text{4-valent} \\ \text{forested map}}} z^{\#\text{faces}}\, u^{\#\text{components}-1}$$

* $u = 1$: spanning forests

* $u = 0$: spanning trees [Mullin, 1967]

* $u = -1$: root-connected acyclic orientations on (dual) quadrangulations.

A COMBINATORIAL DECOMPOSITION



Forested map = map where each vertex is weighted by a tree.

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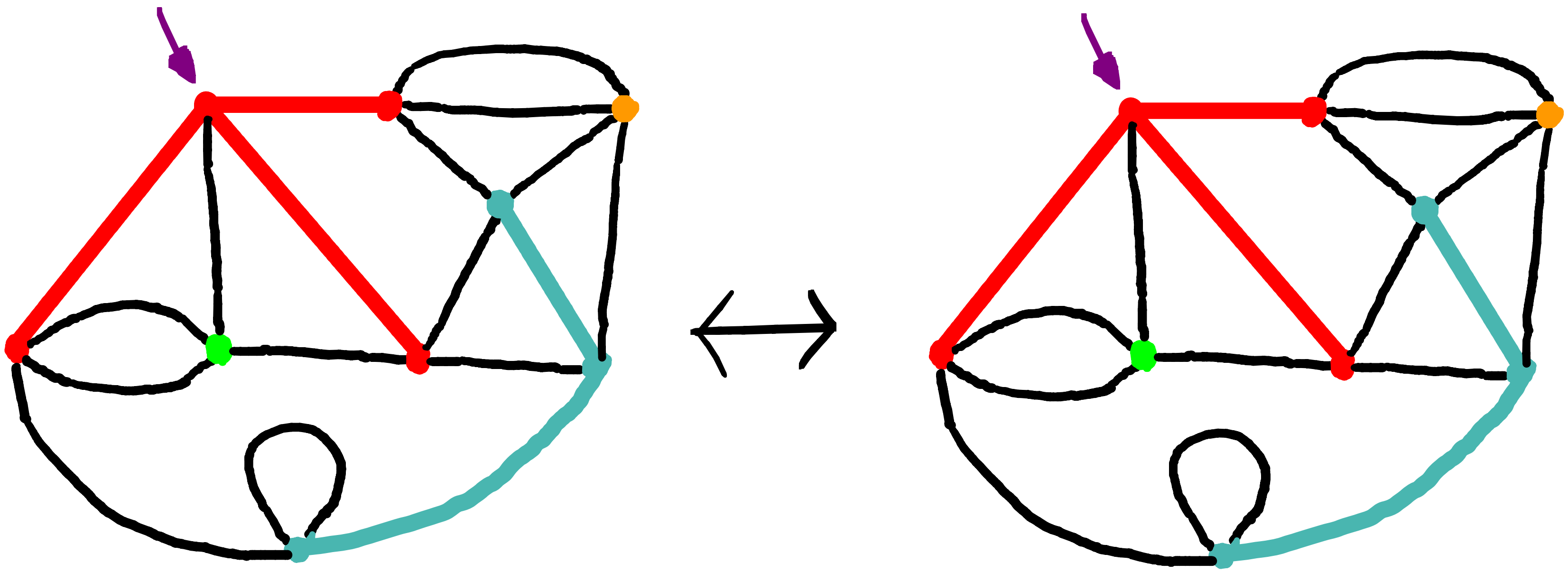

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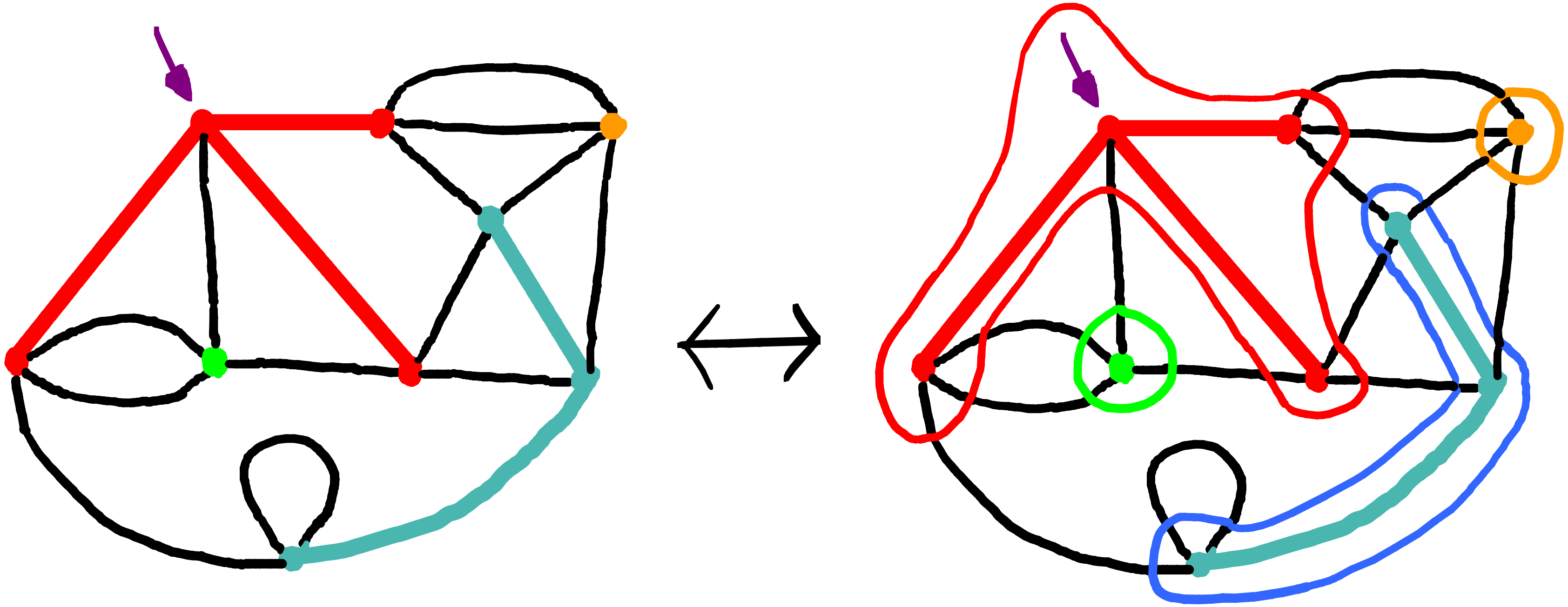

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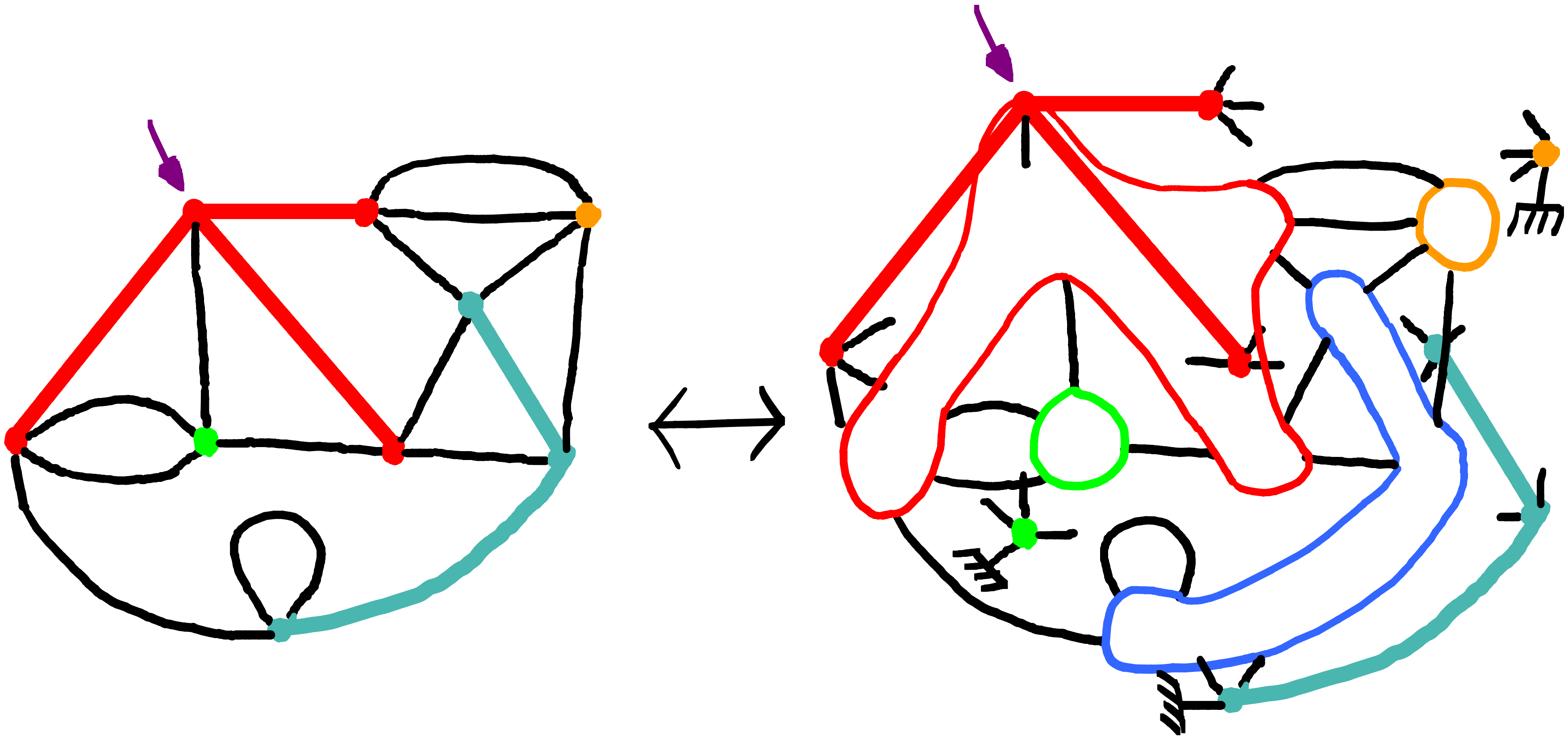

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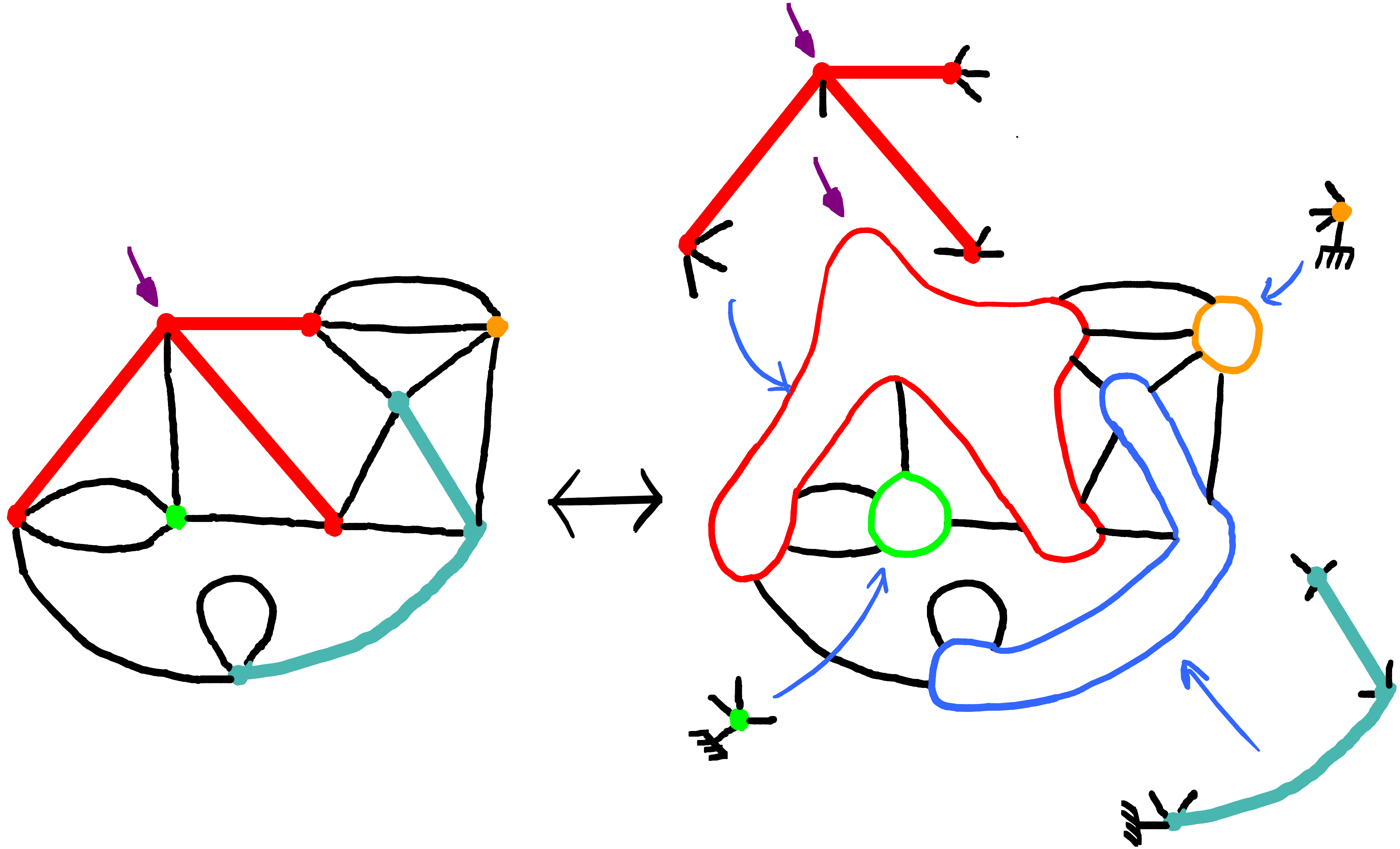

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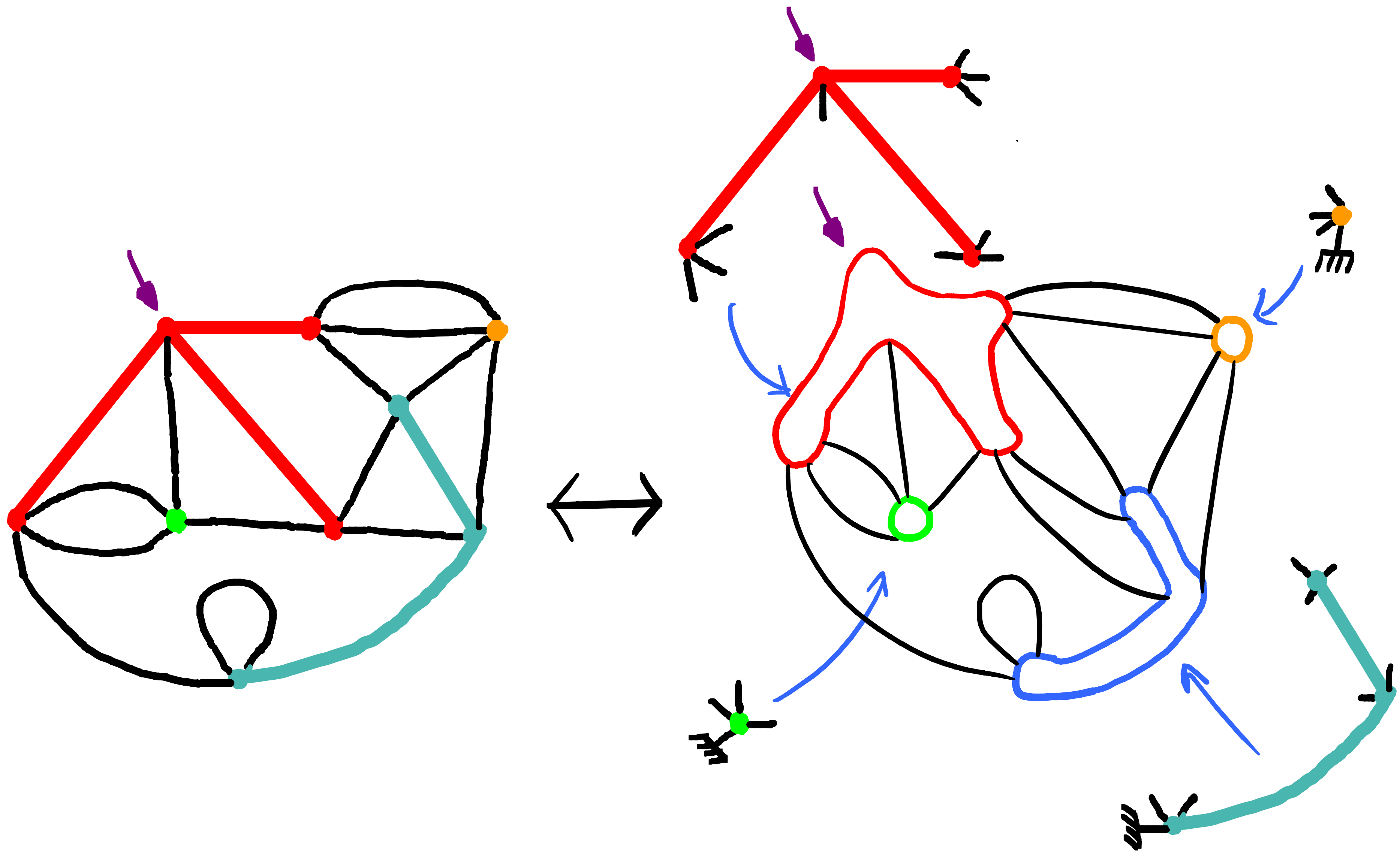

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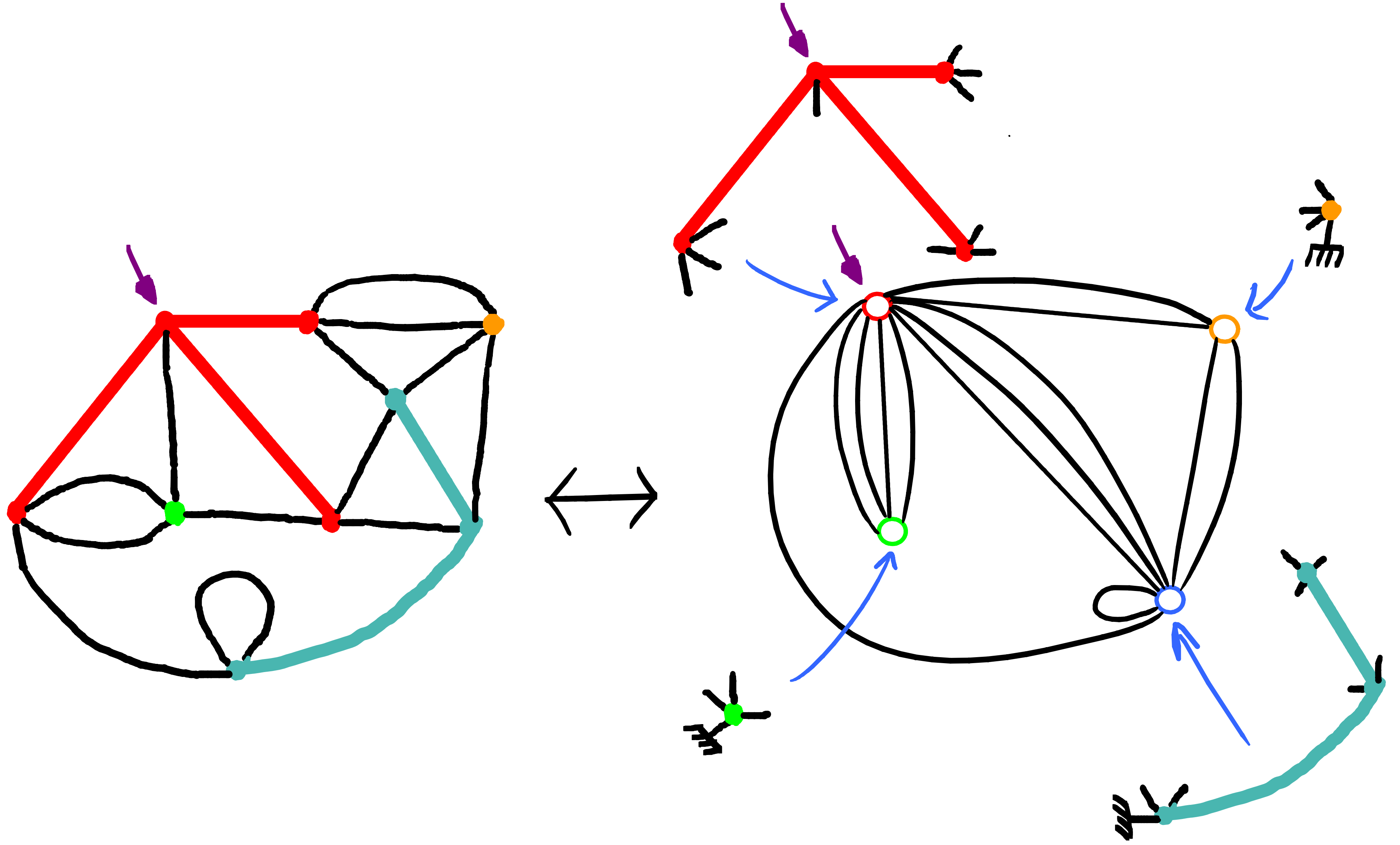

Forested map = map where each vertex is weighted by a tree.

THE GENERATING FUNCTION OF FORESTED MAPS

Theorem

There exists a unique series R in z with coefficients in $\mathbb{Q}[u]$ such that

$$R = z + u \sum_{i \geq 2} \frac{(3i-3)!}{(i-1)!^2\, i!} R^i$$

Then :

$$F' = 4 \sum_{i \geq 2} \frac{(3i-2)!}{(i-2)!\; i!^2} R^i$$

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For $u=0$, [Mullin]

$$R = z \quad \text{and} \quad F' = 4 \sum_{i \geq 2} \frac{(3i-2)!}{(i-2)! \, i!^2} z^i .$$

Phase Transition At 0

$f_n(u) = [z^n]\, F(z,u)$, u is fixed.

$-1 \leq u < 0$

$$f_n(u) \sim \frac{c_u\, \rho_u^{-n}}{n^3 \ln^2 n}$$

New "Universality class" for maps

$u = 0$

$$f_n(u) \sim \frac{c_u\, \rho_u^{-n}}{n^3}$$

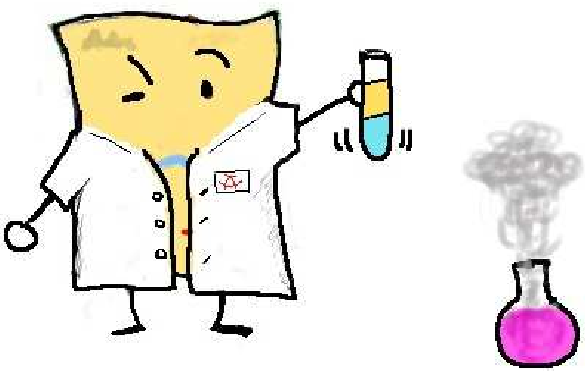
maps with a spanning tree

[Mullin]

$0 < u$

$$f_n(u) \sim \frac{c_u\, \rho_u^{-n}}{n^{5/2}}$$

standard



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D-finite

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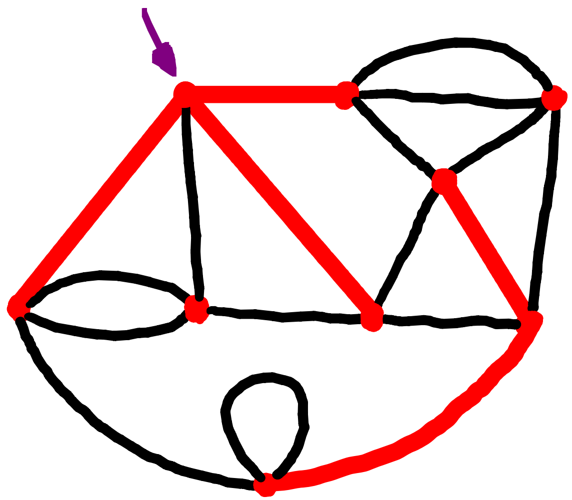
standard

Cor For $u \in [-1, 0)$, $F(z,u)$ is <u>not</u> D-finite, i.e F satisfies <u>no</u> linear differential equation.

SOME PROBABILITY RESULTS

Fix $n \in \mathbb{N}$,
consider a random forested
map with n faces.
(under uniform distribution)



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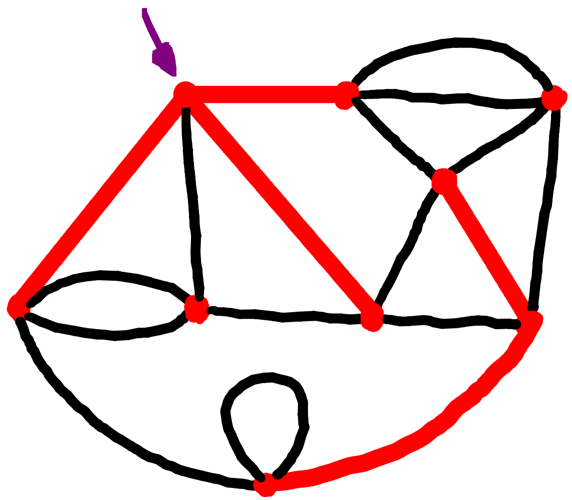

C_n = r.v that counts the number of components

Th $C_n \xrightarrow[\text{distribution}]{}$ Gaussian law with linear mean
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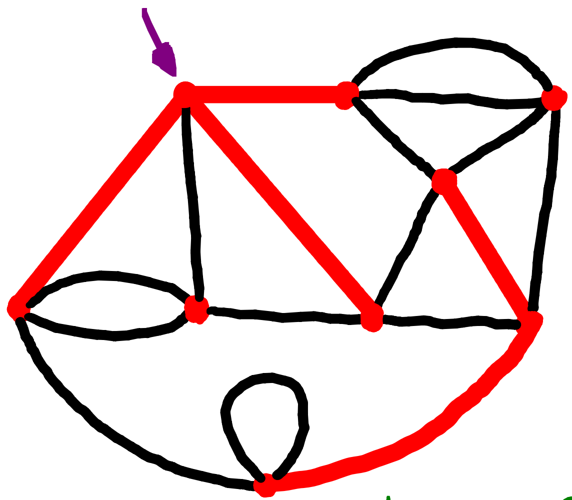

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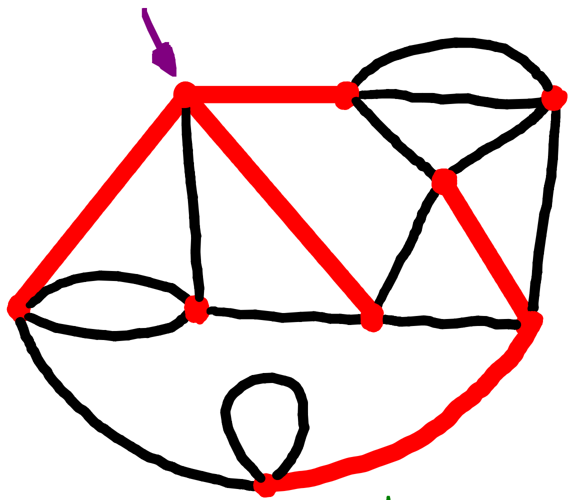


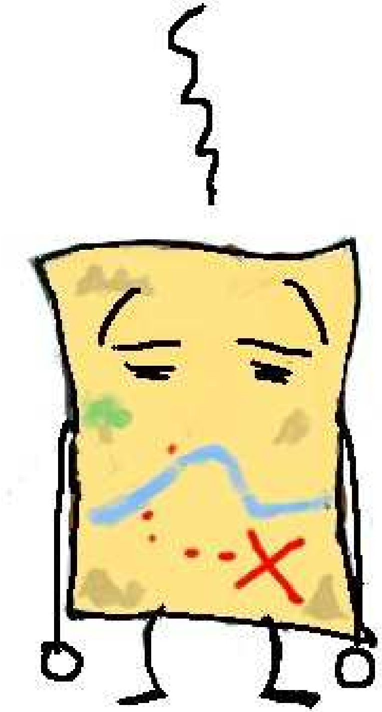
Here $S_n = 4$

S_n = size of the root component (number of vertices)

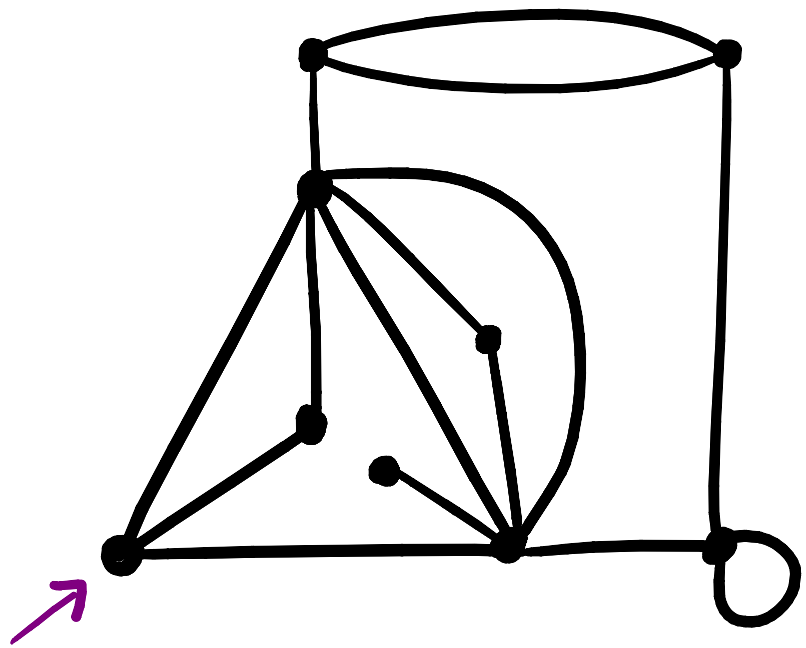
Th

$$\lim_{n \to +\infty} \mathbb{P}_n (S_n = k) = \frac{4\,(3k)!}{(k-1)!\,k!\,(k+1)!} \frac{z_1^{\,k}}{\phi'(z_1)}$$

The next step: Percolation



BOND PERCOLATION ON MAPS



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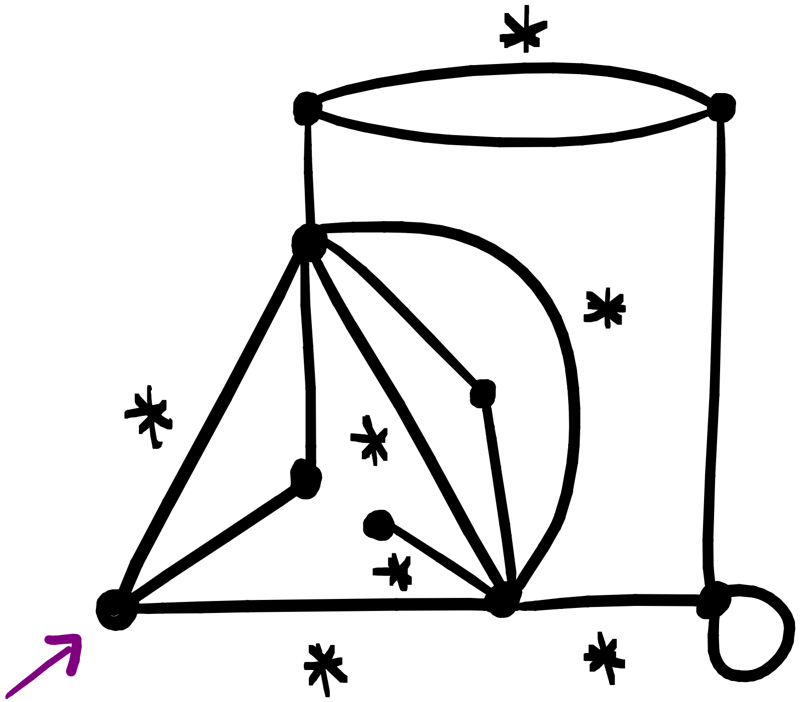

* = active with probability p

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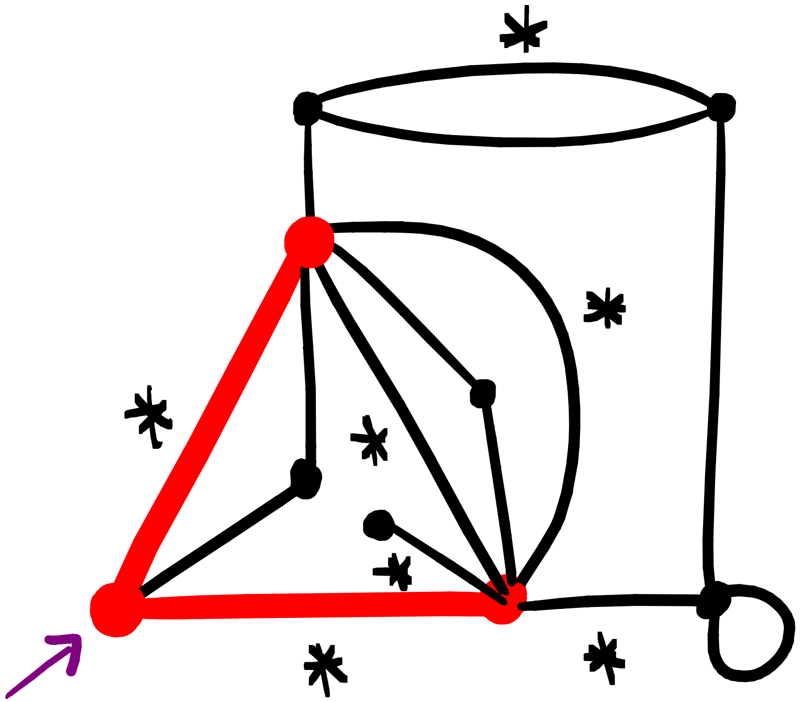

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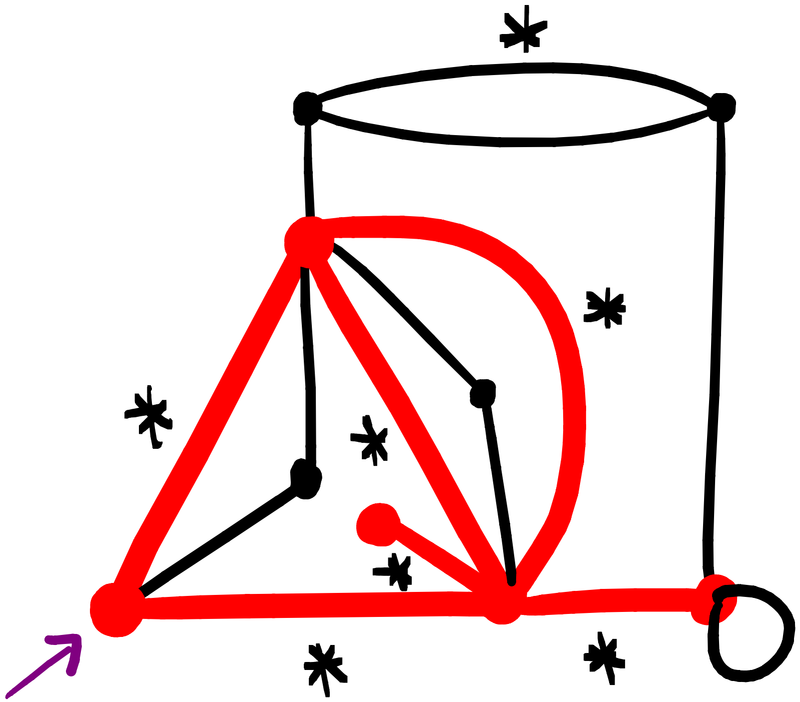

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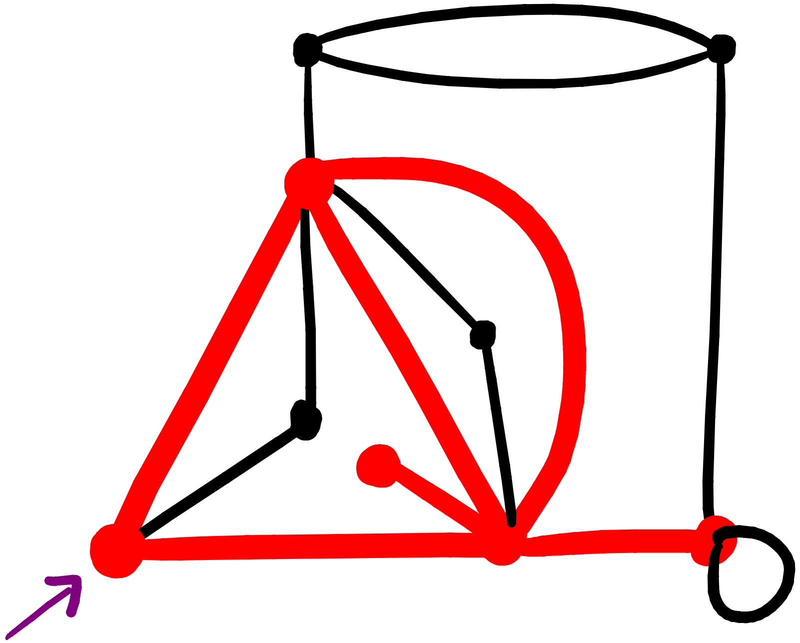
Bond Percolation on Maps

bond animal = connected subgraph containing the root.
(not necessarily spanning)



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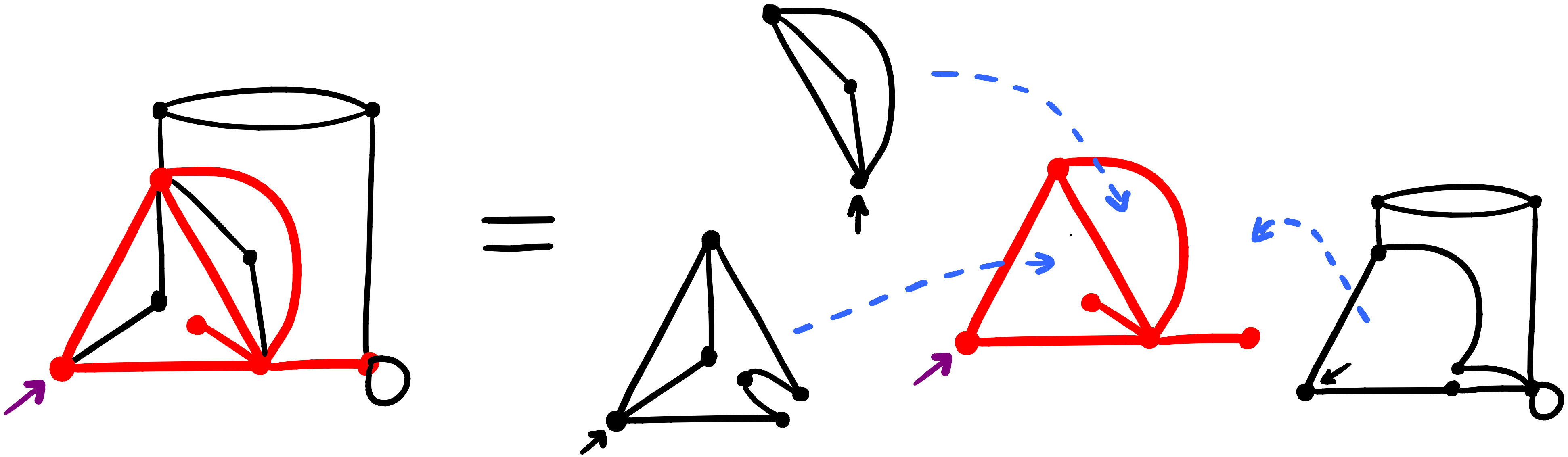

Map equipped with a bond animal =
map where each face is weighted by a map.


THANK YOU!