ENUMERATION OF PLANAR MAPS
WITH ADDITIONAL STRUCTURES

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PLANAR MAPS IN STATISTICAL PHYSICS
Planar map = connected graph + embedding of this graph in the plane, considered up to continuous deformation.
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We root every planar map at an outer corner.
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We root every planar map at an outer corner.
LARGE MAPS
UNIVERSALITY CLASS

Standard asymptotic behaviour of the number of planar maps:

$$\sim c \rho^{-n} n^{-\frac{5}{2}}$$

Examples:
- General planar maps: $$\frac{2}{\sqrt{\pi n}} 12^n n^{-5/2}$$
- 2-connected planar maps: $$\frac{\sqrt{3}}{24\pi n} (\frac{27}{\sqrt{\pi}})^n n^{-5/2}$$
- Planar triangulations: $$\frac{\sqrt{16}}{32\pi n} (\frac{256}{27})^n n^{-5/2}$$

To be compared with the standard asymptotic behaviour of the number of plane trees:

$$\sim c' \rho^{-n} n^{-\frac{3}{2}}$$
ADDITIONAL STRUCTURES
Additional structures: Spanning trees, colourings, percolation, Ising/Potts model, self-avoiding walks... [Tutte, Mullin, Kazakov, Borot, Bouttier, Guitter, Sportiello, Eynard, Duplantier, Bousquet-Mélou, Schaeffer, Bernardi, Angel...]
THE POTTS MODEL
THE POTTIS MODEL
FORESTED MAPS

with Mireille BOUSQUET-MÉLOU (Bordeaux)
Spanning forest of $M$ = graph $F$ such that:
- $V(F) = V(M)$
- $E(F) \leq E(M)$ has no cycle.

Forested map $(M, F) =$ Rooted map $M$ with a spanning forest $F$.

$$F(\varphi_3, \mu) = \sum_{(M, F) \text{ 4-valent forested map}} \varphi_3^{\text{# faces}} \mu^{\text{# components} - 1}$$
Spanning forest of $M$ = graph $F$ such that:
- $V(F) = V(M)$
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Forest map $(M,F) =$ Rooted map $M$ with a spanning forest $F$.

$$F(z_0, w) = \sum_{(M,F) \in \text{4-valent forested map}}^{\text{$M$}} \text{# faces} \times \text{# components} - 1$$
WHAT DOES IT MODEL?

1) Limit $q \to 0$ of the Potts model

2) Tutte polynomial $T_M(\mu + 1, 1)

3) Sandpile model

3)

$$F(g, \mu) = \sum_{\text{quadrangulation with recurrent configuration } C} g^\# \text{ vertices} \mu^{\text{level}(C)}$$
WHAT DOES IT MODEL?

1) Limit $q \to 0$ of the Potts model.

2) Tutte polynomial $T_M(u+1, 1)$.

3) Sandpile model

\[
F(g, u) = \sum_{\text{quadrangulation with recurrent configuration } C} g^\# \text{ vertices} \cdot (u+1)^{\text{level}(C)}
\]
WHAT DOES IT MODEL?

1) Limit \( q \to 0 \) of the Potts model.

2) Tutte polynomial \( \overline{T}_M (\mu + 1, 1) \)

3) Sandpile model

3)\[ F (g, \mu) = \sum \text{quadrangulation with recurrent configuration } C \# \text{ vertices}^{(\mu + 1) \text{level}(C)} \]
WHAT DOES IT MODEL?

1) Limit $q \to 0$ of the Potts model.
2) Tutte polynomial $T_M(\mu + 1, 1)$
3) Sandpile model

3)

$$F(\beta, \mu) = \sum_{\text{quadrangulation with recurrent configuration } C} \beta^\# \text{ vertices } \sim \mu \text{ level}(C)$$

\[\sim \mu = \mu + 1\]
WHAT DOES IT MODEL?

1) Limit $q \to 0$ of the Potts model.

2) Tutte polynomial $T_M (\mu + 1, 1)$

3) Sandpile model

$F(g, \mu) = \sum_{\text{quadrangulation with recurrent configuration } C} \# \text{ vertices } \tilde{\mu} \text{ level}(C)$

$\tilde{\mu} = \mu + 1$
WHAT DOES IT MODEL?

1) Limit \( q \to 0 \) of the Potts model.
2) Tutte polynomial \( T_M(\mu+1, 1) \)
3) Sandpile model

\[ F(\mathcal{G}, \mu) = \sum \text{ # vertices } \tilde{\mu} \text{ level}(C) \]

Natural domain \( \mu \in [-1, +\infty) \)

\[ \tilde{\mu} = \mu + 1 \]
SPECIAL VALUES OF $\mu$

\[
F(x, \mu) = \sum_{(M,F) \text{ 4-valent forested map}} \begin{array}{c}
\text{# faces} \\
\text{# components} - 1
\end{array}
\]

* $\mu = 1$: spanning forests

* $\mu = 0$: spanning trees [Mullin, 1967]

* $\mu = -1$: root-connected acyclic orientations on (dual) quadrangulations.
A combinatorial decomposition

Forest map = map where each vertex is weighted by a tree.
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A COMBINATORIAL DECOMPOSITION

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A COMBINATORIAL DECOMPOSITION

Forested map = map where each vertex is weighted by a tree.
The Generating Function of Forested Maps

**Theorem**

There exists a unique series $R$ in $\mathbb{z}$ with coefficients in $\mathbb{Q}[u]$ such that

$$R = z_8 + u \sum_{i \geq 2} \frac{(3i-3)!}{(i-1)!^2 i!} R^i$$

Then:

$$F' = 4 \sum_{i \geq 2} \frac{(3i-2)!}{(i-2)! i!^2} R^i$$
**Theorem**

There exists a unique series $R$ in $z_8$ with coefficients in $\mathbb{Q}[w]$ such that

$$R = z_8 + \omega \sum_{i \geq 2} \frac{(3i-3)!}{(i-1)!^2 i!} R^i$$

Then:

$$F' = 4 \sum_{i \geq 2} \frac{(3i-2)!}{(i-2)! i!^2} R^i$$

For $\omega = 0$, [Mullin]

$$R = z_8 \quad \text{and} \quad F' = 4 \sum_{i \geq 2} \frac{(3i-2)!}{(i-2)! i!^2} z_8^i.$$
Phase Transition At 0

\( f_n(w) = [z^n] F(z, w), \ w \text{ is fixed} \)

\(-1 \leq w < 0\)

\( f_n(w) \sim \frac{cw \bar{w}^{-n}}{n^3 \ln^2 n} \)

New "Universality class" for maps

\( w = 0 \)

\( f_n(w) \sim \frac{cw \bar{w}^{-n}}{n^3} \)

maps with a spanning tree

\( [\text{Mullin}] \)

\( 0 < w \)

\( f_n(w) \sim \frac{cw \bar{w}^{-n}}{n^{5/2}} \)

standard
**Phase Transition At 0**

\( f_n(u) = \left[ z^n \right] F(z, u), \ u \) is fixed.

\[ -1 \leq u < 0 \]

\( f_n(u) \sim \frac{c u^{\frac{1}{u}}}{n^3 \ln^2 n} \)

"Universality class" for maps

\[ 0 < u \]

\( f_n(u) \sim \frac{c u^{\frac{1}{u}}}{n^{5/2}} \)

maps with a spanning tree [Mullin]

**Cor**

For \( u \in (-1, 0) \), \( F(z, u) \) is not D-finite, i.e., \( F \) satisfies no linear differential equation.
Fix $n \in \mathbb{N}$, consider a random forested map with $n$ faces - (under uniform distribution)
SOME PROBABILITY RESULTS

Fix $n \in \mathbb{N}$, consider a random forested map with $n$ faces - (under uniform distribution)

$C_m = r.v.$ that counts the number of components

$\text{Th}$

$C_m \xrightarrow{\text{distribution}}$ Gaussian law with linear mean & linear variance.
Some probability results

Fix $n \in \mathbb{N}$, consider a random forested map with $n$ faces - (under uniform distribution)

$C_m = \text{r.v. that counts the number of components}$

$\text{Th}$

$C_m \rightarrow \text{Gaussian law with linear mean }$ & $\text{linear variance}$
SOME PROBABILITY RESULTS

Fix \( n \in \mathbb{N} \), consider a random forested map with \( n \) faces.
(under uniform distribution)

\[ S_n = \text{size of the root component (number of vertices)} \]

\[
\lim_{n \to +\infty} P_n(S_n = k) = \frac{4}{(k-1)!k!(k+1)!} \frac{\binom{k}{2}}{\phi(2)}
\]
The next step: Percolation
BOND PERCOLATION ON MAPS
Bond Percolation on Maps

* = active with probability \( p \)
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bond animal = connected subgraph containing the root. (not necessarily spanning)
**BOND PERCOLATION ON MAPS**

bond animal = connected subgraph containing the root,
(not necessarily spanning)

Map equipped with a bond animal = map where each face is weighted by a map.
THANK YOU!