Understanding Dyson-Schwinger equations via chord diagrams

Julien COURTIEL
Univ. de Caen Normandie

Philippe Flajolet seminar
Joint work with Karen YEATS (Univ. of Waterloo)
In this work, analytic combinatorics applied to Q.F.T.
In this work, analytic combinatorics applied to QUANTUM FIELD THEORY
Part I

Background about Q.F.T.
In this work, analytic combinatorics applied to

QUANTUM FIELD THEORY
In this work, analytic combinatorics applied to Quantum Feline Theory Field.
In this work, analytic combinatorics applied to **Quantum Feline Theory Field**
In this work, analytic combinatorics applied to Quantum Feline Theory Field
PHYSICAL BACKGROUND

Yukawa theory

PAWRTICLE ACCELERATOR

fermion

meson
PHYSICAL BACKGROUND

Yukawa theory

Feynman diagram

fermion

meson
PHYSICAL BACKGROUND

Yukawa theory

Feynman diagrams

fermion

meson
PHYSICAL BACKGROUND

Yukawa theory

Feynman diagrams
PHYSICAL BACKGROUND

Yukawa theory

Feynman diagrams

renormalized Feynman integrals

\( M \quad (L) \quad M \quad (L) \quad M \quad (L) \quad \ldots \)
PHYSICAL BACKGROUND

Yukawa theory

Feynman diagrams
renormalized Feynman integrals
perturbative series (Green function)

\[ G(x, L) = \sum_{D \text{ Feynman diagram}} M_D(L) \times \text{# vertices}(D) \]

\[ L = \log \text{ of momentum} \]
\[ x = \text{coupling constant} \]
\[ G(x, L) = \text{probability amplitude} \]
The equation (which is a Dyson-Schwinger equation)

\[ G(x, L) = 1 - \sum_{k \geq 1} x^k G(x, \delta_{-\rho})^{1-2k} (e^{-L \rho} - 1) F_{2k}(\rho) \bigg|_{\rho=0} \]

where \( F_{2k}(\rho) \) is the regularized Feynman integral of the primitive graphs of size \( k \).

\( F_1(\rho) = \text{contribution of } \quad F_2(\rho) = \text{contribution of } \)

has for solution

\[ G(x, L) = \sum_{\text{D Feynman diagram}} \mathcal{M}_D(L) x^\# \text{ vertices}(D) \]
PHYSICAL BACKGROUND
Yukawa theory is not the only theory!
PHYSICAL BACKGROUND

Yukawa theory: not the only theory!

Quantum Electrodynamics

Scalar $\phi^3$ theory
The equation (which is a Dyson-Schwinger equation)

\[ G(x, L) = 1 - \sum_{k \geq 1} x^k G(x, \Delta - \rho) \left( e^{-L\rho - 1} \right) F_k(\rho) \bigg|_{\rho=0} \]

where \( F_k(\rho) \) is the regularized Feynman integral of the primitive graphs of size \( k \)

and \( \Delta = \) insertion growth number

has for solution

\[ G(x, L) = \sum_{\text{Feynman diagram}} M_\Delta(L) x^{\# \text{ vertices}(\Delta)} \]

QED

\[ d=1 \]

\[ \begin{aligned} &\text{\begin{tikzpicture} [baseline] \draw [line width=1pt] (0,0) circle (0.5cm); \draw [line width=1pt] (0,0) -- (0.5,0.5); \draw [line width=1pt] (0,0) -- (0.5,-0.5); \end{tikzpicture}} \\
&\text{\begin{tikzpicture} [baseline] \draw [line width=1pt] (0,0) circle (0.5cm); \draw [line width=1pt] (0,0) -- (0.5,0.5); \draw [line width=1pt] (0,0) -- (0.5,-0.5); \draw [line width=1pt] (0,0) -- (0.5,0); \end{tikzpicture}} \\
\end{aligned} \]

\[ d=2 \]

\[ \text{Yukawa} \]

\[ d=3 \]

\[ \text{Scalar } \phi^3 \]
The equation (which is a Dyson-Schwinger equation)

\[ G(x, L) = 1 - \sum_{k \geq 1} x^k G(x, \phi) \left( e^{-L \phi} - 1 \right) F_k(\phi) \bigg|_{\phi=0} \]

where \( F_k(\phi) \) is the regularized Feynman integral of the primitive graphs of size \( k \)

and \( \Delta = \) insertion growth number

has for solution

\[ G(x, L) = \sum_{\text{\textit{\scriptsize Feynman diagram}}} M_D(L) x^\# \text{ vertices}(D) \]

\( d=1 \)  
QED

\( d=2 \)  
Yukawa

\( d=3 \)  
Scalar \( \phi^3 \)
Part II

Solving Dyson–Schwinger equations in terms of generating functions
The solution of

$$G(x, L) = 1 - \sum_{k \geq 1} \alpha_k \frac{G(x, \partial_{-\rho})^{1+\Delta k}}{(e^{-L\rho} - 1) \bar{F}_k(\rho)} |_{\rho=0}$$

is
The solution of

\[ G(x, L) = 1 - \sum_{k \geq 1} x^k G(x, d_p) \left( e^{-Lp} - 1 \right) \frac{F_k(p)}{\rho = 0} \]

is

\[ F_k(p) = \frac{b_{k,0}}{p} + b_{k,1} + b_{k,2} p + b_{k,3} p^3 + \ldots \]
The solution of
\[ G(x, L) = 1 - \sum_{k \geq 1} x^k G(x, t^k_r) \left( e^{-L} - 1 \right) F_{k_r}(p) \bigg|_{p=0} \]
is the weighted generating function
\[ G(x, L) = 1 - \sum_{\text{Co connected chord diagram}} \left( \sum_{i=1}^{k_1} b(d(t_i), t_i, \frac{L}{t_i}) \right) \prod_{i=1}^{k_1} b(d(t_i), t_i, t_i, \ldots, x) \]
such that \( x_1 < x_2 < \ldots < x_{k_r} \)
are the positions of the terminal chords
for the intersection order

where \( F_{k_r}(p) = \frac{b_{k_r,0}}{p} + b_{k_r,1} + b_{k_r,2} p + b_{k_r,3} p^3 + \ldots \)
The solution of

\[ G(x, L) = 1 - \sum_{k \geq 1} x^k \left( \sum_{i=1}^{k} \frac{b_d(t_i) \cdot \binom{L}{i}}{i!} \right) \prod_{i} b_d(t_i) \cdot \prod_{i \text{ non terminal}} b_d(t_i), t_i \cdots \cdot x \]

is the weighted generating function

\[ G(x, L) = 1 - \sum_{C \text{ w-marked} \text{ decorated connected chord diagram}} \left( \sum_{i=1}^{k} \frac{b_d(t_i) \cdot \binom{L}{i}}{i!} \right) \prod_{i} b_d(t_i) \cdot \prod_{i \text{ non terminal}} b_d(t_i), t_i \cdots \cdot x \]

where \( F_k(p) = \frac{b_{k,0}}{p} + b_{k,1} + b_{k,2} p + b_{k,3} p^3 + \ldots \).
chord diagram = perfect matching of \( \{1, 2, \ldots, 2n\} \)

<table>
<thead>
<tr>
<th>Smallest Chord Diagrams</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 1 )</td>
</tr>
<tr>
<td><img src="image1" alt="Diagram 1" /></td>
</tr>
<tr>
<td>( n = 2 )</td>
</tr>
<tr>
<td><img src="image2" alt="Diagram 2" /></td>
</tr>
</tbody>
</table>
chord diagram = perfect matching of \( \{1, 2, \ldots, 2n\} \)

number of diagrams with \( n \) chords = \((2n - 1)!!\)
= \((2n-1) \times (2n-3) \times \ldots \times 3 \times 1\)
The solution of

\[ G(x,L) = 1 - \sum_{k \geq 1} x^k \ G(x, d_p) \ (e^{-Lp} - 1) \ F_k(p) \bigg|_{p=0} \]

is the weighted generating function

\[ G(x,L) = 1 - \sum_{\text{C WD-marked decorated connected chord diagram}} \left( \sum_{i=1}^{k_1} \ b_{d(t_i), t_i} \ (\frac{-L}{i!})^i \right) \prod_{\text{non terminal}} b_{d(i), 0} \prod_{i=1}^{k-1} b_{d(t_i), t_i + \cdots + t_i} \ x \]

such that \( k_1 < k_2 < \ldots < k_k \) are the positions of the terminal chords for the intersection order

where

\[ F_k(p) = \frac{b_{k,0}}{p} + b_{k,1} + b_{k,2} p + b_{k,3} p^3 + \ldots \]
The solution of

$$G(x, L) = 1 - \sum_{k \geq 1} x^k G(x, \delta_{-\rho})^{1-\delta_k} (e^{-L}\rho - 1) F_{k\rho}(\rho) \big|_{\rho = 0}$$

is the weighted generating function

$$G(x, L) = 1 - \sum_{\text{C w-marked connected chord diagram}} \left( \sum_{i=1}^{k_1} \frac{b(\delta(t_i), t_i, \cdots)}{i!} \right) \prod_{i} b(\delta(t_i), t_i, t_{i+1}, \cdots) x$$

such that $k_1 < k_2 < \cdots < k_{k_1}$ are the positions of the terminal chords for the intersection order.

where $F_{k\rho}(\rho) = \frac{b_{k,0}}{\rho} + b_{k,1} + b_{k,2} \rho + b_{k,3} \rho^2 + \cdots$
Connected chord diagram =
diagram “in one block” =
diagram not of the form

NOT CONNECTED

3 connected components

CONNECTED

Smallest connected chord diagrams:

\[ n = 1 \]
\[ n = 2 \]
\[ n = 3 \]
The solution of

\[ G(x, L) = 1 - \sum_{k \geq 1} x^k \sum_{\text{decorated connected chord diagram}} (\prod_{i=1}^{k-1} \frac{bd(t_i), t_i - i}{i!} (\frac{-L}{i})^i) \prod_{\text{non terminal}} bd(t_i) \rho + \sum_{\text{terminal}} \prod_{i=1}^{k-1} bd(t_i), t_i - t_{i-1}, x \]

is the weighted generating function

\[ G(x, L) = 1 - \sum_{G \in \text{marked connected chord diagram}} (\sum_{i=1}^{k_1} \frac{bd(t_i), t_i - i}{i!} (\frac{-L}{i})^i) \prod_{\text{non terminal}} bd(t_i) \rho + \sum_{\text{terminal}} \prod_{i=1}^{k-1} bd(t_i), t_i - t_{i-1}, x \]

such that \( k_1 < k_2 < \ldots < k_k \) are the positions of the terminal chords for the intersection order.

where

\[ F_k(p) = \frac{bd(0)}{p} + \frac{bd}{p} + \frac{bd^2}{2} p + \frac{bd^3}{3} p^3 + \ldots \]
The solution of

\[ G(x, L) = 1 - \sum_{k \geq 1} x^k \left( e^{-Lp_0} - 1 \right) F_k(p) \bigg|_{p=0} \]

is the weighted generating function

\[ G(x, L) = 1 - \sum_{\text{C wo-marked, decorated connected chord diagram}} \left( \sum_{i=1}^{k_1} \frac{b_{d(t_i), t_i-i}}{i!} \right) \prod_{i=1}^{k_{-1}} b_{d(i)} \prod_{i=1}^{k_{-1}} b_{d(t_i), t_i-t_{i-1}, x} \]

such that \( k_1 < k_2 < \ldots < k_{-1} \)
are the positions of the terminal chords
for the intersection order.

where \( F_k(p) = \frac{b_{k,0}}{p} + b_{k,1} + b_{k,2} p + b_{k,3} p^3 + \ldots \)
decorated chord diagram =
diagram where each chord carries a positive integer
= \text{“decoration”}

\begin{tabular}{|c|c|}
\hline
\text{Smallest decorated chord diagrams} & \\
\hline
\text{n = 1} & \includegraphics[width=0.2\textwidth]{image1} \\
\text{n = 2} & \includegraphics[width=0.2\textwidth]{image2} \\
\text{n = 3} & \includegraphics[width=0.2\textwidth]{image3} \\
\hline
\end{tabular}
decorated chord diagram = diagram where each chord carries a positive integer = "decoration"
**Decorated Chord Diagrams**

- Decorated chord diagram = diagram where each chord carries a positive integer = “decoration”

- 7 chords but size = 11

- Size of a decorated diagram = sum of the decorations
The solution of

\[ G(x, L) = 1 - \sum_{k \geq 1} x^k \frac{G(x, d_{-p})^{1 - \Delta k}}{k!} (e^{-Lp} - 1) \left. F_k(p) \right|_{p=0} \]

is the weighted generating function

\[ G(x, L) = 1 - \sum_{C \text{ wo-marked}} \left( \sum_{i=1}^{k_1} \frac{bd(t_i), t_i - i}{i!} \right) \prod_{c \text{ non-terminal}} \frac{b_{d(c), 0}}{b_{d(c), t_i - t_{i-1}}} x^{k_k} \]

such that \( k_1 < k_2 < \ldots < k_k \)

are the positions of the terminal chords

for the intersection order

where \( F_k(p) = \frac{b_{k, 0}}{p} + b_{k, 1} + b_{k, 2} p + b_{k, 3} p^3 + \ldots \)
The solution of

\[ G(x, L) = 1 - \sum_{k \geq 1} x^k \left( \sum_{\text{marked connected chord diagram}} \frac{(L)}{\prod_{i=1}^{k} b_d(t_i), t_i - 1} \right) \prod_{i=1}^{k-1} b_d(t_i), t_i - 1 \]

is the weighted generating function

\[ G(x, L) = 1 - \sum_{\text{marked connected chord diagram}} \left( \sum_{i=1}^{k_1} \right) \frac{(L)}{\prod_{i=1}^{k} b_d(t_i), t_i - 1} \prod_{i=1}^{k-1} b_d(t_i), t_i - 1 \]

such that \( k_1 < k_2 < \ldots < k_k \) are the positions of the terminal chords for the intersection order.

where \( F_k(p) = \frac{b_{k,0}}{p} + b_{k,1} + b_{k,2} p + b_{k,3} p^3 + \ldots \)
**TERMINAL CHORDS**

terminal chord = chord \((a, b)\) such that there is no intersecting chord \((c, d)\) with \(b < d\)

If terminal, then forbidden

![Diagram of terminal chords]
Terminal Chords

Terminal chord = chord \((a, b)\) such that there is no intersecting chord \((c, d)\) with \(b < d\).

If terminal, then forbidden.

3 terminal chords.
**Terminal Chords**

terminal chord = chord \((a, b)\) such that there is no intersecting chord \((c, d)\) with \(b < d\)

If terminal, then forbidden

3 terminal chords
the rightmost chord must be terminal
The solution of
\[ G(x, L) = 1 - \sum_{k \geq 1} \frac{x^k}{k!} G(x, d_{-\rho}) \left( e^{-L\rho} - 1 \right) F_k(\rho) \big|_{\rho=0} \]

is the weighted generating function
\[ G(x, L) = 1 - \sum_{\text{C w-o-marked}} \left( \sum_{i=1}^{k_1} b_d(t_i, t_i - i) \frac{(-L)^i}{i!} \right) \prod_{b \in \text{non-terminal}} b_d(i, 0) \prod_{i=1}^{k_{-1}} b_d(t_i, t_i - t_{i-1}) x \]

such that \( k_1 < k_2 < \ldots < k_{k-1} \) are the positions of the terminal chords

for the intersection order

where \( F_k(\rho) = \frac{b_{k,0}}{\rho} + b_{k,1} + b_{k,2} \rho + b_{k,3} \rho^3 + \ldots \)
INTERSECTING ORDER

RULE

root chord

smallest

then recursively

these chords,

then these chords

finally these chords

1 2 3 4 5 6 7 8 9 10 11 12 13 14
INTERSECTING ORDER

RULE

root chord smallest

then recursively these chords,
then these chords
finally these chords

1 2 3 4 5 6 7 8 9 10 11 12 13 14
INTERSECTING ORDER

RULE

root chord smallest

then recursively these chords,

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INTERSECTING ORDER

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root chord smallest

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finally these chords

1
2
3 4 5 6 7 8 9 10 11 12 13 14
INTERSECTING ORDER

RULE

root chord smallest

then recursively these chords,

then these chords

finally these chords

1 2 3

4 5 6 7 8 9 10 11 12 13 14
INTERSECTING ORDER

RULE

root chord
smallest

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1 2 3 4 5 6 7 8 9 10 11 12 13 14
INTERSECTING ORDER

RULE

root chord smallest

then recursively these chords,

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finally these chords

1 2 3 4 5 6 7 8 9 10 11 12 13 14
INTERSECTING ORDER

RULE

root chord smallest

then recursively these chords,

then these chords

finally these chords

1 3 4

2 7 5 6
INTERSECTING ORDER

RULE

root chord smallest

then recursively these chords,

then these chords

finally these chords

left-to-right order ≠ intersection order
From now on, chords will be identified with their positions for the intersecting order.
The solution of
\[ G(x, L) = 1 - \sum_{k \geq 1} x^k \left( e^{-Lp} - 1 \right) F_k(p) |_{p=0} \]
is the weighted generating function
\[ G(x, L) = 1 - \sum_{C \text{ wo-marked}} \left( \sum_{i=1}^{k_1} b_{d(t_i), t_i} \left( e^{-L} \right)^i \right) \prod_{i=1}^{k} b_{d(i), 0} \prod_{i=1}^{k} b_{d(t_i), t_i} x \]
where \( F_k(p) = \frac{b_{k,0}}{p} + b_{k,1} + b_{k,2} p + b_{k,3} p^3 + ... \)
The solution of

\[ G(x, L) = 1 - \sum_{k \geq 1} \frac{x^k}{k!} \left( G(x, d_\rho) \right)^{1 - \Delta k} \left( e^{-L \rho} - 1 \right) F_k(\rho) |_{\rho=0} \]

is the weighted generating function

\[ G(x, L) = 1 - \sum_{C \text{ \omega}_0\text{-marked} \atop \check{\text{decorated}} \check{\text{connected}} \check{\text{chord diagram}}} \left( \sum_{i=1}^{k^*} b_d(t_i), t_i - i \left( \frac{-L}{i!} \right)^i \right) \prod_{i=1}^{k^*} b_d(i) \prod_{i=1}^{k^*} b_d(t_i), t_i - t_i \cdot x \]

where \( F_k(\rho) = \frac{b_{k,0}}{\rho} + b_{k,1} + b_{k,2} \rho + \frac{b_{k,3}}{3!} \rho^3 + ... \)
ALGO

For each chord $i$ (in the intersection order)
Label each interval below $i$ by $i$
(erase the previous label if needed)
**Algorithm**

For each chord $i$ (in the intersection order)
Label each interval below $i$ by $i$
(erase the previous label if needed)
For each chord $i$ (in the intersection order), label each interval below $i$ by $i$ (erase the previous label if needed)
For each chord $i$ (in the intersection order)
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(erase the previous label if needed)
ALGO

For each chord $i$ (in the intersection order)
Label each interval below $i$ by $i$
(erase the previous label if needed)
**Definition**

For any integer $\Delta \geq 2$

$\omega_\Delta$-marked diagram = diagram where

- each interval may contain marks (= crosses), horizontally arranged
- for each chord $c$, the intervals covered by $c$ must contain $\Delta \times d(c) - 2$ marks in total.

**Example for $\Delta = 2$**
**DEFINITION**

For any integer $\Delta \geq 2$

$\omega_\Delta$-marked diagram = diagram where

- each interval may contain marks (=crosses), horizontally arranged
- for each chord $c$, the intervals covered by $c$ must contain $\Delta \times d(c) - 2$ marks in total.

E.g. for $\Delta = 2$

$(\Delta = 1$ is a bit more complex$)$
The solution of
\[ G(x,L) = 1 - \sum_{k \geq 1} x^k G(x, d_p) - \delta_k (e^{-Lp-1}) \frac{F_k(p)}{p=0} \]
is the weighted generating function
\[ G(x,L) = 1 - \sum_{C \omega\text{-marked decorated connected chord diagram}} \left( \sum_{i=1}^{k_i} b_{d(k_i),i} \frac{(-L)^i}{i!} \right)^{\text{terminal}} \prod_{i=1}^{k_{i,t}} b_{d(t_i),t_i} x \]
such that \( k_1 < k_2 < \ldots < k_k \)
are the positions of the terminal chords
for the intersection order

where \( F_k(p) = \frac{b_{k,0}}{p} + b_{k,1} + b_{k,2} p + b_{k,3} p^3 + \ldots \)

E.g:
The solution of
\[ G(x, L) = 1 - \sum_{k \geq 1} x^k G(x, d_\rho) \left( e^{-L \rho} - 1 \right) F_k(\rho) \bigg|_{\rho=0} \]
is the weighted generating function
\[ G(x, L) = 1 - \sum_{k \geq 1} \left( \sum_{i=1}^{\lambda_k} \prod_{j=0}^{1} b_d(t_i, t_{i-1}) \frac{(-L)^i}{i!} \right)^{\text{non terminal}} \prod_{i=1}^{\lambda_k-1} b_d(t_i, t_{i-1}) x \]
such that \( \lambda_1 < \lambda_2 < \ldots < \lambda_k \)
are the positions of the terminal chords
for the intersection order.

where \( F_k(\rho) = \frac{b_{k,0}}{\rho} + b_{k,1} + b_{k,2} \rho + b_{k,3} \rho^3 + \ldots \)

\[ \begin{align*}
\lambda_1 &= 4 & d(t_1) &= 2 \\
\lambda_2 &= 6 & d(t_2) &= 1 \\
\lambda_3 &= 7 & d(t_3) &= 2
\end{align*} \]
The solution of
\[ G(x, L) = 1 - \sum_{k \geq 1} x^k G(x, d_{-\rho})^{1-\Delta k} (e^{-L \rho} - 1) F_k(\rho) \big|_{\rho=0} \]
is the weighted generating function
\[ G(x, L) = 1 - \sum_{C \in \omega, \text{marked connected chord diagram}} \left( \sum_{i=1}^{\lambda_i} \frac{b_d(k_i, t_i-i) (-L)^i}{i!} \right)^{\text{non terminal}} \prod_{i=1}^{\lambda_i} b_d(t_i, 0) x^{t_i-t_i-1} \]
such that \( \lambda_1 < \lambda_2 < \ldots < \lambda_k \) are the positions of the terminal chords for the intersection order.

where \( F_k(\rho) = \frac{b_{k,0}}{\rho} + b_{k,1} + b_{k,2} \rho + b_{k,3} \rho^2 + \ldots \)

\[ (b_{2,0} \frac{L^4}{4!} - b_{2,1} \frac{L^3}{3!} + b_{2,2} \frac{L^2}{2} - b_{2,3} L) b_{4,0} b_{3,0} b_{3,2} b_{2,1} x^{11} \]
Roadmap of the exploration of "Solving Dyson-Schwinger Equations as Generating Functions"

Starting Point

- [Marie Yeats 2012]
  D-S eq. restricted to non-crossing Feynman graphs (Yukawa theory)
  Solution: GF of connected diagrams

- [Hihn Yeats 2016]
  D-S eq. from this talk
  Solution: 6F of decorated connected diagrams with J parameter

- [Courtiel Yeats Zeilberger 2019]
  Solution: 6F of decorated bridgeless maps counted by outgoing edges from the root with rightmost DFS

- [Nabergall Mahmoud 2021]
  D-S eq. stemming from some Hopf algebra
  Solution: GF of connected diagrams (avoiding ??)

System of D-S eq. ??

D-S eq. with non-symmetric insertion places ??

Terra Incognita
The solution of
\[ G(x, L) = 1 - \sum_{k \geq 1} x^k G(x, \delta_L)^{1 - \Delta_k} (e^{-L \rho} - 1) F_{k\rho}(\rho) \bigg|_{\rho = 0} \]
is the weighted generating function
\[ G(x, L) = 1 - \sum_{\text{C \omega\text{-marked}}}
\left( \sum_{i=1}^{k_1} \text{bd}(t_i), t_{i-1} \left( \frac{-L}{i!} \right)^i \right) \prod_{i=1}^{k_{\text{non terminal}}} \text{bd}(i), 0 \prod_{i=1}^{k_{\text{terminal}}} \text{bd}(t_i), t_{i-1}, t_{i-2}, \ldots, x \]
such that \( k_1 < k_2 < \ldots < k_k \)
are the positions of the terminal chords for the intersection order.

where \[ F_{k\rho}(\rho) = \frac{b_{k,0}}{\rho} + b_{k,1} + b_{k,2} \rho + b_{k,3} \rho^3 + \ldots \]

What to do with this (awful?) formula?
Part III

Leading-log expansions
LEADING-LOG EXPANSIONS

\[ G(x, L) = 1 - \sum_{C} \left( \sum_{i=1}^{k_i} \mathbf{b}_d(t_i, t_i - i) \frac{(-L)^i}{i!} \right) \prod_{i=1}^{l_{\text{non terminal}}} \mathbf{b}_d(t_i, 0) \prod_{i=1}^{l_{\text{terminal}}} \mathbf{b}_d(t_i, t_i - t_i - 1) x \]

\[ \text{decorated connected chord diagram} \]
\[ \text{such that } k_1 < k_2 < \ldots < k_R \]
\[ \text{are the positions of the terminal chords} \]

is of the form

\[ G(x, L) - 1 = g_{1,1} xL + g_{2,1} x^2L + g_{2,2} x^2L^2 + g_{3,1} x^3L + g_{3,2} x^3L^2 + g_{3,3} x^3L^3 + \ldots \]
\[ G(\infty, L) = 1 - \sum_{C \omega_0\text{-marked \ connected \ chord \ diagram \ such \ that \ } k_1 < k_2 < \ldots < k_k} \left( \sum_{i=1}^{k_1} \text{bd}(k_i, t; i, \frac{(-L)^i}{i!}) \right)^{C_{\text{non terminal}}} \prod_{i=1}^{k} \text{bd}(k_i, t; -k; i) x \]

is of the form

\[ G(x, L) - 1 = g_{1,1} x L + g_{2,1} x^2 L + g_{2,2} x^2 L^2 + g_{3,1} x^3 L + g_{3,2} x^3 L^2 + g_{3,3} x^3 L^3 + \ldots \]

We can write

\[ G(x, L) = 1 + \sum_{k \geq 1} H_k(x, L) \cdot x^k \]
LEADING-LOG EXPANSIONS

\[ G(\infty, L) = 1 - \sum_{\text{decorated connected chord diagram}} \left( \sum_{i=1}^{k_1} b_{d(t_i), t_i} \left( \frac{-L}{\lambda_i^2} \right) \right) \prod_{i=1}^{k_1} b_{d(t_i), 0} \prod_{i=1}^{a-1} b_{d(t_i), t_i - t_i - 1} x \]

such that \( k_1 < k_2 < \ldots < k_a \)
are the positions of the terminal chords

is of the form
\[ G(\infty, L) - 1 = g_{1,1} \frac{\alpha L}{x} + g_{2,1} \frac{x^2 L}{x} + g_{2,2} \frac{\alpha^2 L^2}{x^2} + g_{3,1} \frac{x^3 L}{x} + g_{3,2} \frac{x^3 L^2}{x^2} + g_{3,3} \frac{x^3 L^3}{x^3} + \ldots \]

We can write
\[ G(\infty, L) = 1 + \sum_{k \geq 1} H_k(\infty) x^k \]

**DEFINITION**

\[ H_0(\infty) = \text{leading-log expansion} \]
LEADING-LOG EXPANSIONS

\[ G(x, L) = 1 - \sum_{\text{decorated connected chord diagram}} \left( \sum_{i=1}^{k_i} \prod_{i=1}^{b_d(i),i-t-i} (-L)^i \right) \prod_{i=1}^{b_d(i),t-t-1} x \]

such that \( k_1 < k_2 < \ldots < k_k \)
are the positions of the terminal chords

is of the form

\[ G(x, L) - 1 = g_{1,1} xL + g_{2,1} x^2 L + g_{2,2} x^2 L^2 + g_{3,1} x^3 L + g_{3,2} x^3 L^2 + g_{3,3} x^3 L^3 + \ldots \]

We can write

\[ G(x, L) = 1 + \sum_{k \geq 1} H_k (x, L) \times x^k \]

**DEFINITION**

\[ H_0 (x) = \text{leading-log expansion} \]

\[ H_1 (x) = \text{next-to leading-log expansion} \]
LEADING-LOG EXPANSIONS

\[ G(\alpha, L) = 1 - \sum_{\mathcal{C} \omega_3 \text{-marked}} (\sum_{i=1}^{k_1} \frac{\mathcal{B}d(k_i, t_1, i)}{i!} (-L)^{i-1}) \prod_{i=1}^{k_1} \mathcal{B}d(t_i, 0) \prod_{i=1}^{k_1-1} \mathcal{B}d(t_i, t_i-t_{i+1}) \alpha \]

decorated connected chord diagram
such that \( k_1 < k_2 < \ldots < k_r \)
are the positions of the terminal chords

is of the form

\[ G(\alpha, L) - 1 = g_{1,1} \alpha L + g_{2,1} \alpha^2 L^2 + g_{3,1} \alpha^3 L^3 + \ldots + H_2 \alpha^2 L^2 + H_3 \alpha^3 L^3 + \ldots \\
\]

We can write

\[ G(\alpha, L) = 1 + \sum_{k>1} H_k(\alpha L) \alpha^k \]

**Definition**

\[ H_0(\alpha L) = \text{leading-log expansion} \]
\[ H_1(\alpha L) = \text{next-to leading-log expansion} \]
\[ H_{k+1}(\alpha L) = \text{next-to next-to \ldots next-to, leading-log expansion}. \]

[Krüger, Kreimer 2015]
LEADING LOG EXPANSIONS IN TERMS OF CHORD DIAGRAMS

\[
\text{Leading-log-expansion} (g) = - F_{\text{one-term-chord}} (- f_{1,0} g)
\]

where \( F_{\text{one-term-chord}} (g) \) = exponential generating function of \( \varnothing \)-marked connected diagrams with only one terminal chord (all decorations = 1)
LEADING LOG EXPANSIONS IN TERMS OF CHORD DIAGRAMS

\[
\text{Leading-log-expansion } (g) = - \text{Fone-term-chord } (- f_{1,0} g)
\]

where \( \text{Fone-term-chord } (g) = \) exponential generating function of \( \triangle \)-marked connected diagrams with only one terminal chord (all decorations = 1)

\[
\text{next-to-leading-log-expansion } (g) = \\
\frac{f_{1,1} - f_{1,1}}{\partial g} \frac{\partial \text{Fone-term-chord } (- f_{1,0} g)}{\partial g} - \frac{f_{2,0}}{f_{1,0}} \frac{\partial \text{Fone-term-of-dec-2 } (- f_{1,0} g)}{\partial g} \\
- \frac{f_{1,1}}{f_{1,0}} \frac{\partial \text{Ftwo-consec-term } (- f_{1,0} g)}{\partial g} - \frac{f_{2,0}}{f_{1,0}} \frac{\partial \text{Fone-dec-2-and-one-term } (- f_{1,0} g)}{\partial g}
\]

where \( \text{Fone-term-of-dec-2} \), \( \text{Ftwo-consec-term} \), \( \text{Fone-dec-2-and-one-term} \) are exponential generating functions of \( \triangle \)-marked decorated connected diagrams:

→ for \( \text{Fone-term-of-dec-2} \): with only 1 terminal chord, this chord has decoration 2

→ for \( \text{Ftwo-consec-term} \): with 2 terminal chords, last and before last for intersection order

→ for \( \text{Fone-dec-2-and-one-term} \): with only 1 terminal chord (decoration 1) and only one chord of decoration 2.

**the other chords have decoration 1**
LEADING LOG EXPANSIONS IN TERMS OF CHORD DIAGRAMS

\[ \text{Leading-log-expansion} (g) = -F_{\text{one-term chord}} (-f_{1,0} g) \]

where \( F_{\text{one-term chord}} (g) \) is the exponential generating function of \( \omega \) -marked connected diagrams with only one terminal chord (all decorations = 1).

\[ \text{next-to-leading-log-expansion} (g) = \]
\[ f_{1,1} - f_{1,1} \frac{\partial F_{\text{one-term chord}} (-f_{1,0} g)}{\partial g} - \frac{f_{2,0}}{f_{1,0}} \frac{\partial F_{\text{one-term of dec-2}} (-f_{1,0} g)}{\partial g} \]
\[ - \frac{f_{2,1}}{f_{1,0}} \frac{\partial F_{\text{two-consec term}} (-f_{1,0} g)}{\partial g} - \frac{f_{2,0}}{f_{2,0}} \frac{\partial F_{\text{one-dec-2 and one term}} (-f_{1,0} g)}{\partial g} \]

where \( F_{\text{one-term of dec-2}}, F_{\text{two-consec term}}, F_{\text{one-dec-2 and one term}} \) are exponential generating functions of \( \omega \) -marked decorated connected diagrams:

- for \( F_{\text{one-term of dec-2}} \): with only 1 terminal chord, this chord has decoration 2.
- for \( F_{\text{two-consec term}} \): with 2 terminal chords, last and before last for intersection order.
- for \( F_{\text{one-dec-2 and one term}} \): with only 1 terminal chord (decoration 1) and only one chord of decoration 2.

**\***: the other chords have decoration 1.

Similar formulas exist for the next-to leading log expansions.
WITH ONE TERMINAL CHORD ($\Delta = 2$)
WITH ONE TERMINAL CHORD ($\Delta = 2$)

In general,

Recursive description of $\omega_2$-marked connected chord diagrams with 1 terminal chord (there is no mark)

size $n+1$ = only one component

only one terminal chord

at least 1 terminal chord here

also here

and here
WITH ONE TERMINAL CHORD ($\Delta = 2$)

In general,

Recursive description of $\omega_2$-marked connected chord diagrams with 1 terminal chord (there is no mark)

size $n+1$ = size $m$ + only one component

insert a root chord in an inner interval
remove the root chord
WITH ONE TERMINAL CHORD ($\Delta = 2$)

insert a root chord in an inner interval

remove the root chord
WITH ONE TERMINAL CHORD ($\Delta = 2$)

Insert a root chord in an inner interval.

Remove the root chord.

If $a_n =$ number of $\omega_2$-marked connected diagrams with $n$ chords, only one of which is terminal,

$$a_{n+1} = (2n - 1) a_n$$
WITH ONE TERMINAL CHORD ($\Delta = 2$)

Insert a root chord in an inner interval.
Remove the root chord.

If $a_n =$ number of $\omega_2$-marked connected diagrams with $n$ chords, only one of which is terminal,

$$a_{n+1} = (2n-1) a_n$$

$$(a_n) = (2n-3)!!$$
WITH ONE TERMINAL CHORD ($\Delta = 3$)

What do we do when $\Delta = 3$?

The intervals covered by any chord must contain exactly 1 mark...
WITH ONE TERMINAL CHORD ($\Delta = 3$)

What do we do when $\Delta = 3$?
The intervals covered by any chord must contain exactly 1 mark...

The same decomposition applies!

Recursive description of $\omega_3$-marked connected chord diagrams with 1 terminal chord

insert a root chord in an inner interval

remove the root chord

+ the mark in the first interval
WITH ONE TERMINAL CHORD ($\Delta = 3$)

What do we do when $\Delta = 3$?
The intervals covered by any chord must contain exactly 1 mark...

The same decomposition applies!

Recursive description of $\omega_3$-marked connected chord diagrams with 1 terminal chord

How many ways?

insert a root chord in an inner interval
remove the root chord + the mark in the first interval
WITH ONE TERMINAL CHORD ($\Delta = 3$)

How many ways?

insert a root chord in an inner interval
remove the root chord
+ the mark in the first interval

size $n$

size $n+1$
WITH ONE TERMINAL CHORD \( (\Delta = 3) \)

- Insert a root chord in an inner interval
- Remove the root chord

How many ways?

Size \( n \)

Size \( n+1 \)

+ The mark in the first interval
WITH ONE TERMINAL CHORD ($\Delta = 3$)

- Insert a root chord in an inner interval
- Remove the root chord
- The mark in the first interval

How many ways?
WITH ONE TERMINAL CHORD ($\Delta = 3$)

insert a root chord in an inner interval

remove the root chord + the mark in the first interval

How many ways?
WITH ONE TERMINAL CHORD ($\Delta = 3$)

How many ways?

insert a root chord in an inner interval

remove the root chord + the mark in the first interval

If $b_n =$ number of $\omega_3$-marked connected diagrams with $n$ chords, only one of which is terminal,

$$b_{n+1} = (3n - 1) b_n$$
WITH ONE TERMINAL CHORD ($\Delta = 3$)

- Insert a root chord in an inner interval
- Remove the root chord and the mark in the first interval

If $b_n = \text{number of } \omega_3\text{-marked connected diagrams with } n \text{ chords, only one of which is terminal},$

$$b_{n+1} = (3n - 1) b_n$$

$$b_n = (3n - 4) !!! = (3n - 4) \times (3n - 7) \times \ldots \times 5 \times 2$$
In general, for any $\Delta \geq 1$, the number of ways to insert a root chord of decoration $j$ in a $\infty^{\Delta}$-marked decorated connected diagram of size $n$ is always

$$\Delta \times n - 1$$

(= number of gaps between a dot/mark and a dot/mark)
In general, for any $\Delta \geq 1$, the number of ways to insert a root chord of decoration $\mathfrak{g}$ in a $\omega_{\Delta}$-marked decorated connected diagram of size $n$ is always

$$\Delta \times n - 1$$

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In general, for any $\Delta \geq 1$, the number of ways to insert a root chord of decoration $\Delta$ in a $\omega_\Delta$-marked decorated connected diagram of size $n$ is always $\Delta \times n - 1$

(= number of gaps between a dot/mark and a dot/mark)
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\[
\text{Leading-log-expansion}\ (g) = -F_{\text{one-term-chord}}\ (-f_{1,0} g)
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where \( F_{\text{one-term-chord}}\ (g) \) = exponential generating function of \( \omega \)-marked connected diagrams with only one terminal chord (all decorations = 1)

\[
\text{next-to-leading-log-expansion}\ (g) =
\]
\[
= f_{1,1} - f_{1,1} \frac{\partial}{\partial g} F_{\text{one-term-chord}}\ (-f_{1,0} g) - \frac{f_{2,0}}{f_{1,0}} \frac{\partial}{\partial g} F_{\text{one-term-of-dec-2}}\ (-f_{1,0} g)
\]
\[
- \frac{f_{2,0}}{f_{1,0}} \frac{\partial}{\partial g} F_{\text{two-consec-term}}\ (-f_{1,0} g) - \frac{f_{2,0}}{f_{2,0}} \frac{\partial}{\partial g} F_{\text{one-dec-2-and-one-term}}\ (-f_{1,0} g)
\]

where \( F_{\text{one-term-of-dec-2}}, F_{\text{two-consec-term}}, F_{\text{one-dec-2-and-one-term}} \) are exponential generating functions of \( \omega \)-marked decorated connected diagrams:

\( \rightarrow \) for \( F_{\text{one-term-of-dec-2}} \) : with only 1 terminal chord, this chord has decoration 2

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**\(*\) : the other chords have decoration 1

**Similar formulas exist for the next-to leading log expansions**
$c_n = \text{number of } \omega_0\text{-marked connected diagrams with } n \text{ chords with 2 terminal chords: last and before last for the intersection order}$

$$c_{n+1} = (\delta n - 1) c_n + (\delta - 1) \times (\delta m - \delta - 1) \frac{!! \cdots !}{\delta}$$
\( \mathcal{C}_n = \) number of \( \omega_\Delta \)-marked connected diagrams with \( n \) chords with 2 terminal chords: last and before last for the intersection order

\[ \mathcal{C}_{n+1} = (\Delta n - 1) \mathcal{C}_n + (\Delta - 1) \times (\Delta n - \Delta - 1)!! \cdots ! \]

\( \mathcal{D}_n = \) number of \( \omega_\Delta \)-marked decorated connected diagrams with \( n \) chords with 1 terminal chord and where the only chord of decoration \( \neq 1 \) has decoration 2 and is not terminal.

\[ \mathcal{D}_{n+1} = (\Delta n - 1) \mathcal{D}_n + (\Delta n - \Delta - 1) \times (\Delta n - 2\Delta - 1)!! \cdots ! \]
Part IV

Full results and conclusion
Computing the Next-to-Leading Log Expansions

All these decompositions can be used to compute the next-to-leading log expansions.

Strategy: Decomposition \rightarrow Recurrence \rightarrow Differential equations for the exp. generating function \rightarrow Solve them

<table>
<thead>
<tr>
<th>Expansion</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_0(z)$</td>
<td>$\sqrt{1 + 2 f_{1,0} z} - 1$</td>
</tr>
<tr>
<td>$H_1(z)$</td>
<td>$\frac{- (f_{2,0} + f_{1,1} f_{1,0}) \ln (1 + 2 z f_{1,0})}{2 f_{1,0} \sqrt{1 + 2 z a_{1,0}}}$</td>
</tr>
<tr>
<td>$H_2(z)$</td>
<td>$\frac{(f_{2,0} + f_{1,1} f_{1,0})^2 (\ln (1 + 2 z f_{1,0}))^2}{8 (1 + 2 z f_{1,0})^{3/2} f_{1,0}^2}$ $- \frac{(f_{2,0} + f_{1,1} f_{1,0})^2 \ln (1 + 2 z f_{1,0})}{2 (1 + 2 z f_{1,0})^{3/2} f_{1,0}^2}$ $- z \frac{(-f_{2,0}^2 + 3 f_{1,0}^3 f_{1,2} + 3 f_{2,1} f_{1,0}^2 - f_{1,1} f_{2,0} f_{1,0} + f_{3,0} f_{1,0})}{(1 + 2 z f_{1,0})^{3/2} f_{1,0}}$</td>
</tr>
</tbody>
</table>

\(\Delta = 2\)

Every next-to-leading log expansion can be automatically computed. (We did it in Maple.)

[Krüger Kreimer 2015]
Computing the Next-to-\(L^k\) Log Expansions

All these decompositions can be used to compute the next-to-\(L^k\) leading log expansions.

\[\text{Strategy: Decomposition} \rightarrow \text{Recurrence} \rightarrow \text{Differential equations for the exp. generating function} \rightarrow \text{Solve them}\]

<table>
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<td>(H_0(z))</td>
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<td>(-\frac{\ln (1 + z f_{1,0}) f_{2,0}}{f_{1,0}})</td>
</tr>
<tr>
<td>(H_2(z))</td>
<td>(-\frac{f_{2,0}^2 \ln (1 + z f_{1,0})}{f_{1,0}^2 (1 + z f_{1,0})} - \frac{\left(-f_{2,0}^2 + f_{3,0} f_{1,0} + f_{2,1} f_{1,0}^2\right) z}{f_{1,0} (1 + z f_{1,0})})</td>
</tr>
</tbody>
</table>
| \(H_3(z)\) | \(-\frac{2 f_{1,0}^4 f_{2,2} + 2 f_{1,0}^3 f_{3,1} - f_{1,0}^2 f_{2,0} f_{2,1} + f_{1,0}^2 f_{4,0} - 2 f_{1,0} f_{2,0} f_{3,1} - 2 f_{2,0} f_{3,0} - f_{2,0}^2}{2 f_{1,0}^3}\) 
\(-\frac{f_{2,0} \left(f_{1,0}^2 f_{2,1} + f_{1,0} f_{3,0} - f_{2,0}^2\right)}{f_{1,0}^3 (z f_{1,0} + 1)} + \frac{f_{2,0}^3 (\ln (z f_{1,0} + 1))^2}{2 (z f_{1,0} + 1)^2 f_{1,0}^3}\) 
\(+ \frac{-2 f_{1,0} f_{2,0} f_{3,0} - 2 f_{1,0}^2 f_{2,0} f_{2,1} \ln (z f_{1,0} + 1)}{2 (z a_{1,0} + 1)^2 f_{1,0}^3}\) 
\(+ \frac{2 f_{1,0}^4 f_{2,2} + 2 f_{1,0}^3 f_{3,1} + f_{1,0}^2 f_{4,0} + f_{1,0} f_{2,0} f_{2,1} - f_{2,0}^3}{2 (z f_{1,0} + 1)^2 f_{1,0}^3}\) |

\(\Delta = 1\)

Recovering (and extending) (and correcting) [Krüger Kreimer 2015]

Every next-to-\(L^k\) leading log expansion can be automatically computed. (We did it in maple.)
**ASYMPTOTIC DOMINANCE**

\( k \) fixed and size \( n \to +\infty \)

The diagrams that have a positive asymptotic impact in the next-to-\( k \) leading log expansion have:

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<th>( \Delta = 1 )</th>
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<td>(- k_1 ) terminal chords, consecutive and last for the intersection order</td>
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<td>(- k_1 + k_2 = k )</td>
<td>In this case,</td>
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In this case,

number of such diagrams of size \( n \) \( \sim \) \( \frac{(\Delta - 1)^{k_1} \binom{k}{k_1} \ln(n)^{k} \ m^{-\frac{1}{k}-1} \ m^{-k-1} \ n!}{\Gamma(1-\Delta) k!} \)

number of such diagrams of size \( n \) \( \sim \) \( \frac{1}{(k-1)!} \ln(n)^{k-1} \ m^{-2} \times n! \)
THEOREM

$k$ fixed and size $n \to +\infty$

The diagrams that have a positive asymptotic impact in the
next-to-$k$ leading log expansion have:

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<td>In this case, number of such diagrams of size $n$ $\sim \frac{(\Delta - 1)^{k_1}}{\Gamma(1-\Delta) k_1 !} \left(\begin{array}{c} k \ k_1 \end{array}\right) \ln(n) \frac{k}{m} \frac{1}{\Delta - 1} \frac{n - k - 1}{n} \frac{n!}{m!}$</td>
<td>number of such diagrams of size $n$ $\sim \frac{1}{(k-1)!} \ln(n) \frac{k}{m} \frac{2}{n} \ln(n) \frac{k-1}{m} \frac{n!}{m!}$</td>
</tr>
</tbody>
</table>

Dichotomy between $\Delta = 1$ and $\Delta \geq 2$ ?!
ASYMPTOTIC ESTIMATE IN THE NEXT-TO-LEADING-LOG EXPANSION

**Theorem**

Suppose \( b_{1,0} \neq 0 \) and \( b_{2,0} + (\delta - 1) b_{1,1} b_{1,0} \neq 0 \)

Then the \( n \)th coefficient in the next-to-\( \delta \) leading-log expansion (\( \delta \) fixed) is equivalent to:

\[
\sim \frac{(-1)^n}{\Gamma(1-\frac{1}{\delta}) k!} \ b_{1,0}^{n-k} (b_{2,0} + (\delta - 1) b_{1,1} b_{1,0})^k \ln(n)^k \ m^{\delta-1} \ n^{\delta-1} \quad (\delta \geq 2)
\]

\[
\sim \frac{(-1)^n}{(k-1)!} b_{1,0}^{n-k} b_{2,0}^k \ln(n)^{k-1} m^{k-2} \quad (\delta = 1)
\]

Only \( b_{1,0}, b_{1,1} \) and \( b_{2,0} \) asymptotically matter!
OPEN QUESTIONS

→ Direct interpretation of the solution of $G(x, L)$ in terms of Feynman graphs?

→ Are there better combinatorial objects than $\omega_\triangledown$-marked decorated connected chord diagrams?

→ Studying $[x^n L^m] G(x, L)$ in other directions than $n = m + k$ ($k$ fixed)
Open Questions

→ Direct interpretation of the solution of $G(x, L)$ in terms of Feynman graphs?

→ Are there better combinatorial objects than $\omega^\triangle$-marked decorated connected chord diagrams?

→ Studying $[x^m L^n] G(x, L)$ in other directions than $n = m + k \; (k \, \text{fixed})$