SOLVING MORTIMER & PRELLBERG’S CONJECTURE:

* BIJECTION(S) BETWEEN MOTZKIN PATHS AND TRIANGULAR WALKS

Julien COURTIEL (GREYC, Univ. Caen Normandie, France)

in collaboration with Andrew ELVEY-PRICE (Tours, France)
and Irène MARCOVICI (Nancy, France)
Walking in Trianglattan

Part 1
TWO WAYS OF WALKING IN A TRIANGLE

TRIANGLATTAN

DIRECTIONS: \[ \rightarrow \]

NATTAIGNAIRT

DIRECTIONS: \[ \uparrow \rightarrow \]
Two ways of walking in a triangle

Directions: \( \rightarrow \rightarrow \) = forward paths

Directions: \( \leftarrow \leftarrow \) = backward paths
Theorem [Courtiel Elvey-Price Marcovici]

number of forward paths of length \( n \) starting from \( \gamma \)

= 

number of backward paths of length \( n \) starting from \( \gamma \)

(for any \( \gamma \), \( n \) and \( L \))

Isn't it obvious?

Caption: If two points are outlined in blue/red then the numbers of forward/backward paths with fixed length starting from these points are equal.
TRANSITIONAL OBJECTS

FORWARD PATHS

? 

BACKWARD PATHS
Definitions

**General path** = path in Trianglattian using as steps:

\[
\rightarrow, \uparrow, /, \leftarrow, \downarrow, \uparrow.
\]

**Direction vector** of a general path of length $n$

= word $\in \{F, B\}^n$ such that the $i^{th}$ letter $= \begin{cases} F & \text{if } i^{th} \text{ step } \in \rightarrow \downarrow \\ B & \text{if } i^{th} \text{ step } \in \leftarrow \uparrow \end{cases}$
THE BIJECTION

Rewriting Rules

Last-Step Flips
\[ b_1 \leftrightarrow \overline{b}_2 \]
\[ b_2 \leftrightarrow \overline{b}_4 \]
\[ b_3 \leftrightarrow \overline{b}_2 \]

Swapping Flips
\[ b_i \overline{b}_i \leftrightarrow \overline{b}_i b_i \]
\[ i \neq i \]
\[ b_1 \overline{b}_1 \leftrightarrow \overline{b}_3 b_3 \]
\[ b_2 \overline{b}_1 \leftrightarrow \overline{b}_1 b_1 \]
\[ b_3 \overline{b}_3 \leftrightarrow \overline{b}_2 b_2 \]
THE BIJECTION

REWIRITING RULES

LAST-STEP FLIPS
\[ \delta_1 \leftrightarrow \bar{\delta}_2 \]
\[ \delta_2 \leftrightarrow \bar{\delta}_4 \]
\[ \delta_3 \leftrightarrow \bar{\delta}_2 \]

SWAPPING FLIPS
\[ \delta_i \bar{\delta}_i \leftrightarrow \bar{\delta}_i \delta_i \]
\[ \delta_1 \bar{\delta}_1 \leftrightarrow \bar{\delta}_3 \delta_3 \]
\[ \delta_2 \bar{\delta}_2 \leftrightarrow \bar{\delta}_1 \delta_1 \]
\[ \delta_3 \bar{\delta}_3 \leftrightarrow \bar{\delta}_2 \delta_2 \]
Why is there uniqueness?

Set of Nine Tiles
WHY IS THERE UNIQUENESS?

SET OF NINE TILES
Answering Mortimer & Prellberg's Question

Part 2
A NEW FAMILY

**FORWARD PATHS**

- Forward path starting from origin (= bottom-left corner)
- $L = 2H + 1$

**MOTZKIN PATHS**

- Motzkin path = path using:
  - Increasing steps $\uparrow$
  - Horizontal steps $\to$
  - Decreasing steps $\downarrow$
- Starting at height $= 0$
- Staying at height $\geq 0$
- Ending at height $= 0$
Theorem [Courtiel Elvey-Price Marcovici]

number of forward paths of length $n$ in a triangle of size $2H+1$

= number of Motzkin paths of length $n$ with height $\leq H$

(for any $n$ and $H$) And there is a bijection.
**Theorem** [Courtiel Elvey-Price Marcovici]

number of forward paths of length $n$ in a triangle of size $2H+1$

= number of Motzkin paths of length $n$ with height $\leq H$

(for any $n$ and $H$) And there is a bijection.

---

**Actual Mortimer and Prellberg’s conjecture**

Is there a bijection explaining:

number of general paths of length $n$ in a triangle of size $2H+1$

= number of 2-coloured Motzkin paths of length $n$ with height $\leq H$?
MORTIMER & PRELLBERG'S QUESTION

**Theorem** [Courtiel Elvey-Price Marcovici]

Number of forward paths of length $n$ in a triangle of size $2H+1$

= number of Motzkin paths of length $n$ with height $\leq H$

(for any $n$ and $H$) And there is a bijection.

---

Actual Mortimer and Prellberg's conjecture

Is there a bijection explaining:

Number of general paths of length $n$ in a triangle of size $2H+1$

= number of 2-coloured Motzkin paths of length $n$ with height $\leq H$?

---

It's a corollary of our two theorems!

General paths $\xleftrightarrow{bij} F$ Forward paths + direction vector

$FBBF$
Theorem [Courtiel Elvey-Price Maruvici]

number of forward/general paths of length $n$ in a triangle of size $2H+1$

= number of 1-coloured/2-coloured Motzkin paths of length $n$ with height $\leq H$

(for any $n$ and $H$) And there is a bijection.
MORTIMER & PRELLBERG'S QUESTION

Theorem [Courtiel Elvey-Price Marcovici]

\[(*)\] \{number of forward/general paths of length \(n\) in a triangle of size \(2H+1\) = number of 1-coloured/2-coloured Motzkin paths of length \(n\) with height \(\leq H\) (for any \(n\) and \(H\))\} And there is a bijection.

WHAT WAS KNOWN?

→ \((*)\) was proved by [Mortimer Prellberg 2014] thanks to the Kernel method with \(2\) catalytic variables.

→ the existence of a bijection was an open question.

→ When \(H = +\infty\), many bijections exist:
  - Between forward paths & Motzkin paths
    [Gouyou-Beauchamps 89, Eu 10, Chyzak Yeats 20, …]
  - Between general paths & 2-coloured Motzkin paths
    [Yeats 14]
Even Case?

**Theorem** [Courtiel Elvey-Price Marcovici]

The number of forward paths of length \( n \) in a triangle of size \( 2H \)
equals the number of Motzkin paths of length \( n \) with height \( \leq H \)
with no horizontal step at height \( = H \)
(for any \( n \) and \( H \)) also explained by a bijection.
AN ELEMENTARY PROOF

Lemma

The number of forward paths of length \( n \) starting at \( k \leq H \) is

\[
\sum_{i=0}^{k} m_m(i)
\]

where \( m_m(i) \) is the number of Motzkin paths of length \( m \) starting at height \( i \) and with height \( \leq H \).
Lemma

Number of forward paths of length \( m \) starting at \( \ell \leq H \)

\[
= \sum_{i=0}^{k} m_m(i)
\]

where \( m_m(i) \) = number of Motzkin paths of length \( m \)
starting at height = \( i \)
and with height \( \leq H \)

FORWARD PATHS

MOTZKIN PATHS

- Starting at height = 0
- Starting at height = 1
Lemma

number of forward paths of length $n$ starting at

\[ f_m(k) \]

\[ \text{origin} \]

\[ \Rightarrow \]

\[ \text{height} = k \]

\[ (k \leq H) \]

\[ \sum_{i=0}^{k} m_m(i) \]

where \( m_m(i) \) = number of Motzkin paths of length $m$ starting at height = $i$ and with height $\leq H$.

(Half of a) proof:
AN ELEMENTARY PROOF

Lemma:

number of forward paths of length \( n \) starting at \( f_m(k) \)

\[
= \sum_{i=0}^{k} m_m(i)
\]

where \( m_m(i) \) = number of Motzkin paths of length \( m \) starting at height = \( i \) and with height \( \leq H \)

(Half of a) proof:

We want to show \( f_m(k) - f_m(k-1) = m_m(k) \)

(In this talk, we assume \( 0 < k < H \))
AN ELEMENTARY PROOF

Lemma

number of forward paths of length $n$ starting at $f_m(k)$

\[ f_m(k) = \sum_{i=0}^{k} m_m(i) \]

where $m_m(i)$ = number of Motzkin paths of length $m$ starting at height $i$ and with height $\leq H$

(Half of a) proof:

We want to show $f_m(k) - f_m(k-1) = m_m(k)$

(In this talk, we assume $0 << k << H$)

We have $m_m(k) = m_{m-1}(k-1) + m_{m-1}(k) + m_{m-1}(k+1)$

\[ \uparrow k \downarrow \]
**AN ELEMENTARY PROOF**

Lemma

The number of forward paths of length \( m \) starting at \( f_{m}(k) \) can be expressed as:

\[
\sum_{i=0}^{k} m_{m}(i) \quad \text{where} \quad m_{m}(i) = \text{number of Motzkin paths of length } m \text{ starting at height } i \text{ and with height } \leq H
\]

\((k \leq H)\)

(Half of a) Proof:

We want to show \( f_{m}(k) - f_{m}(k-1) = m_{m}(k) \)

(In this talk, we assume \( 0 < k < H \))

We have \( m_{m}(k) = m_{m-1}(k-1) + m_{m-1}(k) + m_{m-1}(k+1) \)

Let us prove that \( f_{m}(k) - f_{m}(k-1) \) satisfies the same recurrence.
AN ELEMENTARY PROOF

Lemma

number of forward paths of length $m$ starting at

\[ f_m(k) = \sum_{i=0}^{k} m_m(i) \]

where

$\forall m \leq H$

$m_m(i) = \text{number of Motzkin paths of length } m \text{ starting at height } i$

and with height $\leq H$

(Half of a) proof:

\[ f_m(k) = f_{m-1}(k+1) + \diamond \]
AN ELEMENTARY PROOF

Lemma

number of forward paths of length \( n \) starting at \( f_m(k) \)

\[
f_m(k) = \sum_{i=0}^{k} m_m(i)
\]

where \( m_m(i) = \) number of Motzkin paths of length \( m \)

starting at height \( = i \)

and with height \( \leq H \)

(Half of a) proof:

\[
f_n(k-1) = b_n(k-1) = b_{n-1}(k-2) + ∆
\]

\[
f_n(k) = b_{n-1}(k+1) + ∆
\]
AN ELEMENTARY PROOF

Lemma

number of forward paths of length $n$ starting at

$$b_n(k) = \sum_{i=0}^{k} m_n(i)$$

where $m_n(i)$ = number of Motzkin paths of length $n$

starting at height $i$

and with height $\leq H$

($(k \leq H)$)

(Half of a) proof:

$$b_n(k) - b_n(k-1) = b_{n-1}(k+1) - b_{n-1}(k-2)$$
AN ELEMENTARY PROOF

Lemma

The number of forward paths of length \( n \) starting at \( (k \leq H) \) is given by:

\[
\begin{align*}
\text{fn}(k) & = \sum_{i=0}^{k} m_{m}(i) \\
\end{align*}
\]

where

\[ m_{m}(i) = \text{number of Motzkin paths of length } m \text{ starting at height } = i \text{ and with height } \leq H \]

(Half of a) proof:

\[
\begin{align*}
\text{fn}(k) & = \text{fn}(k-1) \\
& = \text{fn}(k-1) \text{ (recursive step)}
\end{align*}
\]

\[
\begin{align*}
\text{fn}(k) & = \text{fn}(k-1) + \text{fn}(k-2) \\
& = \text{fn}(k-1) + \text{fn}(k-2) \text{ (sum of previous steps)}
\end{align*}
\]

\[
\begin{align*}
\text{fn}(k) - \text{fn}(k-1) & = \text{fn}(k-1) - \text{fn}(k-2) \\
& = (\text{fn}(k-1) - \text{fn}(k-2)) + (\text{fn}(k-2) - \text{fn}(k-3)) + \ldots + (\text{fn}(1) - \text{fn}(0))
\end{align*}
\]
THE “HEADACHE” BIJECTION

By the previous lemma, there should be a bijection between

\[ \left\{ \text{forward paths starting at } \begin{array}{c} x \\ + \\ 0 \end{array} \right\} \]

and

\[ \left\{ \text{Motzkin paths starting at height } k \right\} \cup \left\{ \text{forward paths starting at } \begin{array}{c} \downarrow \\ \downarrow \end{array} \right\} \]
The "Headache" Bijection

By the previous lemma, there should be a bijection between \( \{ \text{forward paths starting at } \triangle \} \)

and

\( \{ \text{Motzkin paths starting at height } k \} \cup \{ \text{forward paths starting at } \triangle \} \).

Input \( \omega \)

A triangular path \( \omega \) of length \( n \) starting at \( \triangle + ks_1 \).

\( k = 0 \)

Check if \( k \) is null

\( k > 0 \)

Return the only Motzkin path of length 0

\( n = 0 \)

\( n > 0 \)

Decompose \( \omega = \omega_1 \omega' \) where \( \omega_1 \) is the first step

\( \omega_1 = s_1 \)

\( \omega_1 = s_2 \)

In \( \omega' \), replace each \( s_1, s_2, s_3 \) step

with \( s_1, s_3, s_2 \) respectively.

Then apply \( W_{n-1} \).

Call \( \rho \) the obtained path.

\( k < H \)

Compute \( \omega^{[k+1]} := \Omega_{n-1,k+1}(\omega') \)

\( \omega^{[k+1]} \) is a triangular path

\( \omega^{[k+1]} \) is a Motzkin meander

Return \( \omega^{[k+1]} \)

\( k = H \)

Compute \( \omega^{[k]} := \Omega_{n-1,k}(\omega^{[k+1]}) \)

\( \omega^{[k]} \) is a triangular path

\( \omega^{[k]} \) is a Motzkin meander

Return \( \omega^{[k]} \)

\( k > H \)

Compute \( \omega^{[k-1]} := \Omega_{n-1,k-1}(\omega^{[k]}) \)

\( \omega^{[k-1]} \) is a triangular path

\( \omega^{[k-1]} \) is a Motzkin meander

Return \( \omega^{[k-1]} \)

Output \( \Omega_{n,k}(\omega) \)

A Motzkin meander of length \( n \) starting at height \( k \)

or

A triangular path of length \( n \) starting at \( \triangle + (k-1)s_1 \).
Many Other Bijections

Part 3
coordinates of a point \( z \) = \((i, j, k)\) such that

\[
\text{Definition}
\]

profile of a point with coordinates \((i, j, k)\) = vector \((p_0, p_1, \ldots, p_H)\)

\[
\frac{(1-x^{i+1})(1-x^{j+1})(1-x^{k+1})}{(1-x)^2}
\]

\[
= p_0 + p_1 x + \ldots + p_H x^H
+ p_{H+1} x^{H+1} + \ldots + p_{L+1} x^{L+1}
\]

**Ex:** For \((1,1,3)\),

\[
\frac{(1-x^2)^2(1-x^4)}{(1-x)^2} = 1 + 2 x + 6 x^2 - 2 x^4 - x^5 - x^6
\]
THE GENERALISATION OF THE EARLIER LEMMA

Lemma

Number of forward paths of length \( m \) starting at

\[
\sum_{i=0}^{H} \mu_i \times m_m(i)
\]

where\( m_m(i) \) = number of Motzkin paths of length \( m \) starting at height = \( i \) and with height \( \leq H \)

Diagram: Triangle with paths and labels.
The generalisation of the earlier lemma

Lemma

Number of forward paths of length $m$ starting at $\gamma$ with profile $p_0, p_1, \ldots, p_H$

\[
\sum_{i=0}^{H} p_i \times m_m(i)
\]

where $m_m(i)$ is the number of Motzkin paths of length $m$ starting at height $i$ and with height $\leq H$.

So there should be a bijection between

\[
\left\{ (\text{cell of } \gamma, \text{Motzkin path}) \right\}
\]

compatible height

and

\[
\left\{ \text{forward paths starting from } \gamma \right\}
\]
**Definition**

scaffolding = a riot of arrows like this

**Theorem**

For each scaffolding, there is a bijection between forward paths and Motzkin paths
SCAFFOLDING AND BIJECTION

Definition
scaffolding = a riot of arrows like this

Theorem
For each scaffolding, there is a bijection between forward paths and Motzkin paths
GENERALISATION

PART 4
Actually,

Trianglattan of size \( L = \{i \mathbf{e}_1 + j \mathbf{e}_2 + k \mathbf{e}_3 : i + j + k = L\} \)

and step set:

\[
\begin{align*}
\overrightarrow{e_1-e_3}, & \quad \overrightarrow{e_2-e_1}, \quad \overrightarrow{e_3-e_2} \\
\rightarrow & \quad \leftarrow \quad \downarrow
\end{align*}
\]
TRIANGLATTAN IN HIGHER DIMENSION?

Actually,

\[ \text{Trianglattan of size } L = \{ i \mathbf{e}_1 + j \mathbf{e}_2 + k \mathbf{e}_3 : i + j + k = L \} \]

and step set:

\[ \mathbf{e}_1 - \mathbf{e}_3, \mathbf{e}_2 - \mathbf{e}_1, \mathbf{e}_3 - \mathbf{e}_2 \]

↓ shift in higher dimension

Hyperattan of dimension \( d \) =

\[ \{ i_1 \mathbf{e}_1 + i_2 \mathbf{e}_2 + \ldots + i_d \mathbf{e}_d : \sum_{k=1}^{d} i_k = L \} \]

Step set: \( \mathbf{e}_1 - d \mathbf{e}_d, \mathbf{e}_2 - \mathbf{e}_1, \ldots, \mathbf{e}_d - (d-1) \mathbf{e}_d \)

Extension of the previous theorems?

- Symmetry between forward paths & backward paths?
- Bijection with other family of paths?
TRIANGLATTAN IN HIGHER DIMENSION?

Actually,

Trianglattan of size \( L \) =
\[
\{ i e_1 + j e_2 + k e_3 : i + j + k = L \}
\]

and step set:
\[
e_1 - e_3 , e_2 - e_1 , e_3 - e_2
\]

\[\downarrow \text{shift in higher dimension}\]

Hyperattan of dimension \( d \) =
\[
\{ i_1 e_1 + i_2 e_2 + \ldots + i_d e_d : \sum_{k=1}^{d} i_k = L \}
\]

Step set: \( e_1 - e_d , e_2 - e_1 , \ldots , e_d - e_{d-1} \)

Extension of the previous theorems?

- Symmetry between forward paths & backward paths? Yes, EZ
- Bijection with other family of paths?
TRIANGLATTAN IN HIGHER DIMENSION?

Actually,

Trianglattan of size $L = \{ i e_1 + j e_2 + k e_3 : i+j+k=L \}$

and step set:

$$e_1 - e_3, e_2 - e_1, e_3 - e_2$$

$\downarrow$ shift in higher dimension

Hyperattan of dimension $d = \{ i_1 e_1 + i_2 e_2 + \ldots + i_d e_d : \sum_{k=1}^{d} i_k = L \}$

Step set: $e_1 - e_d, e_2 - e_{d-1}, \ldots, e_d - e_1$

Extension of the previous theorems?

• Symmetry between forward paths & backward paths? Yes, EZ
• Bijection with other family of paths? Only dimension 4...
A NEW BIJECTION

\[ \{ \text{"pyramidal" walks} \} \quad \longleftrightarrow \quad \text{bijection} \quad \rightarrow \quad \{ \text{"waffle" walks} \} \]
A new bijection

\[
\{ \text{“pyramidal” walks} \} \quad \longleftrightarrow \quad \text{bijection} \quad \longrightarrow \quad \{ \text{“waffle” walks} \}
\]

The counting of these walks was an open question from [Mortimer Prellberg].
A NEW BIJECTION

\[ P(t) = \frac{1}{(L + 4)^2} \sum_{1 \leq j < k \leq L+3} \frac{(\alpha^k + \alpha^{-k} - \alpha^j - \alpha^{-j})^2(2 + \alpha^j + \alpha^{-j})(2 + \alpha^{-k} + \alpha^k)}{1 - (\alpha^j + \alpha^{-j} + \alpha^k + \alpha^{-k})t} \]

where \( \alpha = e^{\frac{i\pi}{L+4}} \)

The counting of these walks was an open question from [Mortimer Prellberg]

\{ "pyramidal" walks \} \quad \text{bijection} \quad \{ "waffle" walks \}