UNDERSTANDING LATTICE WALKS VIA CENTRAL WEIGHTINGS

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Séminaire Combinatoire
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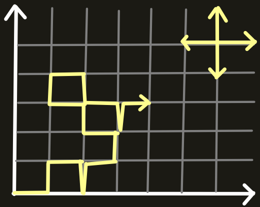
MOTIVATION

Various asymptotic behaviours

$$\sim \frac{4}{\pi} 4^n n^{-1}$$



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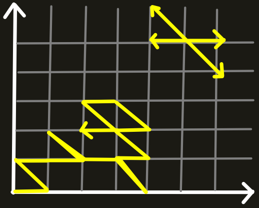
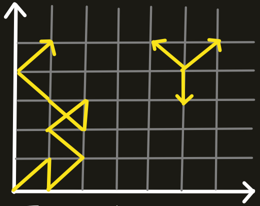
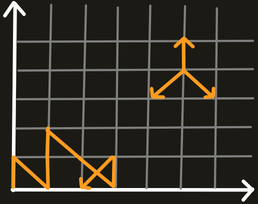




$$\sim \frac{\sqrt{3}}{2\sqrt{\pi}} 3^n n^{-\frac{1}{2}}$$



$$\sim \frac{4A_n}{\pi} (2\sqrt{2})^n n^{-2}$$

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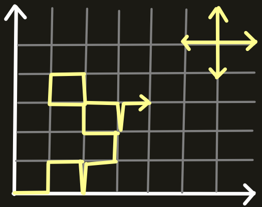

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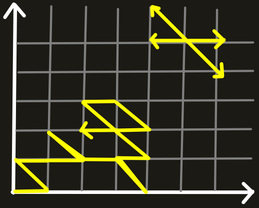

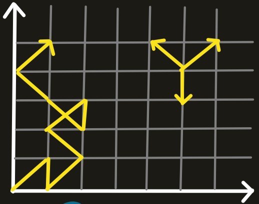

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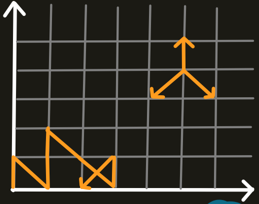

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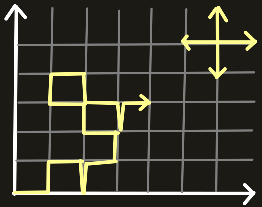

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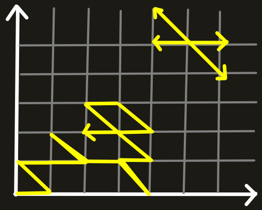

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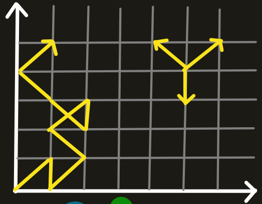
Exponential growth

Critical exponent



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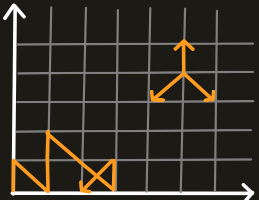

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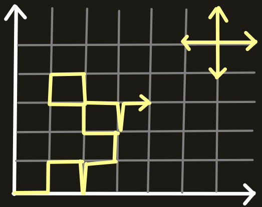

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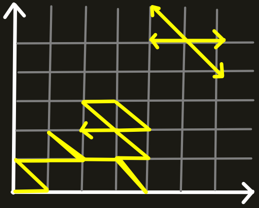

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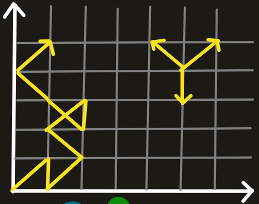

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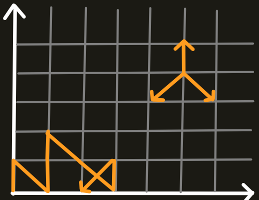

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Motivation: opt for a continuous model to detect the transitions

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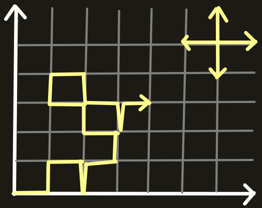

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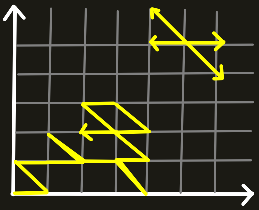

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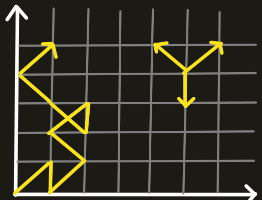

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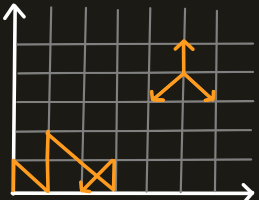
Motivation: opt for a continuous model to detect the transitions

Central weightings



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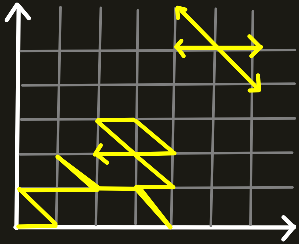
PART 1

Weighted Gouyou-Beauchamps model

GOUYOU-BEAUCHAMPS MODEL



In general: walks starting at $(0,0)$
staying in the positive quadrant
with steps in $\mathcal{S}$
where $\mathcal{S} \subseteq \{\leftarrow, \rightarrow, \uparrow, \downarrow, \nwarrow, \searrow, \nearrow, \swarrow\}$

Here: $\mathcal{S} = \{\leftarrow, \rightarrow, \nwarrow, \searrow\}$
$\rightarrow$ Gouyou-Beauchamps model

Number of walks ending anywhere after...
1 step : 1   2 steps : 3   3 steps : 6
4 steps : 20   5 steps : 50   6 steps : 175

GOUYOU-BEAUCHAMPS MODEL

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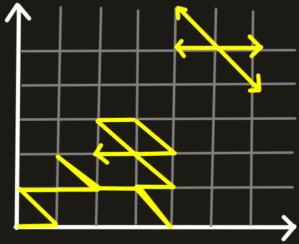
→ Gouyou-Beauchamps model



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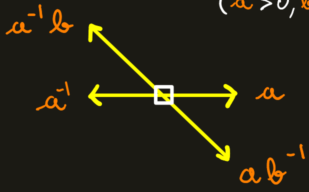
A small connection with probabilities :

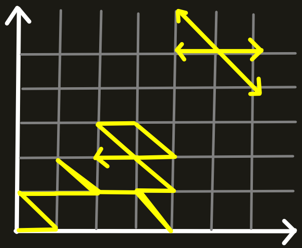
$$\mathbb{P}\left(\begin{array}{c}\text{staying in the}\\ \text{quadrant after}\\ 6 \text{ steps}\end{array}\right) = \frac{175}{4^6}$$

WEIGHTED GOUYOU-BEAUCHAMPS MODEL

A weight to each step:

($a>0$, $b>0$)





Weight of walks ending anywhere after...

1 step : a

2 steps : $1 + b + a^2$

3 steps : $2ab + a^3 + 3a$

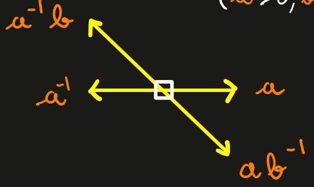
A small connection with probabilities :

$$\mathbb{P}\left(\begin{array}{c}\text{staying in the}\\ \text{quadrant after}\\ 3 \text{ steps}\end{array}\right) = \frac{2ab+a^3+3a}{(a+a^{-1}+ab^{-1}+ba^{-1})^3}$$

WEIGHTED GOUYOU-BEAUCHAMPS MODEL

A weight to each step:

$(a>0, b>0)$



Weight of walks ending anywhere after..

1 step : a $= [z^1]\, Q_{\uparrow}(1,1;z)$

2 steps: $1 + b + a^2$ $= [z^2]\, Q_{\uparrow}(1,1;z)$

3 steps: $2ab + a^3 + 3a = [z^3]\, Q_{\uparrow}(1,1;z)$

$Q_{\uparrow}(x,y;z) :=$ generating function of weighted GB-walks

$$= \sum_{\substack{w \text{ GB-walk} \\ (0,0)\to(i,j) \\ \text{of length } n}} \mathrm{weight}(w)\, x^i y^j z^n$$

$$\mathrm{weight}(w) := \prod_{\substack{s \text{ step in } w \\ \text{(with multiplicity)}}} \mathrm{weight}(s)$$

$Q(x,y;z) =$ GF of unweighted GB-walks

$$= \sum_{\substack{w \text{ GB-walk} \\ (0,0)\to(i,j) \\ \text{of length } n}} x^i y^j z^n$$

Rk: $Q_{\uparrow}(1,1;z) = Q(a,b;z)$

CENTRAL WEIGHTING

Theorem A weight p_s is assigned to each $s \in \{\leftarrow, \rightarrow, \nwarrow, \searrow\}$. There is equivalence between:

(i) The probability of a given walk only depends on its length and its end point.

(ii) There exist constants $a, b, c > 0$ such that

$$p_{\leftarrow} = c\,a^{-1} \qquad p_{\rightarrow} = c\,a \qquad p_{\nwarrow} = c\,a^{-1}b \qquad p_{\searrow} = c\,a\,b^{-1}$$

(iii) $p_{\leftarrow} \times p_{\rightarrow} = p_{\nwarrow} \times p_{\searrow}$

(iv) If $Q_p(x,y;z) :=$ the weighted generating function and $Q(x,y;z) :=$ the unweighted one, then $Q_p(x,y;z) = Q(ax, by; cz)$.

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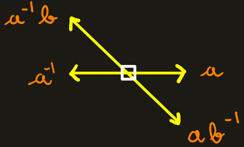
(iii) $\omega_{\leftarrow} \times \omega_{\rightarrow} = \omega_{\nwarrow} \times \omega_{\searrow}$

(iv) If $Q_\omega(x,y;z) :=$ the weighted generating function and $Q(x,y;z) :=$ the unweighted one, then $Q_\omega(x,y;z) = Q(ax, by; cz)$.

Too restrictive?

THE BIG THEOREM

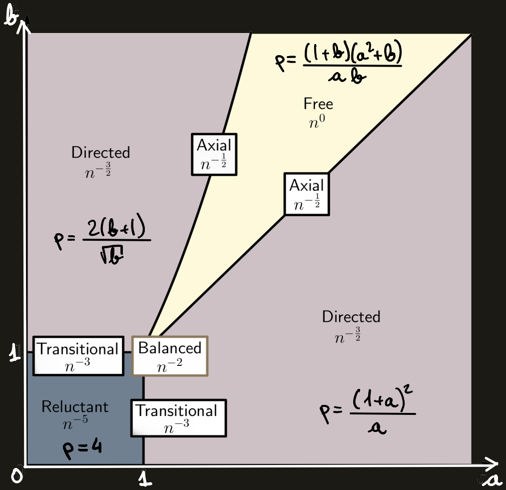


Theorem :

Weight of GB-walks of length n

$$= [z^n]\, Q_{\uparrow}(1,1;t)$$

$$\sim K^{[n]}\ \rho^{n}\ n^{-\alpha}$$



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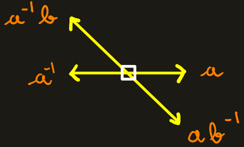
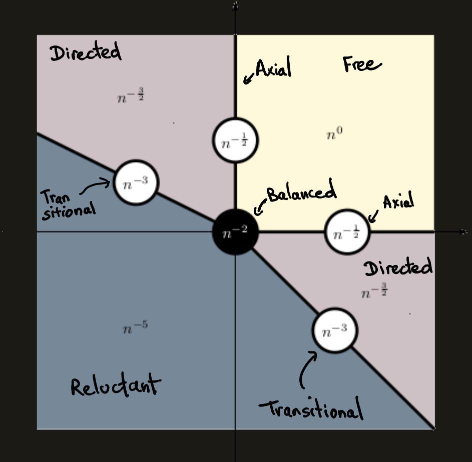


In terms of the drift $= (a - a^{-1} + ab^{-1} - a^{-1}b,\ a^{-1}b - ab^{-1})$



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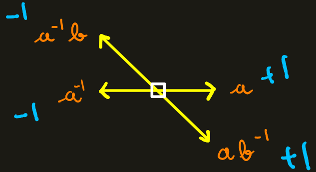

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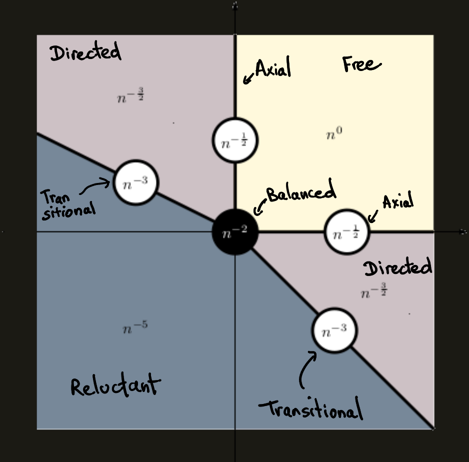

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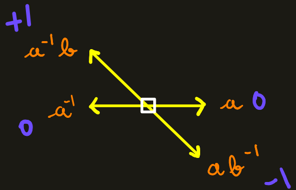

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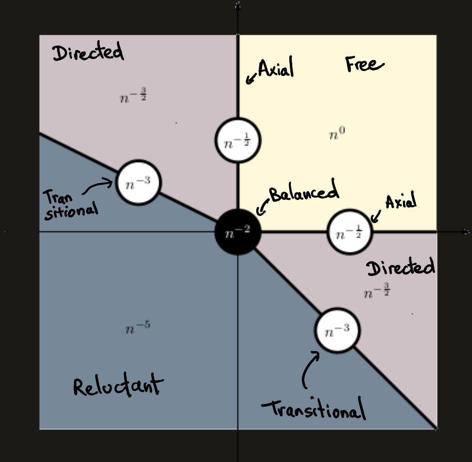


THE BIG THEOREM

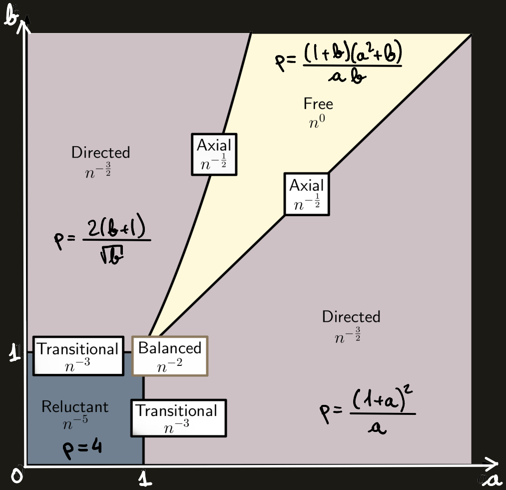
Also works when $(0,0)$ is not the starting point.

Theorem :

Weight of GB-walks of length n starting at (i,j) ending anywhere

$$\sim K \, V^{[n]}(i,j) \, \rho^{n} \, n^{-\alpha}$$

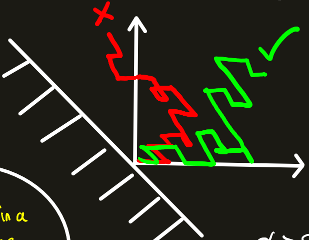
fonction harmonique



APPLICATION : RANDOM GENERATION

① [Lumbroso, Mishna, Ponty]

Generation in $O(n^{\alpha-\frac{1}{2}} \ln n)$ time

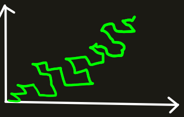


generation in a half-plane

⊕ rejection if not the quadrant

$\alpha > 3$

② Anticipated rejection algorithm from [Bacher, Sportiello]



Linear complexity for free and axial cases

UNDERSTANDING THE PROOF

Main ingredient :

Analytic
Combinatorics
in Several
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Steps:

1. Express the generating function in terms of a diagonal
2. Find the contributing points ← exponential growth
3. Transform the generating function into an integral and apply some theorem ([Hörmander]) ← whole asymptotic estimate

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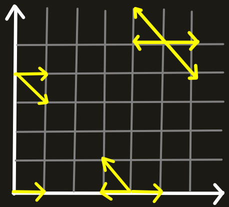
FUNCTIONAL EQUATION

3 important functions:

$S(x,y) = x + x^{-1} + xy^{-1} + x^{-1}y$ INVENTORY

$K(x,y;z) = xy(1 - zS(x,y))$ KERNEL

$Q(x,y;z)$ UNWEIGHTED GENERATING FUNCTION



KERNEL EQUATION:

[Bousquet-Mélou, Mishna]

$$\underbrace{K(x,y;z)Q(x,y;z) = xy}_{\text{Walks without constraints}} - \underbrace{K(x,0;z)Q(x,0;z) - K(0,y;z)Q(0,y;z) + K(0,0;z)Q(0,0;z)}_{\text{Boundary restrictions (Inclusion - exclusion)}}$$

INTO A DIAGONAL EXPRESSION

KERNEL EQUATION:

$$K(x,y;z)\,Q(x,y;z) = xy - \underbrace{K(x,0;z)\,Q(x,0;z) - K(0,y;z)\,Q(0,y;z) + K(0,0;z)\,Q(0,0;z)}_{\text{We want to eliminate this}}$$

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Algebraic operations
(Orbit sum
⊕ Extraction of
positive coefficients)

DIAGONAL EXPRESSION:

$$Q(1,1;z) = \Delta\left(\frac{(1-x)(1-y)(1+x)(x^{-2}-y^{-1})(x-y)(x+y)}{\left(1-xyz\,S(x^{-1},y^{-1})\right)(1-x)\,(1-y)}\right)$$

where $\displaystyle \Delta\left(\sum_{i,j,n\geq 0} f_{i,j,n}\, x^i y^j z^n\right) = \sum_{n\geq 0} f_{n,n,n}\, z^n$

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orbit sum applied to the start point

where $\Delta\left(\sum_{i,j,n\geq 0} f_{i,j,n}\, x^i y^j z^n\right) = \sum_{n\geq 0} f_{n,n,n}\, z^n$

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$$\Big\downarrow$$

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codes the excursions

where $$\Delta\left(\sum_{i,j,n\geq 0} f_{i,j,n}\, x^i y^j z^n\right) = \sum_{n\geq 0} f_{n,n,n}\, z^n$$

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DIAGONAL EXPRESSION:

$$Q(1,1;z) = \Delta\left(\frac{\overbrace{(1-x)(1-y)(1+x)(x^{-2}-y^{-1})(x-y)(x+y)}^{\text{orbit sum applied to the start point}}}{\underbrace{(1-xyz\,S(x^{-1},y^{-1}))}_{\text{codes the excursions}}\underbrace{(1-x)}_{\text{releases the } x\text{-constraint}}\underbrace{(1-y)}_{\text{releases the } y\text{-constraint}}}\right)$$

where $\Delta\left(\sum_{i,j,n\geq 0} f_{i,j,n}\, x^i y^j z^n\right) = \sum_{n\geq 0} f_{n,n,n}\, z^n$

WEIGHTED VERSION

$$Q_{\uparrow}(1,1;z) = \frac{1}{a^4 b^3 z^2} \Delta\left(yz^2 \frac{(y-b)(a-x)(a+x)(a^2y-bx^2)(ay-bx)(ay+bx)}{\left(1-xyzS(x^{-1},y^{-1})\right)(1-x)(1-y)} \right)$$

where

$$S(x,y) = ax + a^{-1}x^{-1} + ab^{-1}xy^{-1} + a^{-1}bx^{-1}y$$ WEIGHTED INVENTORY

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where

$$S(x,y) = ax + a^{-1}x^{-1} + ab^{-1}xy^{-1} + a^{-1}b\,x^{-1}y$$ WEIGHTED INVENTORY

UNDERSTANDING THE PROOF

Main ingredient : Analytic Combinatorics in Several Variables

Steps:

1. Express the generating function in terms of a diagonal ✓
2. Find the contributing points ← exponential growth
3. Transform the generating function into an integral and apply some theorem ([Hörmander]) ← whole asymptotic estimate

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Main ingredient : Analytic Combinatorics in Several Variables

Steps:

blackboard

① Express the generating function in terms of a diagonal ✓

② Find the contributing points ← exponential growth

③ Transform the generating function into an integral and apply some theorem ([Hörmander]) ← whole asymptotic estimate

AN EXPRESSION FOR THE RADIUS

Recall: $$Q_p(1,1;z) = \Delta F(x,y,z) = \Delta \frac{G(x,y,z)}{H(x,y,z)}$$

Quick reasoning:

Let $\mathcal{D}$ = disk of convergence of F, and $(x,y,z) \in \mathcal{D}$.

F is absolutely convergent on (x,y,z)

so $Q_p(1,1;|xyz|) < +\infty$

Hence radius of $Q_p \geq \sup\limits_{x \in \overline{\mathcal{D}}} |xyz|$

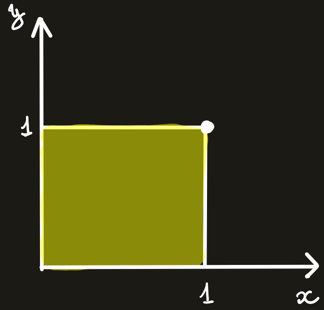
Actually $$\boxed{\text{radius of } Q_p = \sup_{\substack{x \in \overline{\mathcal{D}} \\ H(x,y,z)=0}} |xyz|}$$

AN EXPRESSION FOR THE RADIUS

I should have convinced you that:

$$\left(\text{radius of } Q_p(1,1;z)\right)^{-1} = \min_{\substack{|x|\leq 1\\ |y|\leq 1}} \left|S(x^{-1},y^{-1})\right|$$

$S(x,y)$ = WEIGHTED INVENTORY



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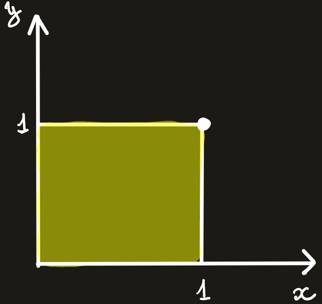


If minimum $\in$ ,

$a < 1$ and $b < 1$

reluctant case

If minimum $\in$  or ,

$a \geq 1$ and $a > b$

or $b \geq 1$ and $b > a^2$.

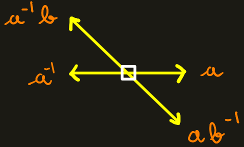
directed case

If minimum $\in$ 

$1 < \sqrt{b} < a < b$

Free case

THE BIG THEOREM

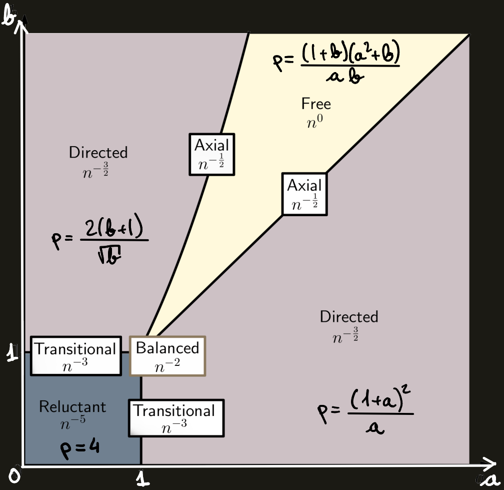


Theorem :

Weight of GB-walks of length n

$$= [z^n]\, Q_{\uparrow}(1,1;t)$$

$$\sim K^{[n]}\, \rho^{n}\, n^{-\alpha}$$



PART 2

General Central Weightings

CENTRAL WEIGHTINGS IN ALL GENERALITY

<u>Theorem</u> Let $\mathcal{S}$ be a non-singular set of integer steps in dimension 2. A weight p_s is assigned to each $s \in \mathcal{S}$. There is equivalence between

(i) The probability of a given walk only depends on its length, its start and end point.

(ii) There exist constants $a, b, c > 0$ such that for every $(x_s, y_s) \in \mathcal{S}$, $p_{(x_s,y_s)} = c \times a^{x_s} \times b^{y_s}$

(iii) Take $|\mathcal{S}| - 2 - 1$ "independant" pairs of paths (w_k, w'_k) such that w_k and w'_k share the same length, the same start and end point.

Then for every k, $\displaystyle\prod_{\substack{s \in w_k \\ \text{with multiplicity}}} p_s = \prod_{\substack{s' \in w'_k \\ \text{with multiplicity}}} p_{s'}$.

(iv) The Kernels are essentially the same.

GENERATING FUNCTIONS

What about (v) There exist constants $a, b, c > 0$ such that

$$\underbrace{Q_p}_{\text{weighted GF}}(x, y; z) = \underbrace{Q}_{\text{unweighted GF}}(ax, by; cz) \quad ?$$

GENERATING FUNCTIONS

What about (v) There exist constants $a, b, c > 0$ such that

$$Q_p(x, y; z) = Q(ax, by; cz) \quad ?$$

weighted GF

unweighted GF

<u>Prop</u>: If a weighting is central, then (v) holds.

<u>Conjecture</u>: If (v) holds, then the weighting is central.

(true for all small 2D-models)

SOME CONSEQUENCES

(v) There exist constants $a, b, c > 0$ such that

$$Q_p(x,y;z) = Q(ax, by; cz)$$

weighted GF

unweighted GF

Consequence 1: $Q_p(x,y;z)$ is D-finite.

$$\Updownarrow$$

$Q(x,y;z)$ is D-finite.

(if the weights are central ⊕ rational)

SOME CONSEQUENCES

(v) There exist constants $a, b, c > 0$ such that

$$Q_{p}(x, y; z) = Q(ax, by; cz)$$

weighted GF unweighted GF

<u>Consequence 1</u>: $Q_{p}(x, y; z)$ is D-finite.

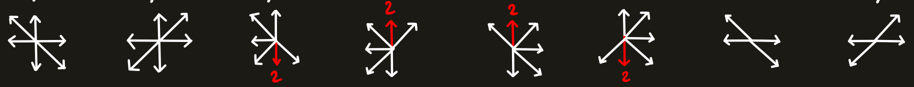
$$\Updownarrow$$

$Q(x, y; z)$ is D-finite.

(if the weights are central ⊕ rational)

Consistant with [Kauers, Yatchak]: systematic search of the D-finite weighted 2D-models

Family 1a Family 1b Family 2a Family 2b Family 3a Family 3b Family 4a Family 4b



SOME CONSEQUENCES

(v) There exist constants $a, b, c > 0$ such that

$$Q_p(x, y; z) = Q(ax, by; cz)$$

weighted GF unweighted GF

Consequence 2: Given a central weighting $p(x_s, y_s) = c \cdot a^{x_s} b^{y_s}$,

Weigth of excursions of size n $= c^n \times$ number of excursions of size n.

SOME CONSEQUENCES

(v) There exist constants $a, b, c > 0$ such that

$$\underset{\text{weighted GF}}{Q_p(x,y;z)} = \underset{\text{unweighted GF}}{Q(ax, by; cz)}$$

<u>Consequence 2</u>: Given a central weighting $p(x_0,y_s) = c \cdot a^{x_s} b^{y_s}$,

Weigth of excursions of size n = c^n x number of excursions of size n.

In general,

$$\boxed{\text{evaluations of } Q(x,y;z)} \Longleftrightarrow \boxed{\text{central weightings}}$$

THE CONJECTURE OF GARBIT, MUSTAPHA & RASCHEL

$S(x,y)$ = WEIGHTED INVENTORY

$$(x^*, y^*) = \operatorname*{argmin}_{\substack{x \geq 1 \\ y \geq 1}} S(x,y)$$

Conjecture (Every estimate is up to a constant.)

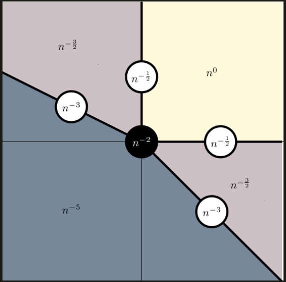
	$\nabla S(x^*,y^*) = 0$	or $\frac{\partial S}{\partial x}(x^*,y^*) > 0$ or $\frac{\partial S}{\partial y}(x^*,y^*) > 0$	and $\frac{\partial S}{\partial x}(x^*,y^*) > 0$ and $\frac{\partial S}{\partial y}(x^*,y^*) > 0$
$(x^*,y^*) = (1,1)$	$\sim S(1,1)^n \, n^{-p/2}$ balanced	$\sim S(1,1)^n \, n^{-1/2}$ axial	$\sim S(1,1)^n \, n^{0}$ free
$x^* = 1$ or $y^* = 1$	$\sim S(x^*,y^*)^n \, n^{-p/2+1}$ transitional	$\sim S(x^*,y^*)^n \, n^{-3/2}$ directed	impossible
$x^* > 1$ and $y^* > 1$	$\sim S(x^*,y^*)^n \, n^{-p-1}$ reluctant	impossible	impossible

$$p = \pi / \arccos(-c)$$

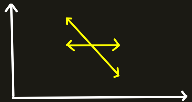
$$c = \frac{\partial^2 S}{\partial x \partial y}(x^*,y^*) \Big/ \sqrt{\frac{\partial^2 S}{\partial x^2}(x^*,y^*) + \frac{\partial^2 S}{\partial y^2}(x^*,y^*)}$$

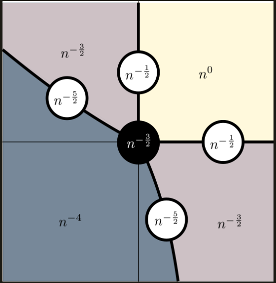
do not depend on the central weights

SOME DRIFT DIAGRAMS

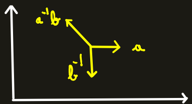


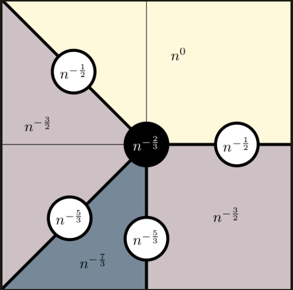
Gouyou-Beauchamps model



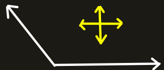


Tandem model





Gessel model in its natural cone.



CONCLUSION

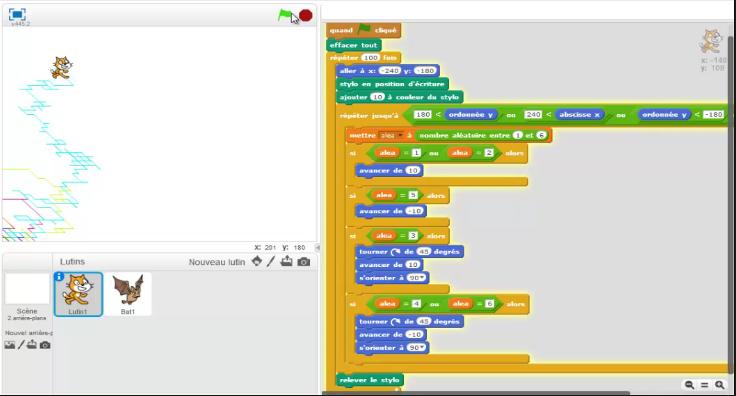
→ Good framework to understand transitions.

→ Many left conjectures:

- A weighting is central iff $Q_p(x, y; z) = Q(ax, by; cz)$?
- Which step sets have "pretty" (like conic) regions in their drift diagrams?
- ...

→ Hope to understand more non-D-finite models.

THANK YOU


x: 201 y: 180
Lutins
Nouveau lutin
Scène
2 arrière-plans
Nouvel arrière-p
Lutin1
Bat1
quand cliqué
effacer tout
répéter 100 fois
aller à x: -240 y: -180
stylo en position d'écriture
ajouter 10 à couleur du stylo
répéter jusqu'à 180 < ordonnée y ou 240 < abscisse x ou ordonnée y < -180
mettre alea à nombre aléatoire entre 1 et 6
si alea = 1 ou alea = 2 alors
avancer de 10
si alea = 5 alors
avancer de -10
si alea = 3 alors
tourner de 45 degrés
avancer de 10
s'orienter à 90
si alea = 4 ou alea = 6 alors
tourner de 45 degrés
avancer de -10
s'orienter à 90
relever le stylo
x: -149
y: 109