UNDERSTANDING LATTICE WALKS VIA CENTRAL WEIGHTINGS

Julien COURTIEL (LIPN, Paris 13)

Co-authors

Stephen MELCZER (Univ. Waterloo/ENS Lyon)
Marni MISHNA (Simon Fraser University)
Kilian RASCHEL (Université de Tours)

Séminaire Combinatoire
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Various asymptotic behaviours
MOTIVATION

Various asymptotic behaviours
Exponential growth
MOTIVATION

Various asymptotic behaviours
Exponential growth
Critical exponent
MOTIVATION

Various asymptotic behaviours
Exponential growth
Critical exponent

Motivation: opt for a continuous model to detect the transitions
MOTIVATION

Various asymptotic behaviours
Exponential growth
Critical exponent

Motivation: opt for a continuous model to detect the transitions

Central weightings
PART 1

Weighted Gouyou-Beauchamps model
In general: walks starting at (0,0) staying in the positive quadrant with steps in $\mathcal{S}$.

Where $\mathcal{S} \subseteq \{<, >, \uparrow, \downarrow, \uparrow\downarrow, \downarrow\uparrow\}$

Here: $\mathcal{S} = \{<, >, \uparrow\downarrow, \downarrow\uparrow\}$

$\rightarrow$ Gouyou-Beauchamps model

Number of walks ending anywhere after...

1 step: 1  2 steps: 3  3 steps: 6
4 steps: 20  5 steps: 50  6 steps: 175
GOUYOU-BEAUCHAMPS MODEL

In general: walks starting at (0,0) staying in the positive quadrant with steps in $\mathcal{S}$

where $\mathcal{S} \subseteq \{\leftarrow, \rightarrow, \uparrow, \downarrow, \swarrow, \searrow\}$

Here: $\mathcal{S} = \{\leftarrow, \rightarrow, \swarrow, \searrow\}$

$\rightarrow$ Gouyou-Beauchamps model

Number of walks ending anywhere after...

1 step: 1  2 steps: 3  3 steps: 6  4 steps: 20  5 steps: 50  6 steps: 175

A small connection with probabilities:

$P(\text{staying in the quadrant after 6 steps}) = \frac{175}{4^6}$
A weight to each step: \(a > 0, b > 0\)

\[ a^{-1}, b \]

\[ a^{-1}, a \]

\[ a \]

\[ a^{-1}, a^{-1} \]

\[ ab^{-1} \]

Weight of walks ending anywhere after...

1 step: \(a\)

2 steps: \(1 + b + a^2\)

3 steps: \(2ab + a^3 + 3a\)

A small connection with probabilities:

\[ P(\text{staying in the quadrant after 3 steps}) = \frac{2ab + a^3 + 3a}{(a + a^{-1} + ab + ba^{-1})^3} \]
WEIGHTED GOUYOU-BEAUCHAMPS MODEL

A weight to each step:
\[(a > 0, b > 0)\]

\[a^{-1} b \quad a \quad a^{-1} b^{-1} \]

Weight of walks ending anywhere after:

1 step: \[a\]

2 steps: \[1 + b + a^2\]

3 steps: \[2ab + a^3 + 3a\]

\[Q_p(x, y; z) := \text{generating function of weighted GB-walks}\]

\[= \sum_{\omega \text{ GB-walk } (0,0) \rightarrow (i,j) \text{ of length } n} \text{weight}(\omega) x^i y^j z^n\]

\[\text{weight}(\omega) := \prod_{s \text{ step in } \omega} \text{weight}(s)\]

\[Q(x, y; z) = \text{GF of unweighted GB-walks}\]

\[= C_{\frac{a}{2}} \sum_{\omega \text{ GB-walk } (0,0) \rightarrow (i,j) \text{ of length } n} x^i y^j z^n\]

\[\text{Rk: } Q_p(1,1; z) = Q(a, b; z)\]
Theorem A weight \( p_d \) is assigned to each \( d \in \{ \leftarrow, \rightarrow, \uparrow, \downarrow \} \).

There is equivalence between:

(i) The probability of a given walk only depends on its length and its endpoint.

(ii) There exist constants \( a, b, c > 0 \) such that
\[
\begin{align*}
p_{\leftarrow} &= c a^{-1} \\
p_{\rightarrow} &= c a \\
p_{\uparrow} &= c a^{-1} b \\
p_{\downarrow} &= c a b^{-1}
\end{align*}
\]

(iii) \( p_{\leftarrow} \times p_{\rightarrow} = p_{\uparrow} \times p_{\downarrow} \)

(iv) If \( Q^p(x, y, z) := \text{the weighted generating function} \)
and \( Q(x, y, z) := \text{the unweighted one,} \)
then \( Q^p(x, y, z) = Q(ax, by, cz) \).
CENTRAL WEIGHTING

**Theorem**  A weight \( p_a \) is assigned to each \( a \in \{<, \rightarrow, \uparrow, \downarrow\} \). There is equivalence between:

(i) The probability of a given walk only depends on its length and its endpoint.

(ii) There exist constants \( a, b, c > 0 \) such that
\[
\begin{align*}
p_\leftarrow &= c \ a^t \\
p_\rightarrow &= c \ a \\
p_\uparrow &= c \ a^t b \\
p_\downarrow &= c ab^{-1}
\end{align*}
\]

(iii) \( p_\leftarrow \times p_\rightarrow = p_\uparrow \times p_\downarrow \)

(iv) If \( Q_p(x,y,z) \) := the weighted generating function and \( Q(x,y,z) := the unweighted one, \) then \( Q_p(x,y,z) = Q(ax,by,cz) \).

Too restrictive?
Theorem:

Weight of GB-walks of length \( n \)

\[ = [g^n] Q_{\rho}(1,1;\lambda) \]

\[ \sim K^n \rho^n n^{-\alpha} \]
Theorem:
Weight of GB-walks of length \( n \) = \( g_n \mathcal{Q}_n(1,1;k) \)
\( \sim K^n \rho^n - \alpha \)
THE BIG THEOREM

In terms of the drift = \((a - a^t + ab^{-1}a^{-1}b^{-1} - ab^{-1}a, ab^{-1}a^{-1}b^{-1} - ab^{-1})\)

Theorem:
Weight of GB-walks of length \(n\) = \([g^n]\mathcal{Q}_n(1,1;k)

\sim K[n] \rho^n \sim n^{-\alpha}

Diagram:
- Directed
- Axial
- Free
- Balanced
- Reluctant
- Transitional
THE BIG THEOREM

In terms of the drift \( (\mathbf{a} - \mathbf{a}^t + \mathbf{a} \mathbf{b} - \mathbf{a}^t \mathbf{b}, \mathbf{a} \mathbf{b} - \mathbf{a} \mathbf{b}^t) \)

**Theorem:**

Weight of GB-walks of length \( n \) = \([y^n] Q_n(1,1;\mathbf{a})\)

\(~ K \mathbf{n}^n \mathbf{p} \mathbf{n}^{-\alpha} ~\)
THE BIG THEOREM

Also works when (0,0) is not the starting point.

Theorem:

Weight of GB-walks of length n starting at (i,j) ending anywhere

$$\sim K \sqrt{(i,j)} \cdot \rho^n \cdot n^{-\alpha}$$

fonction harmonique
APPLICATION: RANDOM GENERATION

1. \([\text{Lumbroso, Mishna, Ponty}]\)
   Generation in \(O(n^{\alpha - \frac{1}{2}} \log n)\) time
   
   \(\alpha > 3\)

2. Anticipated rejection algorithm from \([\text{Bacher, Sportiello}]\)
   Linear complexity for free and axial cases
UNDERSTANDING THE PROOF

Main ingredient:

Analytic Combinatorics in several variables
UNDERSTANDING THE PROOF

Main ingredient: Analytic Combinatorics in several variables

Steps:
1. Express the generating function in terms of a diagonal
2. Find the contributing points
3. Transform the generating function into an integral and apply some theorem ([Hörmander])

 exponential growth
 whole asymptotic estimate
UNDERSTANDING THE PROOF

Main ingredient: Analytic Combinatorics in several variables

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1. Express the generating function in terms of a diagonal
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FUNCTIONAL EQUATION

3 important functions:

\[ S(x, y) = x + x^{-1} + xy^{-1} + x^{-1}y \]  INVENTORY

\[ K(x, y; z) = xy(1 - zS(x, y)) \]  KERNEL

\[ Q(x, y; z) \]  UNWEIGHTED GENERATING FUNCTION

KERNEL EQUATION:

\[ K(x, y; z) Q(x, y; z) = xy - K(x, 0; z) Q(x, 0; z) - K(0, y; z) Q(0, y; z) + K(0, 0; z) Q(0, 0; z) \]

Walks without constraints

Boundary restrictions (Inclusion - exclusion)
Into a diagonal expression

Kernel equation:

\[ K(x, y ; \mathbf{z}) Q(x, y ; \mathbf{z}) = xy - K(x, 0 ; \mathbf{z}) Q(x, 0 ; \mathbf{z}) - K(0, y ; \mathbf{z}) Q(0, y ; \mathbf{z}) + K(0, 0 ; \mathbf{z}) Q(0, 0 ; \mathbf{z}) \]

We want to eliminate this
INTO A DIAGONAL EXPRESSION

KERNEL EQUATION:

$$K(x,y,z)Q(x,y,z) = xy - K(x,0,z)Q(x,0,z) - K(0,y,z)Q(0,y,z) + K(0,0,z)Q(0,0,z)$$

We want to eliminate this

Algebraic operations
- Orbit sum
- Extraction of positive coefficients

DIAGONAL EXPRESSION:

$$Q(1,1,z) = \Delta \left( \frac{(1-x)(1-y)(1+x)(x^2-y^2)(x-y)(x+y)}{(1-xyyzS(x^{-1},y^{-1}))(1-x)(1-y)} \right)$$

where

$$\Delta \left( \sum_{i,\delta,\gamma,n \geq 0} f_i,\delta,\gamma,x^i y^\delta \gamma^n \right) = \sum_{n \geq 0} f_{n,n,n} z^n$$
INTO A DIAGONAL EXPRESSION

KERNEL EQUATION:

\[ K(x,y;i^2) Q(x,y;i^2) = xy - K(x,0;i^2) Q(x,0;i^2) - K(0,y;i^2) Q(0,y;i^2) + K(0,0;i^2) Q(0,0;i^2) \]

We want to eliminate this

Algebraic operations
- Orbit sum
- Extraction of positive coefficients

DIAGONAL EXPRESSION:

\[ Q(1,1;i^2) = \sum \left( \frac{(1-x)(1-y)(1+x)(x^2-y^2)(x-y)(x+y)}{(1-xy^2 S(x^1,y^1))(1-x)(1-y)} \right) \]

where

\[ \Delta \left( \sum_{i,j,n \geq 0} b_{i,j,n} x^i y^j z^n \right) = \sum_{n \geq 0} b_{n,n,n,n} z^n \]
INTO A DIAGONAL EXPRESSION

KERNEL EQUATION:

$$K(x, y; z)Q(x, y; z) = xy - K(x, 0; z)Q(x, 0; z) - K(0, y; z)Q(0, y; z) + K(0, 0; z)Q(0, 0; z)$$

We want to eliminate this

Algebraic operations

- Orbit sum
- Extraction of positive coefficients

DIAGONAL EXPRESSION:

$$Q(1, 1; z) = \Delta \left( \frac{(1-x)(1-y)(1+x)(x^2 - y^2)(x-y)(x+y)}{(1-xyz)^2(x^{-1}, y^{-1})} \right)$$

```
(1-x) (1-y)
```

Orbit sum applied to the start point

codes the excursions

where

$$\Delta \left( \sum_{i, \delta, n \geq 0} b_i \delta, n x^i y^\delta z^n \right) = \sum_{n > 0} b_n n, n, n z^n$$
INTO A DIAGONAL EXPRESSION

**Kernel Equation:**

\[
K(x, y; \tilde{z}) Q(x, y; \tilde{z}) = xy - \underbrace{K(x, 0; \tilde{z}) Q(x, 0; \tilde{z}) - K(0, y; \tilde{z}) Q(0, y; \tilde{z}) + K(0, 0; \tilde{z}) Q(0, 0; \tilde{z})}_{\text{We want to eliminate this}}
\]

**Diagonal Expression:**

\[
Q(1, 1; \tilde{z}) = \Delta \left( \frac{(1-x)(1-y)(1+x)(x^2-y^2)(x-y)(x+y)}{(1-xy\tilde{z}S(x^{-1}, y^{-1}))(1-x)(1-y)} \right)
\]

where

\[
\Delta \left( \sum_{i, j, n \geq 0} b_i j_n x^i y^j z^n \right) = \sum_{n > 0} b_n x^n y^n z^n
\]

Algebraic operations

- Orbit sum
- Extraction of positive coefficients

orbit sum applied to the start point

releases the \( y \)-constraint

codes the excursions

releases the \( x \)-constraint
WEIGHTED VERSION

\[ Q^w_{i,j}(l, x, y) = \frac{1}{a^4 b^3 z^2} \bigtriangleup \left( y^2 \frac{2(y-b)(a-x)(a+x)(a' y - b x')(a y - b x')(a y + b x')} {\left( 1 - x y z S(x^{-1}, y^{-1}) \right)(1-x)(1-y)} \right) \]

where

\[ S(x, y) = ax + a' x^{-1} + ab' x y' + a'b' x' y \]
\[ Q_{\uparrow}(1,1; z) = \frac{1}{a^4 b^3 z} \triangle \left( \frac{2(y-b)(a-x)(a+x)(a'y-bx')(ay-bx)(ay+bx)}{(1-xyz S(x', y'))(1-x)(1-y)} \right) \]

where

\[ S(x, y) = ax + a'x^{-1} + ab'xy' + a'b'x'y' \]

\[ F(x, y, z) = \frac{G(x, y, z)}{H(x, y, z)} \]

WEIGHTED VERSION

WEIGHTED INVENTORY
UNDERSTANDING THE PROOF

Main ingredient:

 Analytic Combinatorics in Several Variables

Steps:
1. Express the generating function in terms of a diagonal
2. Find the contributing points
3. Transform the generating function into an integral and apply some theorem (Hörmander)
UNDERSTANDING THE PROOF

Main ingredient: Analytic Combinatorics in several variables

Steps:
① Express the generating function in terms of a diagonal
② Find the contributing points
③ Transform the generating function into an integral and apply some theorem (Hörmander)
AN EXPRESSION FOR THE RADIUS

Recall: \( Q_p(1,1;z_b) = \Delta F(x,y,z_b) = \Delta \frac{G(x,y,z_b)}{H(x,y,z_b)} \)

Quick reasoning:
Let \( D = \) disk of convergence of \( F \), and \((x,y,z_b) \in D\)

\( F \) is absolutely convergent on \((x,y,z_b)\)
so \( Q_p(1,1; |xyz_b|) < +\infty \)

Hence radius of \( Q_p \geq \sup_{x \in \overline{D}} |xyz_b| \)

Actually radius of \( Q_p = \sup_{x \in \overline{D}} |xyz_b| \)
\( \text{subject to } H(x,y,z_b) = 0 \)
AN EXPRESSION FOR THE RADIUS

I should have convinced you that:

\[
\left( \text{radius of } Q_p(1,1; z) \right)^{-1} = \min_{\|x\| \leq 1, \|z\| \leq 1} |S(x^{-1}; y^{-1})|
\]

\[S(x, y) = \text{WEIGHTED INVENTORY}\]
AN EXPRESSION FOR THE RADIUS

I should have convinced you that:

\[
\left( \text{radius of } Q_p((1,1); y) \right)^{-1} = \min \left| S(x, y) \right| \quad \text{subject to} \quad |x| \leq 1, |y| \leq 1
\]

\[
S(x, y) = \text{WEIGHTED INVENTORY}
\]

If minimum \( \in \) \[
\begin{cases}
\begin{array}{l}
\text{If minimum } \in \text{ \[ \quad \text{reluctant case} \quad \text{and } \quad a < 1 \text{ and } b < 1 \end{array}
\end{cases}
\]

\[
\begin{cases}
\begin{array}{l}
\text{If minimum } \in \text{ \[ \quad \text{directed case} \quad \text{and } \quad a \geq 1 \text{ and } a > b \end{array}
\end{cases}
\begin{array}{l}
\text{or } \quad b \geq 1 \text{ and } b > a^2
\end{array}
\]

\[
\begin{cases}
\begin{array}{l}
\text{If minimum } \in \text{ \[ \quad \text{Free case} \quad \text{and } \quad 1 < \sqrt{ab} < a < b
\end{array}
\end{cases}
\]

\[
\begin{cases}
\begin{array}{l}
\quad \text{Free case}
\end{array}
\end{cases}
\]
**Theorem:**

Weight of GB walks of length $n$

$$= [g^n] Q_p(1,1;\kappa)$$

$$\sim K^n \rho^n -\alpha$$

$$\rho = \frac{2(k+1)}{V^2}$$

$$\rho = \frac{(1+b)(a^2+b)}{ab}$$

$$\rho = \frac{(1+a)^2}{\alpha}$$
PART 2

General Central Weightings
Theorem Let \( S \) be a non-singular set of integer steps in dimension \( 2 \). A weight \( p_\sigma \) is assigned to each \( \sigma \in S \). There is equivalence between

(i) The probability of a given walk only depends on its length, its start and end point.

(ii) There exist constants \( a, b, c > 0 \) such that for every \( (x_0, y_0) \in S \),

\[
P((x_0, y_0)) = c \times a^{x_0} \times b^{y_0}
\]

(iii) Take \( |S| - 2 - 1 \) "independant" pairs of paths \((w_k, w'_k)\) such that \( w_k \) and \( w'_k \) share the same length, the same start and end point.

Then for every \( k \),

\[
\prod_{\sigma \in w_k} p_\sigma = \prod_{\sigma' \in w'_k} p_{\sigma'}
\]

(iv) The kernels are essentially the same.
What about

(b) There exist constants $a, b, c > 0$ such that

$$Q_p(x, y; z) = Q(ax, by; cz)$$

weighted GF

unweighted GF
GENERATING FUNCTIONS

What about (v) There exist constants $a, b, c > 0$ such that

$Q_r(x, y, z) = Q(ax, by, cz)$

Prop: If a weighting is central, then (v) holds.

Conjecture: If (v) holds, then the weighting is central.

(true for all small 2D-models)
(v) There exist constants $a, b, c > 0$ such that

$$Q_p(x, y; z) = Q(ax, by; cz)$$

Consequence 1: $Q_p(x, y; z)$ is D-finite. $\square$

$Q(x, y; z)$ is D-finite. (if the weights are central and rational)
Some consequences

(1) There exist constants $a, b, c > 0$ such that

$$Q_p(x, y, z) = Q(ax, by, cz)$$

Consequence 1: $Q_p(x, y, z)$ is D-finite.

$\Rightarrow$$Q(x, y, z)$ is D-finite.

(if the weights are central rational)

Consistent with [Kauers, Yatchak]: systematic search of the D-finite weighted 2D-models

Family 1a  Family 1b  Family 2a  Family 2b  Family 3a  Family 3b  Family 4a  Family 4b
Some Consequences

There exist constants $a, b, c > 0$ such that

$$Q_n(x, y; z) = Q(ax, by; cz)$$

Consequence 2: Given a central weighting $p(x_0, y_0) = c \cdot a^{x_0} b^{y_0}$, the weight of excursions of size $n = c^n \times$ number of excursions of size $n$. 
Some Consequences

There exist constants $a, b, c > 0$ such that

$$Q_n(x, y; z) = Q(ax, by; cz)$$

Consequence 2: Given a central weighting $\gamma(x_0, y_0) = c \cdot a^{x_0} b^{y_0}$, the weight of excursions of size $n = \gamma_n \times$ number of excursions of size $n$.

In general, evaluations of $Q(x, y; z)$ can be equated to central weightings.
THE CONJECTURE OF GARBIT, MUSTAPHA & RASCHEL

\[ S(x, y) = \text{WEIGHTED INVENTORY} \]

\[ (x^*, y^*) = \text{argmin}_{x \geq 1, y \geq 1} S(x, y) \]

**Conjecture**  
(Every estimate is up to a constant.)

<table>
<thead>
<tr>
<th>( (x^<em>, y^</em>) = (1, 1) )</th>
<th>( \nabla S(x^<em>, y^</em>) = 0 ) or ( \frac{\partial S}{\partial x}(x^<em>, y^</em>) &gt; 0 ) and ( \frac{\partial S}{\partial y}(x^<em>, y^</em>) &gt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>balanced</td>
<td>( S((1, 1)) \sim n^{-p/2} )</td>
</tr>
<tr>
<td>axial</td>
<td>( S(1, 1) \sim n^0 )</td>
</tr>
<tr>
<td>free</td>
<td>( S((1, 1)) \sim n^0 )</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>( x^* = 1 ) or ( y^* = 1 )</th>
<th>( S(x^<em>, y^</em>) ) ( \sim n^{-p+1} )</th>
<th>( S(x^<em>, y^</em>) ) ( \sim n^{-3/2} )</th>
<th>impossible</th>
</tr>
</thead>
<tbody>
<tr>
<td>transitional</td>
<td>directed</td>
<td>impossible</td>
<td></td>
</tr>
</tbody>
</table>

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<thead>
<tr>
<th>( x^* &gt; 1 ) and ( y^* &gt; 1 )</th>
<th>( S(x^<em>, y^</em>) ) ( \sim n^{-p-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>reluctant</td>
<td>impossible</td>
</tr>
</tbody>
</table>

\[ p = \frac{T}{\arccos(c)} \]

\[ c = \frac{\frac{\partial S}{\partial x}(x^*, y^*)}{\sqrt{\frac{\partial^2 S}{\partial x^2}(x^*, y^*) + \frac{\partial^2 S}{\partial y^2}(x^*, y^*)}} \]

[do not depend on the central weights]
SOME DRIFT DIAGRAMS

Gouyou - Beauchamps model

Tandem model

Gessel model in its natural cone.
CONCLUSION

- Good framework to understand transitions.

- Many left conjectures:
  - A weighting is central iff $Q_n(x, y; z) = Q(ax, by; cz)$?
  - Which step sets have "pretty" (like conic) regions in their drift diagrams?
  - ...

- Hope to understand more non-D-finite models.