DIAGRAMMES DE CORDES & CARTES ENRAÇINÉES

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DIS PAPA, ÇA SERT À QUOI UNE BIJECTION ?

Journées Algocomb Normastic Mai 2019
THE PROTAGONISTS

THE PHYSICIST

THE COMPUTER SCIENTIST

THE MATHEMATICIAN
ONCE UPON A TIME IN VANCOUVER

MY PHYSICS EQUATION HAS A COMBINATORIAL SOLUTION. MAYBE YOU CAN HELP ME WITH THAT.

OK.
Karen Yeats

- expert in combinatorics and in perturbative Quantum Field Theory

- What she studies: generating functions of Feynman diagrams weighted by their renormalized Feynman integrals (hard to compute!)
The Starting Point

[Marie, Yeats] [Hihn, Yeats]

The Dyson–Schwinger equation

\[ G(x, L) = 1 - \sum_{k \geq 1} x^k G(x, \varphi_p) (e^{-L \varphi} - 1) F_k(p) \]

has for solution

\[ G(x, L) = 1 - \sum_{C \text{ decorated connected chord diagram}} \omega(C) \left( \sum_{i=1}^{k_1} \int f d(t_i) \cdot \frac{(-L)^i}{i!} \right) \prod_{i=1}^{k_1} f d(c_0) \prod_{i=1}^{k_{n-1}} f d(t_i) \cdot t_i \cdot \cdots \cdot t_{k_{n-1}} \cdot x \]

where \( F_k(p) = b_{k,0} p^{-1} + b_{k,1} + b_{k,2} p + b_{k,3} p^2 + \cdots \) is the regularized Feynman integral of the primitive graphs of size \( k \).
THE STARTING POINT

Theorem [Marie, Yeats] [Hihn, Yeats]

Some physical equation

BAD BAD EQUATION

has for solution

\[ \sum \text{BAD BAD FORMULA} \]

connected chord diagram

terminal chords
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**First Definitions**

Diagram with \( n \) chords

= perfect matching of the set \( \{1, \ldots, 2n\} \)

Connected diagram = "everything is one block."
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3 connected components:

![Diagram with 3 connected components]
diagram with \( n \) chords

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? terminal chords?
**FIRST DEFINITIONS**

**diagram with n chords**

= perfect matching of the set \( \{1, \ldots, 2n\} \)

**connected diagram** = "everything is one block."

**terminal chord** = chord \((a, b)\) such that there is no intersecting chord \((c, d)\) such that \(b < d\).
**FIRST DEFINITIONS**

- **Diagram with n chords**
  - perfect matching of the set \( \{1, \ldots, 2n\} \)

- **Connected diagram**
  - "everything is one block."

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Their proof: The coefficients in the solution of the physical equation and in the generating function of chord diagrams share the same recurrences.
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Their proof: The coeffs in the solution of the physical equation and in the generating function of chord diagrams share the same recurrences. In other words: magic.
ONE YEAR LATER
AT POLYTECHNIQUE ...

... SO THIS IS OUR WONDERFUL RESULT ABOUT CONNECTED CHORD DIAGRAMS THAT WE FOUND WITH KAREN...
Is there a connection between connected chord diagrams and bridgeless maps?

No, I don't think so.
THE NEXT DAY

CRAP, HE'S RIGHT.
• expert in logic (proof theory)

• What he studies: the connections between lambda-calculus and the combinatorics of maps.
**WHAT IS A MAP?**

\[
\text{map} = \text{connected graph where we have cyclically ordered the half-edges around each vertex.}
\]

**Examples:**

\[
\begin{align*}
\text{Circle} & \quad = \quad \text{Cylinder} \quad \neq \quad \text{Torus} \\
\end{align*}
\]

**Why is \( \text{Circle} \) the same as \( \text{Cylinder} \)?**
WHAT IS A MAP?

map = connected graph where we have cyclically ordered the half-edges around each vertex.

Examples:

Why is the same as?
**WHAT IS A MAP?**

\[ \text{map} = \text{connected graph where we have cyclically ordered the half-edges around each vertex.} \]

**Examples:**

Why is \( \square \) different from \( \square \)?

Absent pattern in \( \square \):

\[ a \leftrightarrow a', \quad a \cup b' \]
\[ b \leftrightarrow b', \quad a' \cup b' \]
**WHAT IS A MAP?**

\[ \text{map} = \text{connected graph where we have cyclically ordered the half-edges around each vertex.} \]

**Examples:**

We root every map on a leaf.
**WHAT IS A MAP?**

\[ \text{map} = \text{connected graph where we have cyclically ordered the half-edges around each vertex}. \]

**Examples:**

- **1 edge**
  
- **2 edges**
  
- **3 edges**
WHAT IS A MAP?

\[ \text{bridge (or isthmus)} = \text{edge which disconnects the map when removed} \]

\[ \text{bridgeless map} = \text{map without any bridge (duh)} \]

*: The root is not a bridge.
WHAT IS A MAP?

bridge (or isthmus) = edge which disconnects the map when removed.

bridgeless map = map without any bridge

Question: Which are the bridgeless maps?

1 edge

2 edges

3 edges
WHAT IS A MAP?

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Question: Which are the bridgeless maps?

1 edge

2 edges

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4 edges
Our Theorem

Theorem [Courtiel Yeats Zeilberger]

There are as many connected diagrams with $n$ chords as bridgeless maps with $n$ edges.
Our Theorem

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Why is this surprising?

- The sequence counting the connected chord diagrams (A000639 in OEIS) was actively studied.

  1952 Touchard
  1978 Everett
  2000 Flajolet-Noy
  2016 Karen & I

but no mention of maps!
**OUR THEOREM**

**Theorem** [Courtiel Yeats Zeilberger]

There are as many connected diagrams with \( n \) chords as bridgeless maps with \( n \) edges.

Why is this surprising?

- A bijection between maps and indecomposable diagrams was already known! [Ossona de Mendez-Rosenstiehl] [Cori]
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There are as many connected diagrams with $n$ chords as bridgeless maps with $n$ edges.

Why is this surprising?

- A bijection between maps and indecomposable diagrams was already known! [Ossona de Mendez, Rosenstiehl] [Cori]

indecomposable diagram

= diagram which is not the concatenation of two diagrams

Ex: Counter-ex:

\[
\begin{array}{c}
\includegraphics[scale=0.5]{example1.png} \\
\includegraphics[scale=0.5]{example2.png}
\end{array}
\]
**Our Theorem**

Theorem [Courtiel Yeats Zeilberger]

There are as many connected diagrams with $n$ chords as bridgeless maps with $n$ edges.

Why is this surprising?

- A bijection between maps and indecomposable diagrams was already known! [Ossona de Mendez-Rosenstiehl] [Cori]

Indecomposable diagram

= diagram which is not the concatenation of two diagrams

Ex: Counter-ex:

\[ \text{\includegraphics[width=0.2\textwidth]{diagram}} \]

However, their bijection [OMR] indecomposable diagrams maps indecomposable diagrams [Cori] does not restrict to connected diagrams bridgeless maps.
Theorem [Courtiel Yeats Zeilberger]

There are as many connected diagrams with $n$ chords as bridgeless maps with $n$ edges.
Theorem [Courtiel Yeats Zeilberger]
There are as many connected diagrams with \( n \) chords as bridgeless maps with \( n \) edges.

Let’s show that
the number \( \mathcal{C}_n \) of connected diagrams with \( n \) chords
and the number of bridgeless maps with \( n \) edges
both satisfy
\[
\mathcal{C}_1 = 1 \quad \text{and} \quad \mathcal{C}_n = \sum_{k=1}^{n-1} (2k-1) \times \mathcal{C}_k \times \mathcal{C}_{n-k}
\]
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Let's show that the number $c_n$ of connected diagrams with $n$ chords and the number of bridgeless maps with $n$ edges both satisfy

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CONNECTED DIAGRAMS
**THEOREM**

**Theorem [Courtiel Yeats Zeilberger]**

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& \text{CONNECTED DIAGRAMS}
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**OUR THEOREM**

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C_1 = 1 \quad \text{and} \quad C_n = \sum_{k=1}^{n-1} (2k-1) \cdot C_k \cdot C_{n-k}
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**Theorem** [Courtiel Yeats Zeilberger]

There are as many connected diagrams with $n$ chords as bridgeless maps with $n$ edges.

Let's show that the number $c_m$ of connected diagrams with $n$ chords and the number of bridgeless maps with $n$ edges both satisfy

\[ c_1 = 1 \quad \text{and} \quad c_m = \sum_{k=1}^{n-1} (2k-1) \times c_k \times c_{n-k} \]
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$$\downarrow$$

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**Bridgeless Maps**
**Theorem** [Courtiel Yeats Zeilberger]

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Let's show that the number \( L_n \) of connected diagrams with \( n \) chords and the number of bridgeless maps with \( n \) edges both satisfy

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Theorem [Courtiel, Yeats, Zeilberger]
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Let's show that the number \( \mu_n \) of connected diagrams with \( n \) chords and the number of bridgeless maps with \( n \) edges both satisfy

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![Diagram of a connected diagram with chords and bridgeless maps](image)
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**Bridgeless Maps**
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Bridgeless Maps
Theorem [Courtel Yeats Zeilberger]

There are as many connected diagrams with $\eta$ chords as bridgeless maps with $\eta$ edges.
BETWEEN INDECOMPOSABLE DIAGRAMS AND MAPS
Decomposition of indecomposable diagrams:

\[
\text{indecomposable diagram} = \text{\begin{itemize}
    \item \includegraphics[width=0.08\textwidth]{diagram1.png}
    \item \includegraphics[width=0.1\textwidth]{diagram2.png}
    \item \includegraphics[width=0.15\textwidth]{diagram3.png}
\end{itemize}}
\]
Decomposition of indecomposable diagrams:

\[
\text{indecomposable diagram} = \begin{array}{c}
\text{or} \\
\end{array}
\]

Decomposition of maps:

\[
\text{map} = \begin{array}{c}
\text{or} \\
\end{array}
\]

[Arquès - Béraud]
BETWEEN INDECOMPOSABLE DIAGRAMS AND MAPS

Decomposition of indecomposable diagrams:

\[
\text{indecomposable diagram} = \begin{array}{c}
\text{or} \\
\text{or} \\
\end{array}
\]
Decomposition of indecomposable diagrams:

\[
\text{indecomposable diagram} = \text{Diagram 1} \quad \text{or} \quad \text{Diagram 2} \quad \text{or} \quad \text{Diagram 3}
\]
BETWEEN INDECOMPOSABLE DIAGRAMS AND MAPS

Decomposition of indecomposable diagrams:

indecomposable diagram = \[ \text{diagram} \] or \[ \text{diagram} \] or \[ \text{diagram} \]

Decomposition of maps:

map = \[ \text{map} \] or \[ \text{map} \] or \[ \text{map} \]

[Arques - Beraud]

bridge

not a bridge
Decomposition of indecomposable diagrams:

indecomposable diagram = \textbullet \text{ or } \text{ or }

Decomposition of maps:

map = \text{ or } \text{ or }

[Arquès - Béraud]
Decomposition of indecomposable diagrams:

indecomposable diagram = \[\text{diagram 1}\] or \[\text{diagram 2}\] or \[\text{diagram 3}\]

Decomposition of maps:

map = \[\text{map 1}\] or \[\text{map 2}\] or \[\text{map 3}\]

[Arquès - Béraud]
**Decomposition of indecomposable diagrams:**

indecomposable diagram = \[ \text{Diagram 1} \] or \[ \text{Diagram 2} \] or \[ \text{Diagram 3} \]

**Decomposition of maps:**

map = \[ \text{Diagram 4} \] or \[ \text{Diagram 5} \] or \[ \text{Diagram 6} \]

[Arquès - Béraud]

Same decomposition = same numbers!
Decomposition of indecomposable diagrams:

\[
\text{indecomposable diagram} = \begin{array}{c}
\text{or}\end{array}
\]

Decomposition of maps:

\[
\text{map} = \begin{array}{c}
\text{or}\end{array}
\]

Same decomposition = same numbers!  But where is the bijection?
THE BIJECTION

\[ \text{maps} \quad \phi \quad (\text{indecomposable}) \quad \text{diagrams} \]
THE BIJECTION

maps $\phi$ (indecomposable) diagrams

bridge

not a bridge
THE BIJECTION

maps \( \phi \) (indecomposable) diagrams

bridge

not a bridge
THE BIJECTION

maps \phi \rightarrow (indecomposable) diagrams

bridge

not a bridge
THE BIJECTION

maps \rightarrow (indecomposable) diagrams

\phi

bridge

\phi(C_1) \rightarrow \phi(C_2)

not a bridge

\phi(C')

We label the corners from 1 to n
Some properties of the bijection

[C. Yeats Zeilberger]

maps \quad \leftrightarrow \quad \text{indecomposable diagrams}
Some properties of the bijection

[Charalampos Yeats-Klein, Zeilberger]

Maps \quad \leftrightarrow \quad \text{indecomposable diagrams}

Bridgeless maps \quad \leftrightarrow \quad \text{connected diagrams}
Some properties of the bijection

[C. Yeats Zeilberger]

maps $\leftrightarrow$ indecomposable diagrams

bridgeless maps $\leftrightarrow$ connected diagrams

leaves $\leftrightarrow$ isolated chords
Some properties of the bijection

[C. Yeats Zeilberger]

maps \iff \text{indecomposable diagrams}

bridgeless maps \iff \text{connected diagrams}

leaves \iff \text{isolated chords}

planar maps \iff \text{indecomposable diagrams avoiding } \cdots \cdots \cdots
SOME PROPERTIES OF THE BIJECTION

[C. Yeats Zeilberger]

maps \quad \longleftrightarrow \quad \text{indecomposable diagrams}

bridgeless maps \quad \longleftrightarrow \quad \text{connected diagrams}

leaves \quad \longleftrightarrow \quad \text{isolated chords}

planar maps \quad \longleftrightarrow \quad \text{indecomposable diagrams}

avoiding \quad \text{ terminal chords}
Some properties of the bijection

[C. Yeats Geoffe] maps \leftrightarrow \text{indecomposable diagrams}

bridgeless maps \leftrightarrow \text{connected diagrams}

leaves \leftrightarrow \text{isolated chords}

planar maps \leftrightarrow \text{indecomposable diagrams avoiding}

vertices! \leftrightarrow \text{terminal chords}
Application to Perturbative QFT

**Theorem** [Marie, Yeats] [Hihn, Yeats]

The Dyson-Schwinger equation

\[ G(x, L) = 1 - \sum_{k \geq 1} x^k \ G(x, \mathcal{D}_p) \ (e^{-Lp} - 1) \ F_k(p) \]

has for solution

\[ G(x, L) = 1 - \sum_{C \ \text{decorated connected chord diagram}} w(C) \left( \sum_{i = 1}^{k_r} \ \int \frac{d(t_i) t_i - i}{i!} \right) \prod_{i = 1}^{k_r} \int \frac{d(0_i) t_i.t_i}{t_i.t_i} \]

such that \( k_1 < k_2 < \ldots < k_r \)

are the positions of the terminal chords.

where \( F_k(p) = F_{k,0} p^{-1} + F_{k,1} p + F_{k,2} p^2 + \ldots \) = regularized Feynman integral of the primitive graphs of size \( k \).
APPLICATION TO PERTURBATIVE QFT

**Theorem** [Marie, Yeats] [Hihn, Yeats]

The Dyson-Schwinger equation

\[
G(x, L) = 1 - \sum_{k \geq 1} x^k \left( G(x, \partial_x) \left( e^{-\Delta} - 1 \right) F_k(p) \right)
\]

has for solution

\[
G(x, L) = 1 - \sum_{\text{C decorated bridgeless maps}} \omega(C) \left( \sum_{i=1}^{\ell_1} \int d(t_i) t_i^{-i} \left( \frac{-L}{i!} \right)^i \prod_{c \text{ non terminal}} \int d(c)_0 \prod_{i=1}^{\ell_i} \int d(t_i) t_i t_{i-1}^i x \right)
\]

where \( F_k(p) = b_k, 0 p^{-1} + b_k, 1 + b_k, 2 p + b_k, 3 p^2 + \ldots \) = regularized Feynman integral of the primitive graphs of size \( k \)
APPLICATION TO PERTURBATIVE QFT

Theorem [Marie, Yeats] [Hihn, Yeats]

Some physical equation

BAD BAD EQUATION

has for solution

\[ \sum \text{bridgeless maps} \]

BAD BAD FORMULA

vertices
APPLICATION TO PERTURBATIVE QFT

Theorem \[[\text{Marie, Yeats}] [\text{Hihn, Yeats}]\]

Some physical equation

\[ \sum \text{BAD BAD EQUATION} \]

has for solution

\[ \sum \text{BAD BAD FORMULA} \]

bridgeless maps

vertices

"New" proof: Now the recurrence can be explained combinatorially.
APPLICATION TO PERTURBATIVE QFT

Theorem [Marie, Yeats] [Hihn, Yeats]

Some physical equation

BAD BAD EQUATION

has for solution

\[ \sum \text{ bridgeless maps} \]

\[ \text{vertices} \]

"New" proof: Now the recurrence can be explained combinatorially.

magic = science?
APPLICATION TO ASYMPTOTICS

Ex under the uniform distribution:

Theorem [Stein-Everett]

A diagram is connected with proba \( n \to \infty \) \( \frac{1}{e} \)

can be (almost) straightforwardly translated by

Theorem

A map is bridgeless with proba \( n \to \infty \) \( \frac{1}{e} \)
Ex 2 under the uniform distribution:

**Theorem**

A random map with $n$ edges has $\sim \ln(n)$ vertices.

can be (almost) straightforwardly translated by

**Theorem**

A random connected diagram with $n$ chords has $\sim \ln(n)$ terminal chords.
APPLICATION TO LAMBDA-CALCULUS

ANOTHER STORY...
Ils vécurent heureux et eurent beaucoup de papiers...

THE END

ZZZZZ