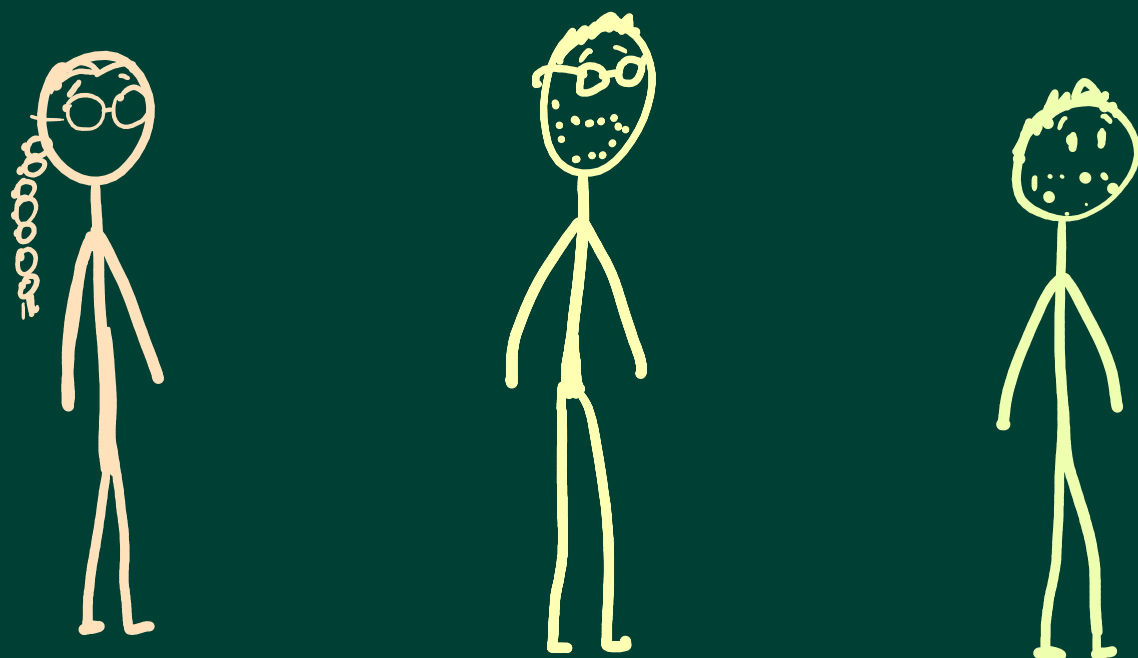
DIAGRAMMES DE CORDES & CARTES ENRACINÉES

Julien COURTIEL (GREYC, Univ. de Caen)

Journées Algocomb Normastic Mai 2019

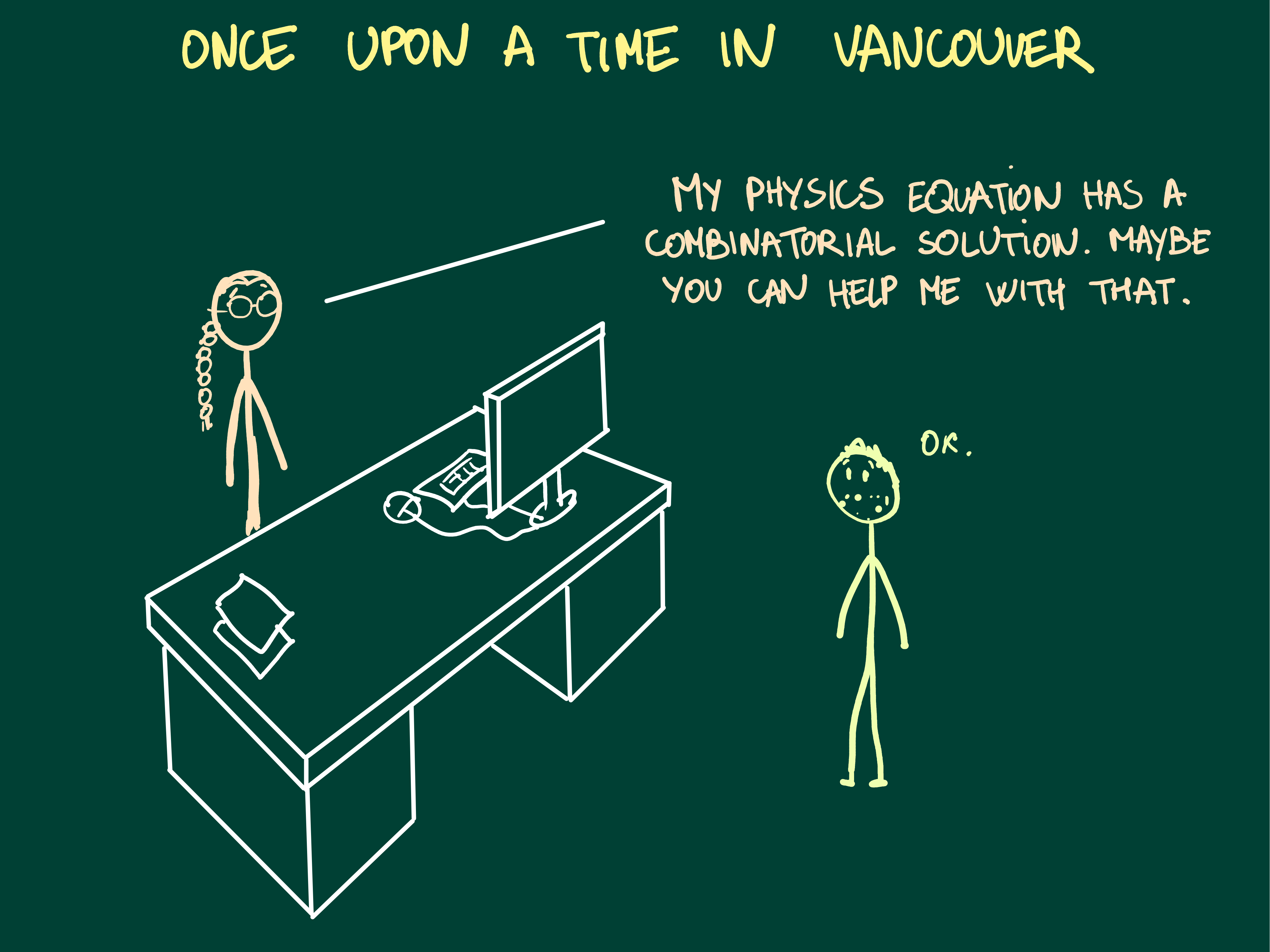
THE PROTAGONISTS



THE PHYSICIST

THE COMPUTER SCIENTIST

THE MATHEMATICIAN


ONCE UPON A TIME IN VANCOUVER
MY PHYSICS EQUATION HAS A COMBINATORIAL SOLUTION. MAYBE YOU CAN HELP ME WITH THAT.
OK.

KAREN YEATS

- expert in combinatorics and in pertubative Quantum Field Theory

- <u>What she studies:</u> generating functions of Feynman diagrams weighted by their renormalized Feynman integrals

  (hard to compute!)

  ≈ proba

THE STARTING POINT

Theorem [Marie, Yeats] [Hihn, Yeats]

The Dyson-Schwinger equation

$$G(x,L) = 1 - \sum_{k\geq 1} x^k \, G(x, \partial_{-\rho})^{1-sk} \left(e^{-L\rho} - 1\right) F_k(\rho)$$

has for solution

$$G(x,L) = 1 - \sum_{\substack{C \text{ decorated} \\ \text{connected chord diagram}}} w(C) \left(\sum_{i=1}^{t_1} f_{d(t_1), t_1 - i} \frac{(-L)^i}{i!} \right) \times \prod_{c \text{ non terminal}} f_{d(c),0} \times \prod_{i=1}^{k-1} f_{d(t_i), t_i - t_{i-1}} \, x^{\|C\|}$$

such that $t_1 < t_2 < \dots < t_k$ are the positions of the **terminal chords**

where $F_k(\rho) = f_{k,0}\,\rho^{-1} + f_{k,1} + f_{k,2}\,\rho + f_{k,3}\,\rho^2 + \dots$ = regularized Feynman integral of the primitive graphs of size k

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<u>Theorem</u> [Marie, Yeats] [Hihn, Yeats]

Some physical equation

BAD BAD EQUATION

has for solution

$$\sum_{\text{connected chord diagram}} \text{terminal chords}$$

BAD BAD FORMULA

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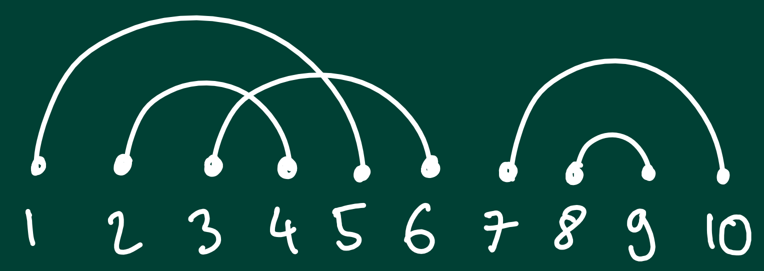
? connected chord diagram ?

terminal chords

FIRST DEFINITIONS

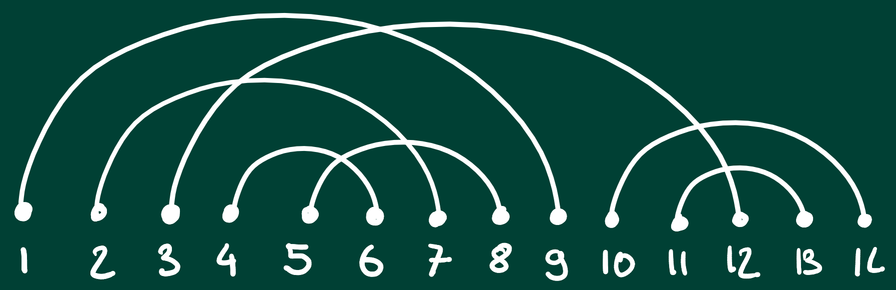
diagram with n chords

$=$ perfect matching of the set $\{1, \ldots, 2n\}$



connected diagram $=$

"everything is one block."



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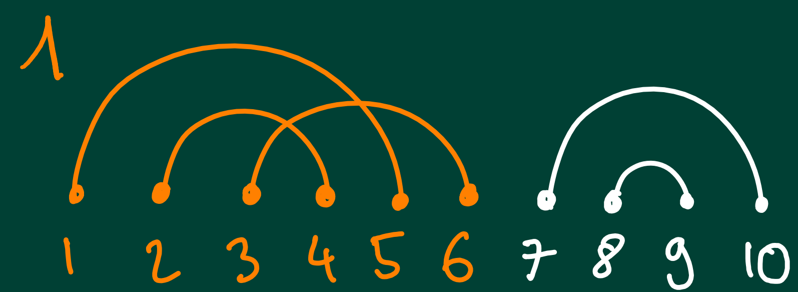

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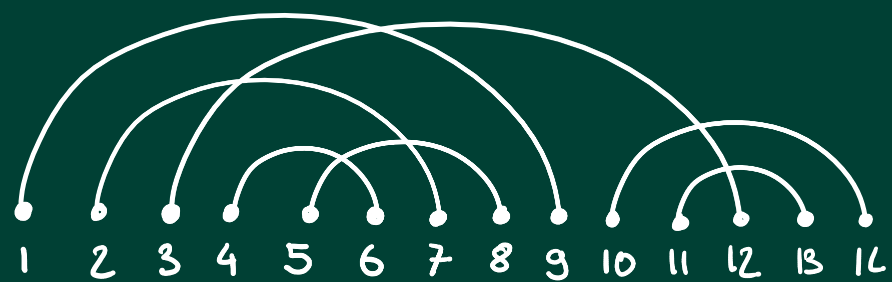

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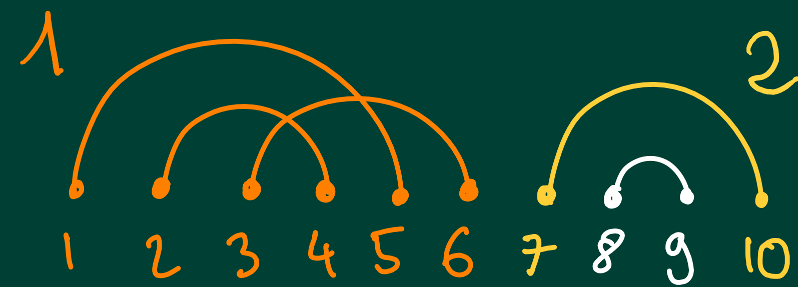

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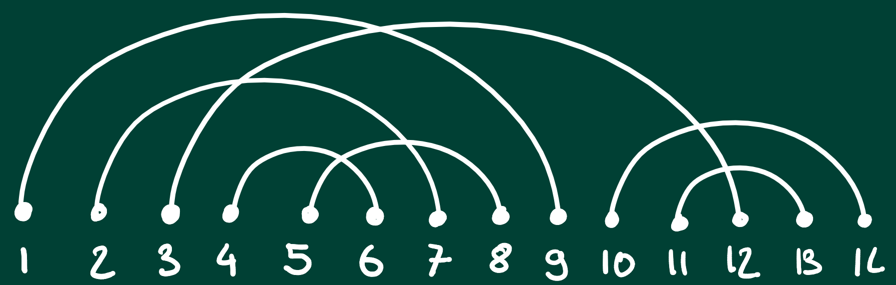

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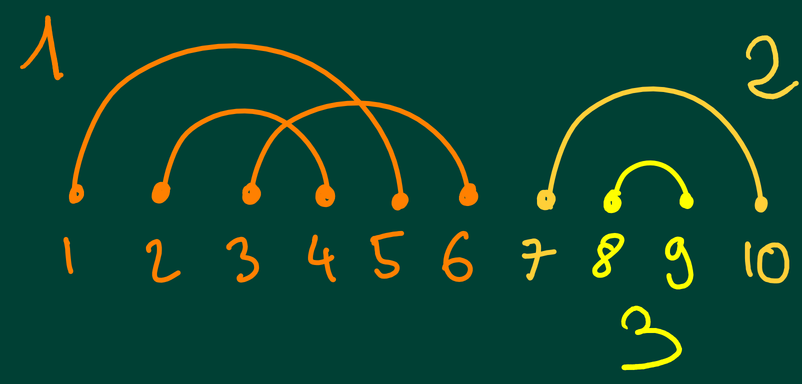

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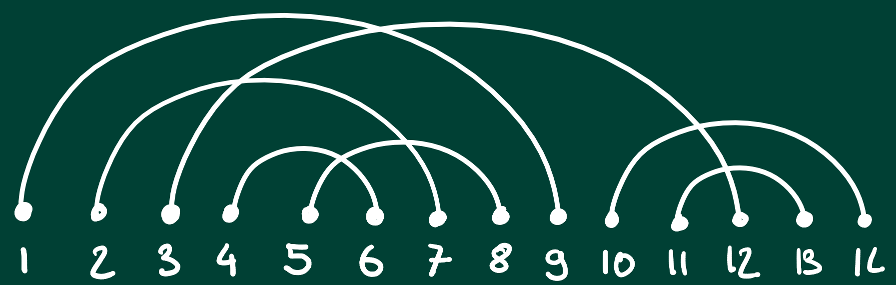

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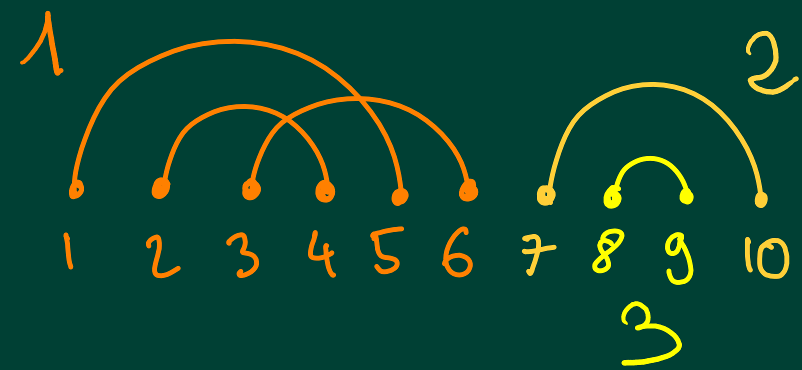

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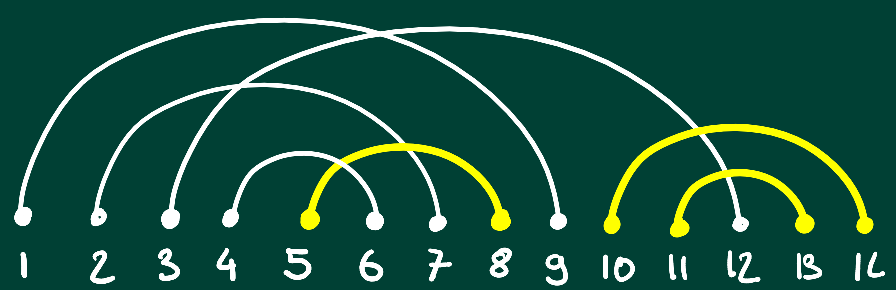
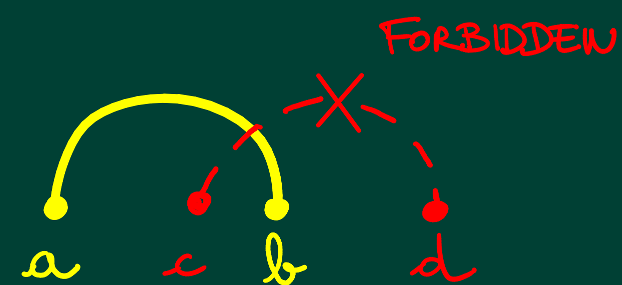


terminal chord =

chord (a, b) such that there is no intersecting chord (c, d) such that $b < d$.



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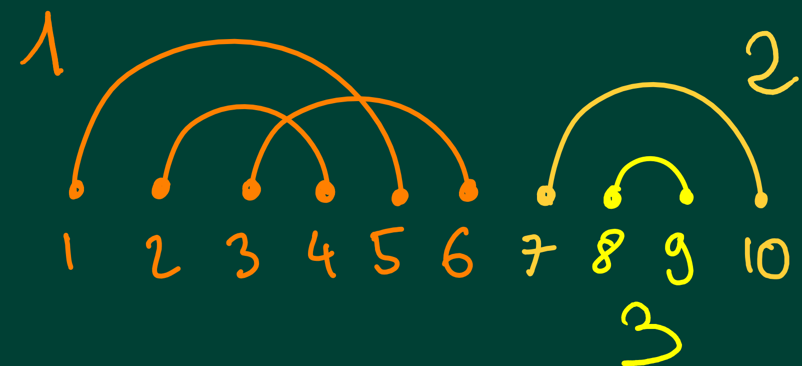

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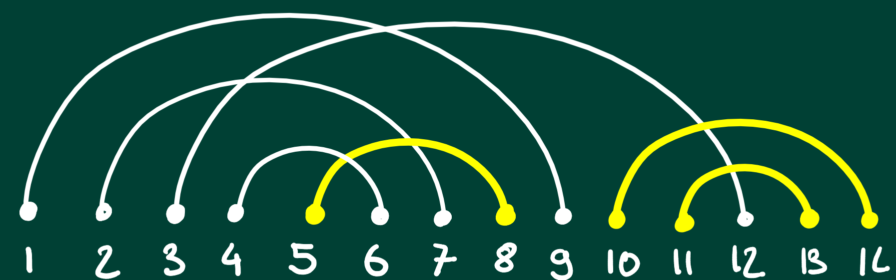

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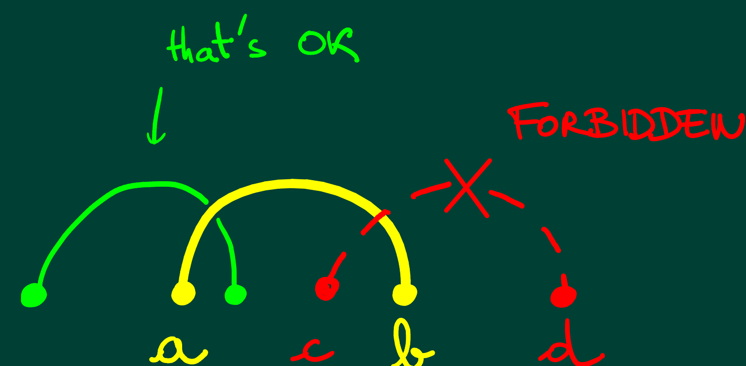

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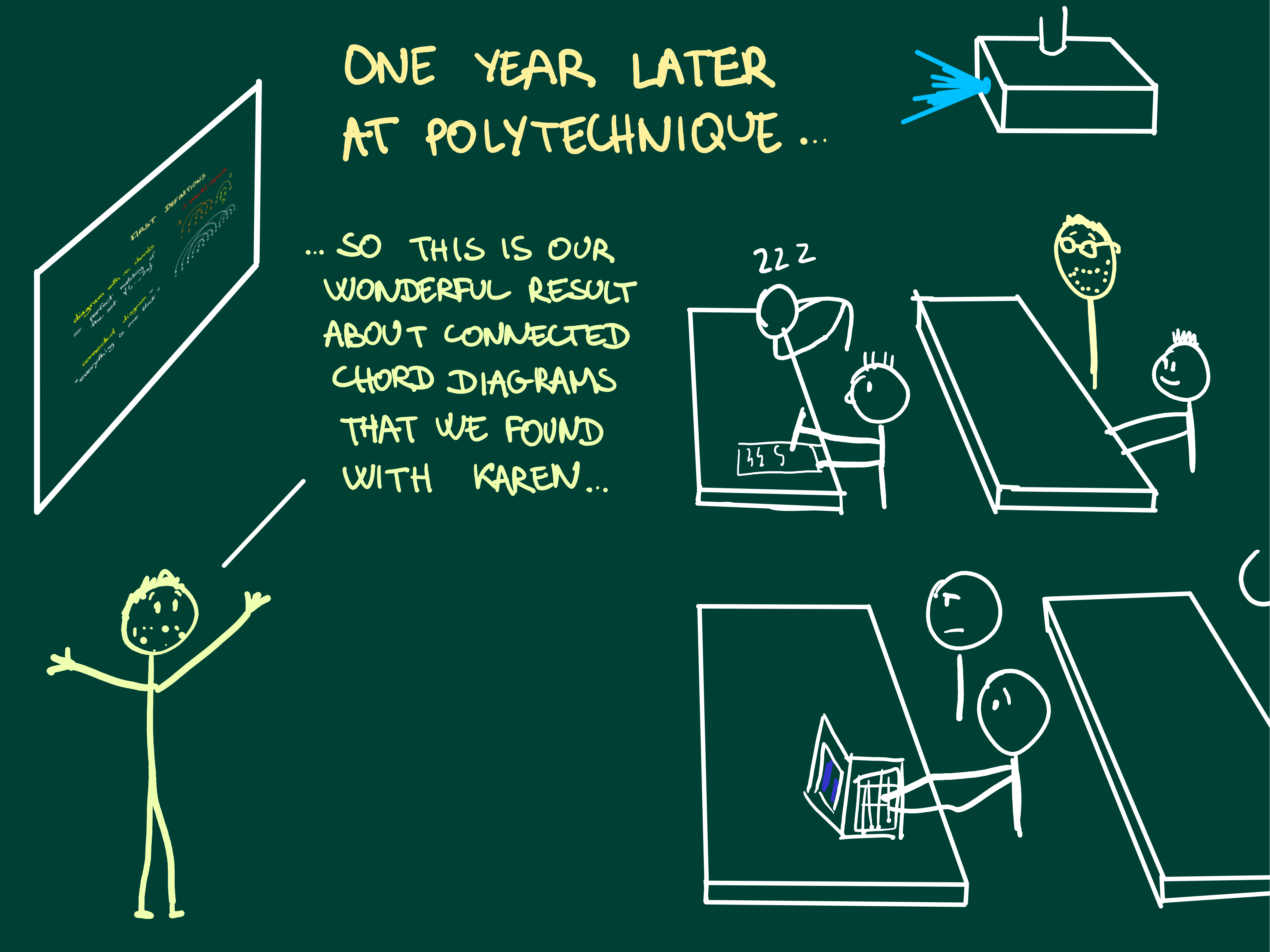
In other words: magic.


ONE YEAR LATER
AT POLYTECHNIQUE...
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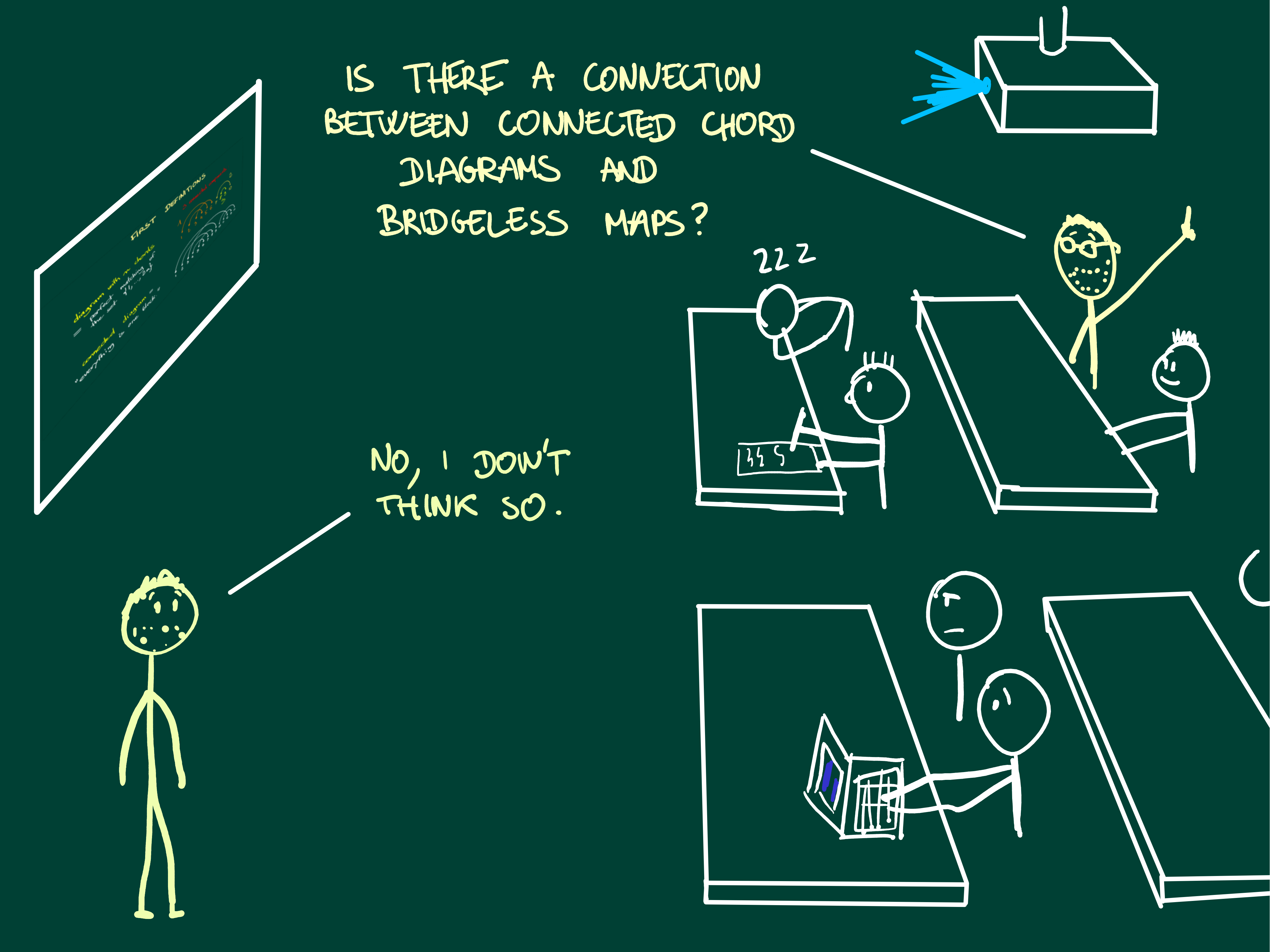
3 connected components
... SO THIS IS OUR WONDERFUL RESULT ABOUT CONNECTED CHORD DIAGRAMS THAT WE FOUND WITH KAREN...
ZZZ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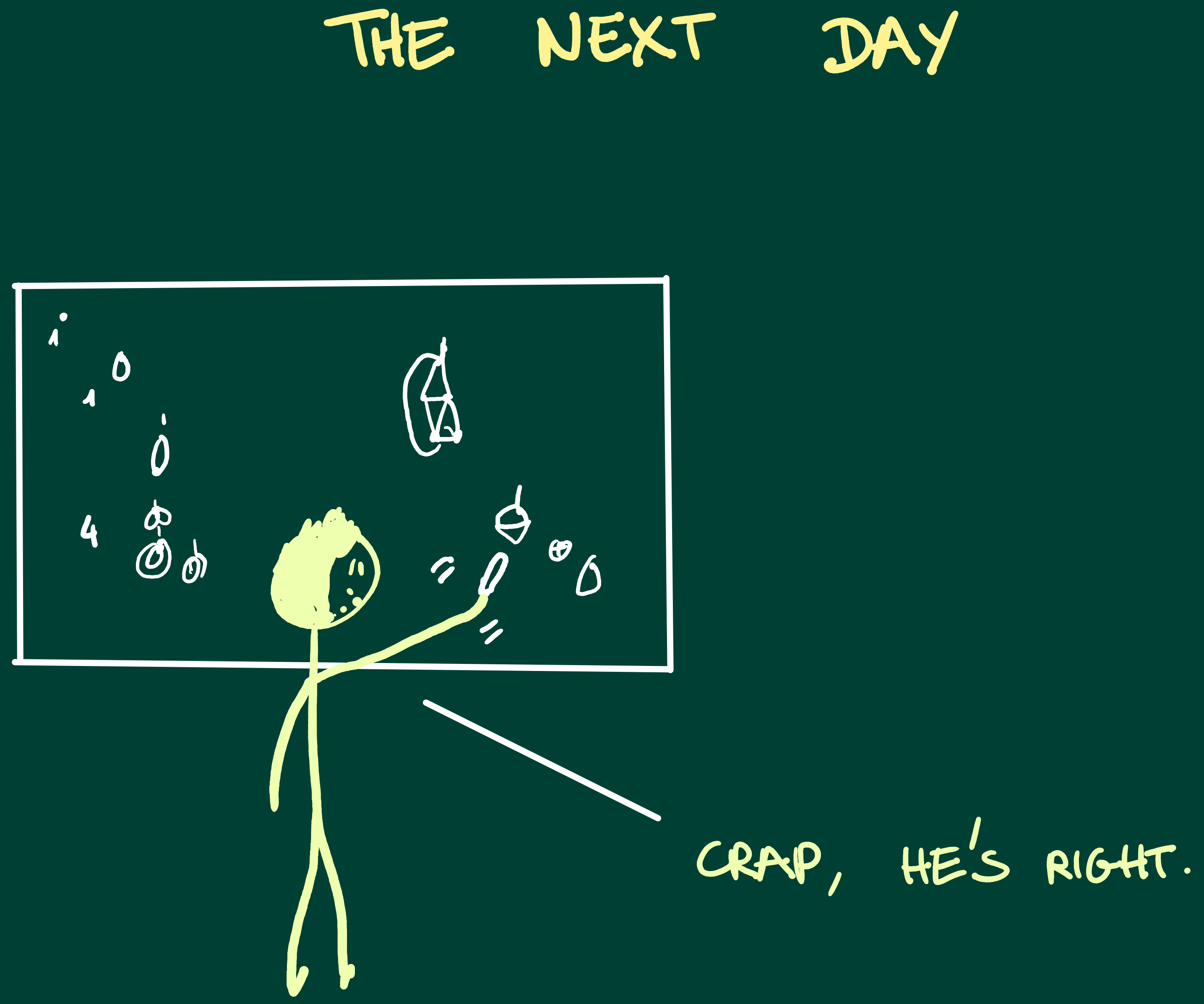
IS THERE A CONNECTION BETWEEN CONNECTED CHORD DIAGRAMS AND BRIDGELESS MAPS?
ZZZ
NO, I DON'T THINK SO.


THE NEXT DAY
CRAP, HE'S RIGHT.

NOAM ZEILBERGER

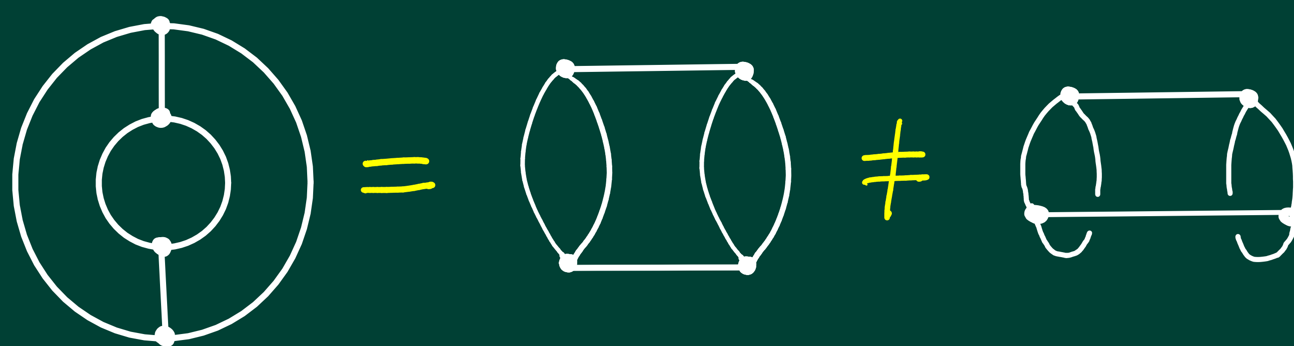
- expert in logic (proof theory)

- <u>What he studies</u>: the connections between lambda-calculus and the combinatorics of maps.

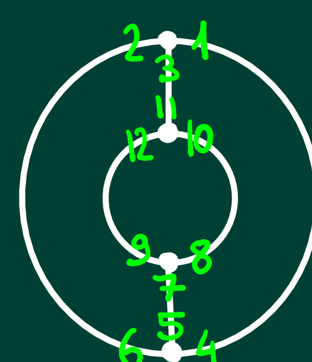
WHAT IS A MAP?

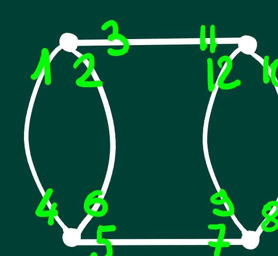
map = connected graph where we have cyclically ordered the half-edges around each vertex.

Examples:



Why is  the same as  ?





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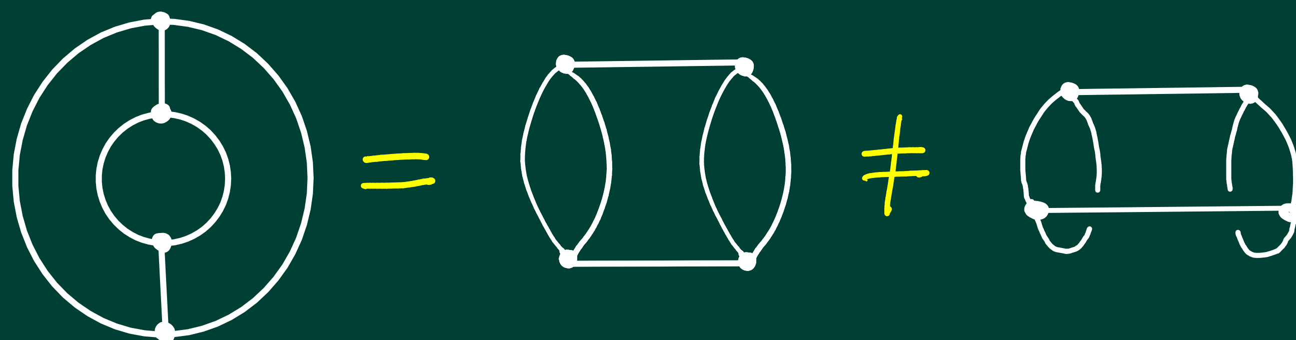

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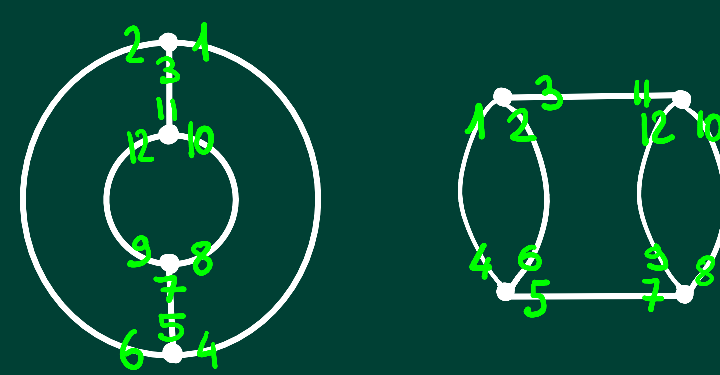

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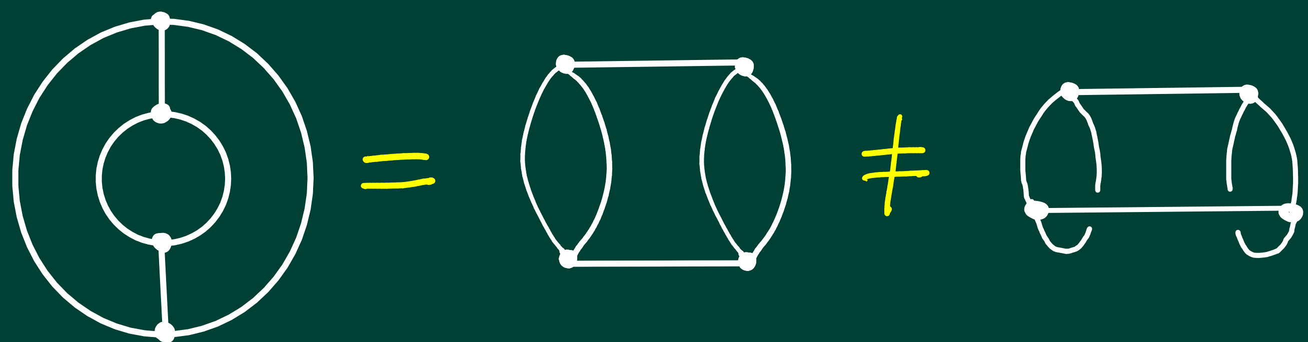
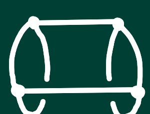
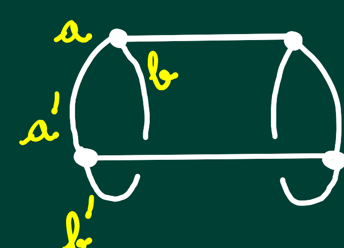


Why is  different from  ?



Absent pattern in  :

$a \leftrightarrow a'$ $\quad a \circlearrowright b$

$b \leftrightarrow b'$ $\quad a' \circlearrowright b'$

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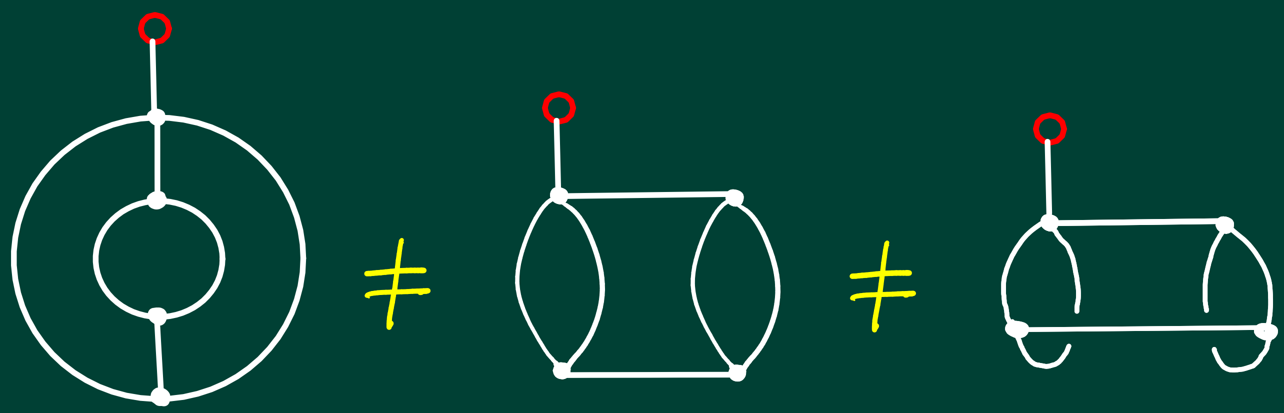


We root every map on a leaf.

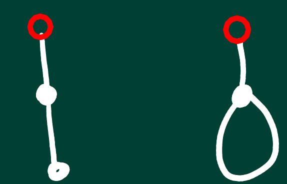
WHAT IS A MAP?

map = connected graph where we have cyclically ordered the half-edges around each vertex.

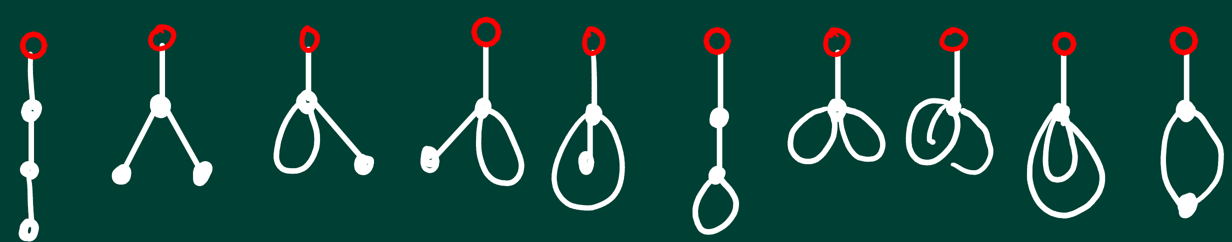
Examples:

1 edge

2 edges

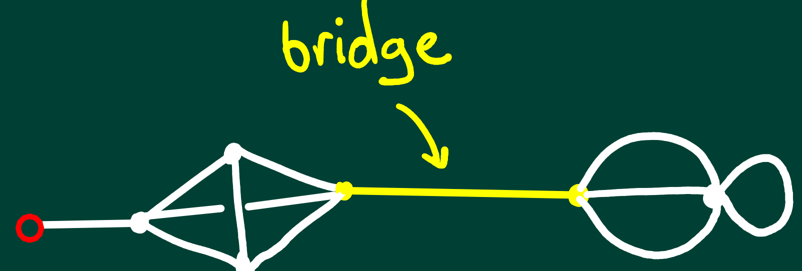


3 edges



WHAT IS A MAP?

bridge (or isthmus) = edge which disconnects the map when removed.*



bridgeless map = map without any bridge (duh)

*: The root is <u>not</u> a bridge.

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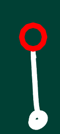
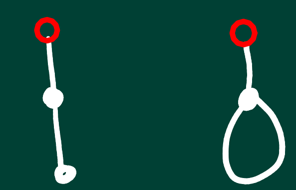
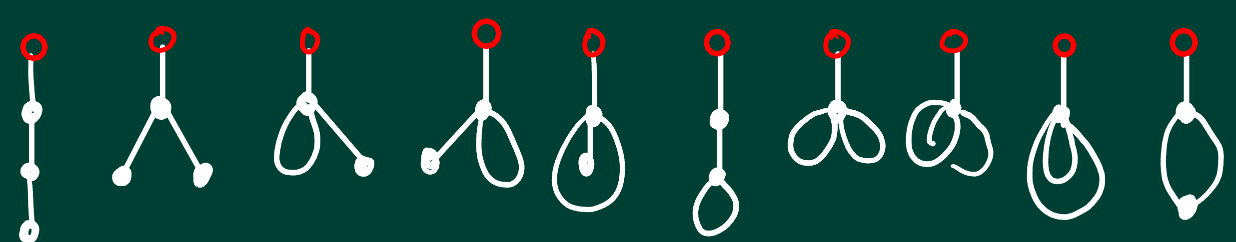
Question: <u>Which are the bridgeless maps?</u>

1 edge 

2 edges 

3 edges 

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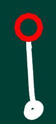
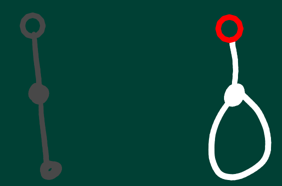
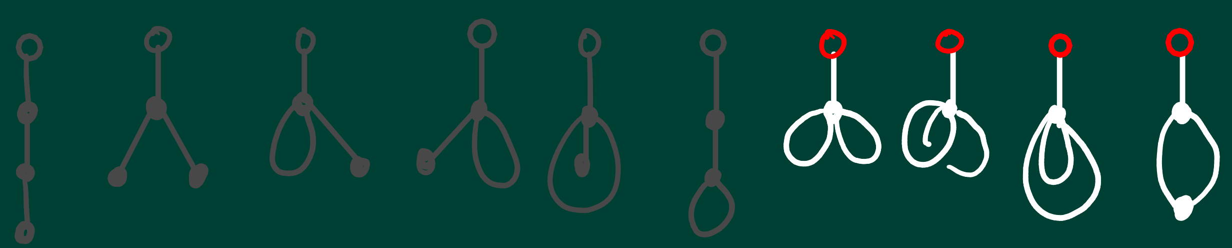
1 edge  1   2 edges  1

3 edges  4

OUR THEOREM

Theorem [Courtiel Yeats Zeilberger]

There are as many connected diagrams with n chords as bridgeless maps with n edges.

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Why is this surprising?

- The sequence counting the connected chord diagrams (A000699 in OEIS) was actively studied

1952	1978	2000	2016
Touchard	Everett	Flajolet-Noy	Karen & I

but no mention of maps!

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Counter-ex:

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However their bijection

indecomposable diagrams $\xleftrightarrow[\text{[Cori]}]{\text{[OMR]}}$ maps

does <u>not</u> restrict to

connected diagrams $\longleftrightarrow$ bridgeless maps.

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<u>Let's show that</u>

the number c_n of connected diagrams with n chords and the number of bridgeless maps with n edges both satisfy

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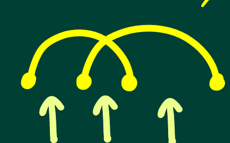

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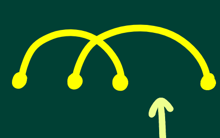

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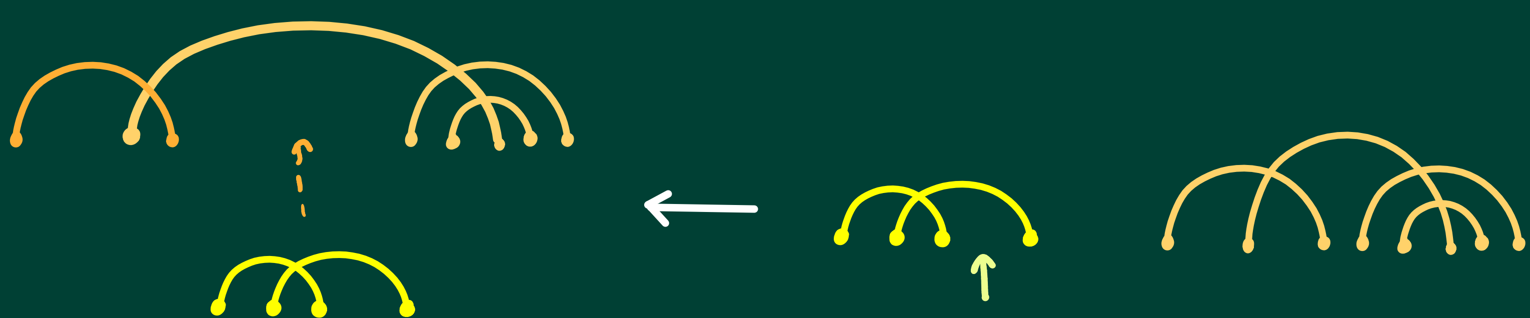

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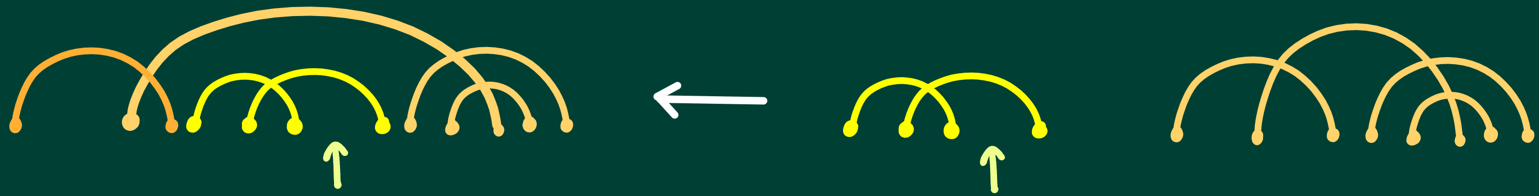

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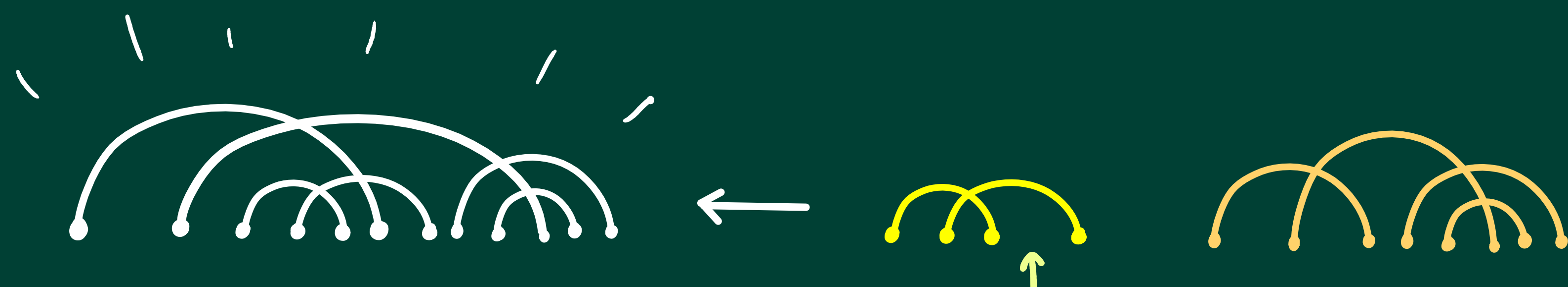

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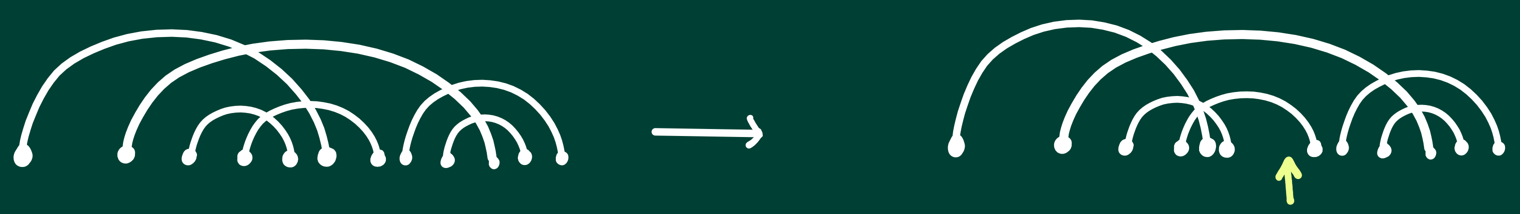

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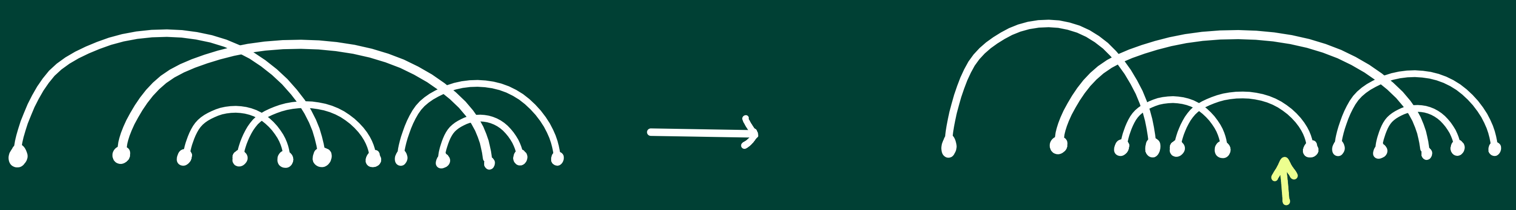

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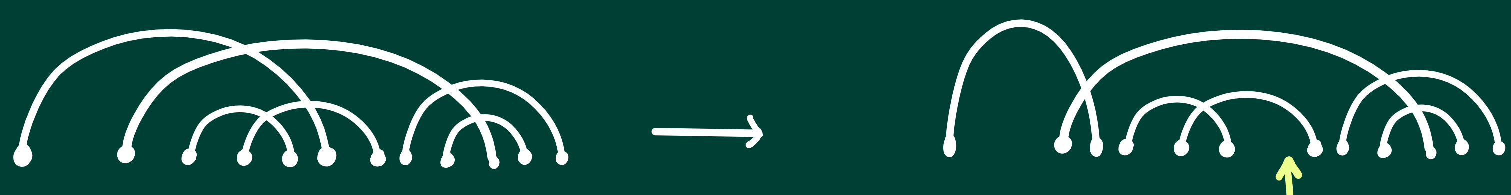

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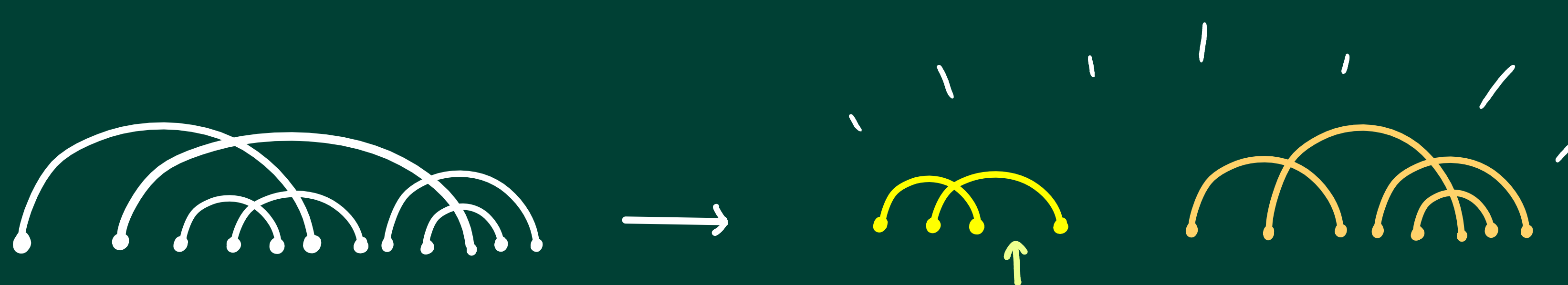

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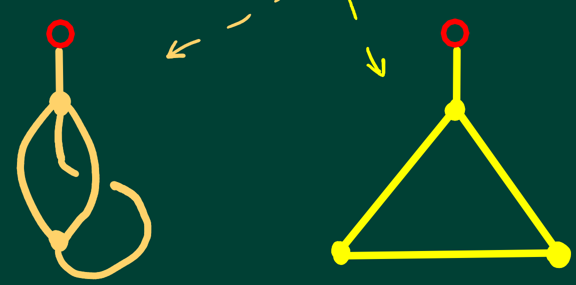

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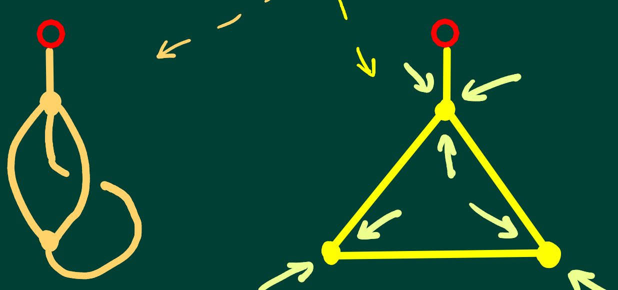

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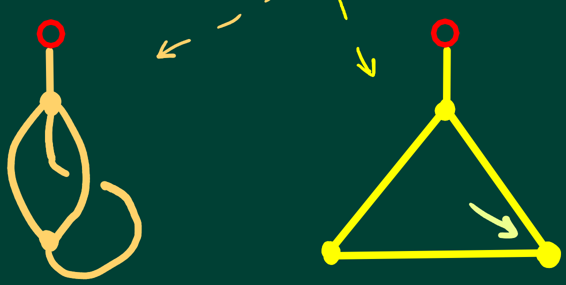

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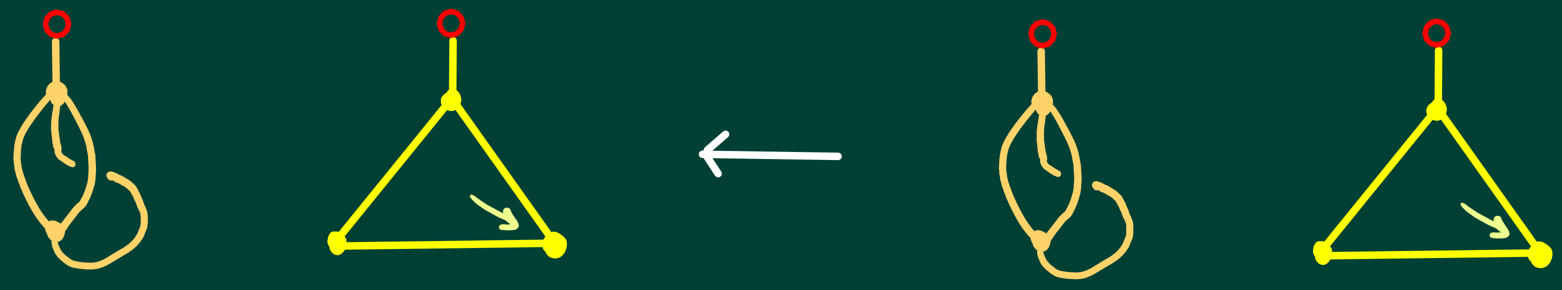

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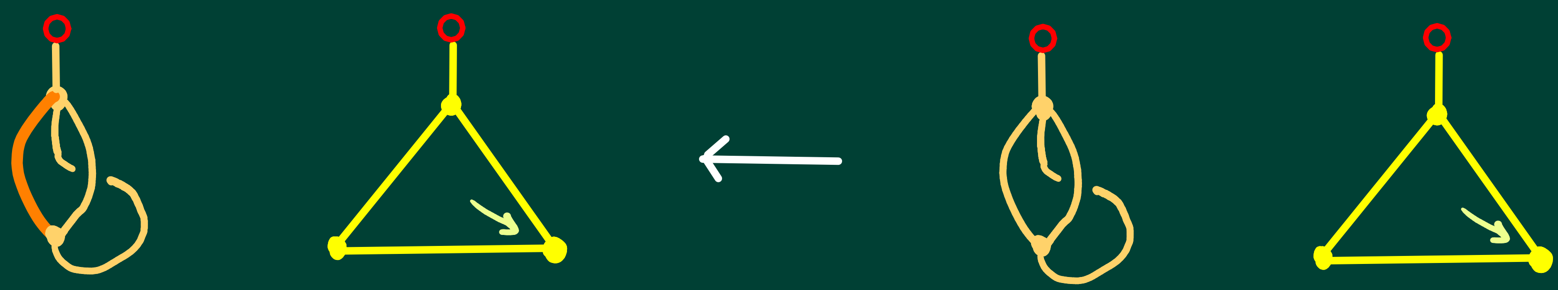

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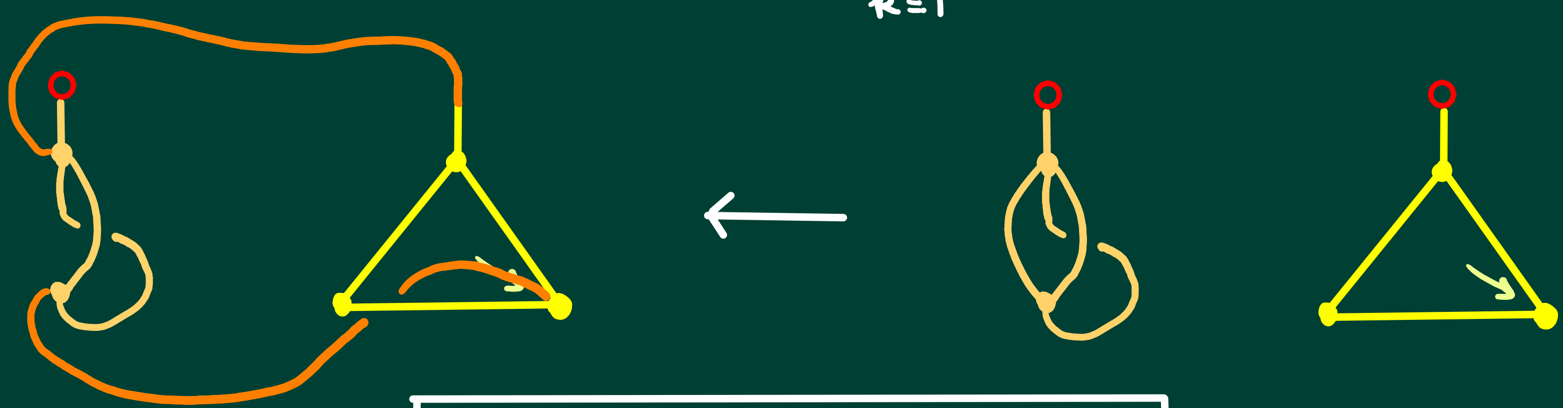

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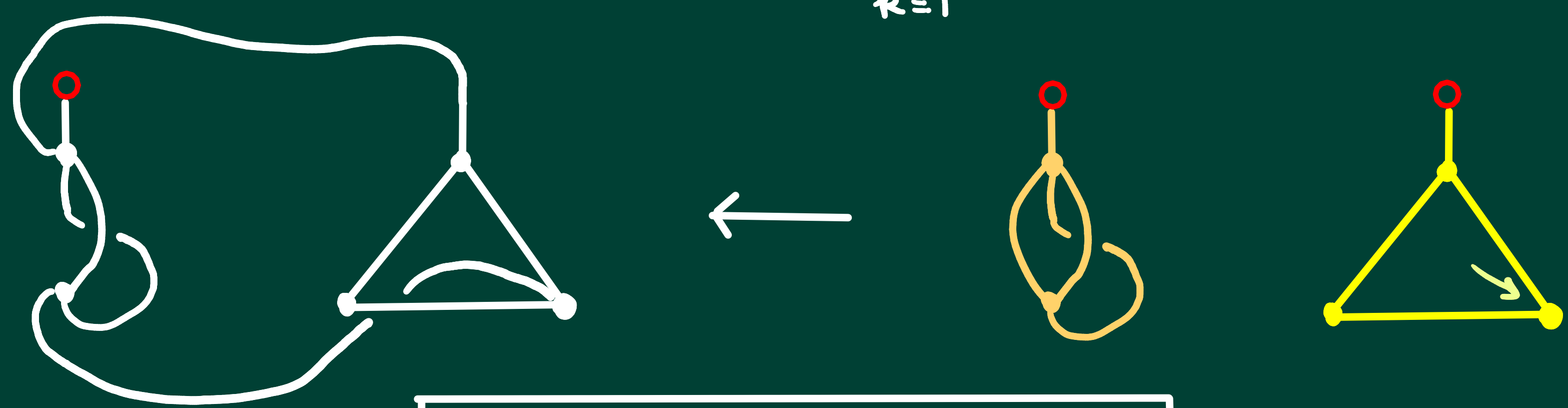

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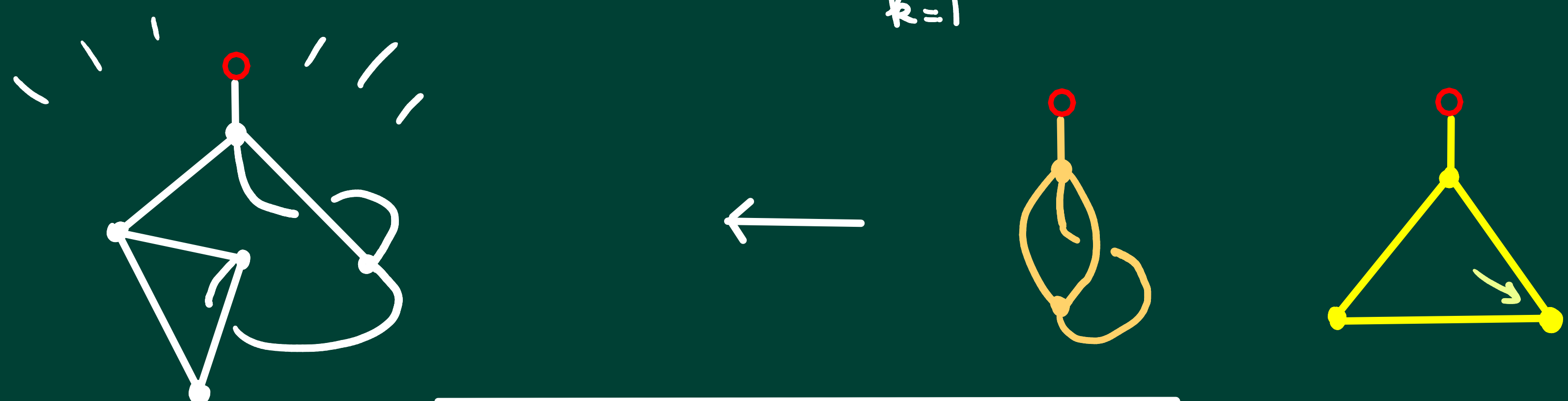

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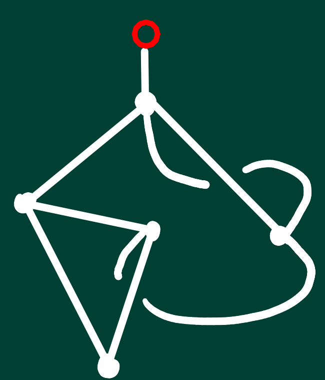

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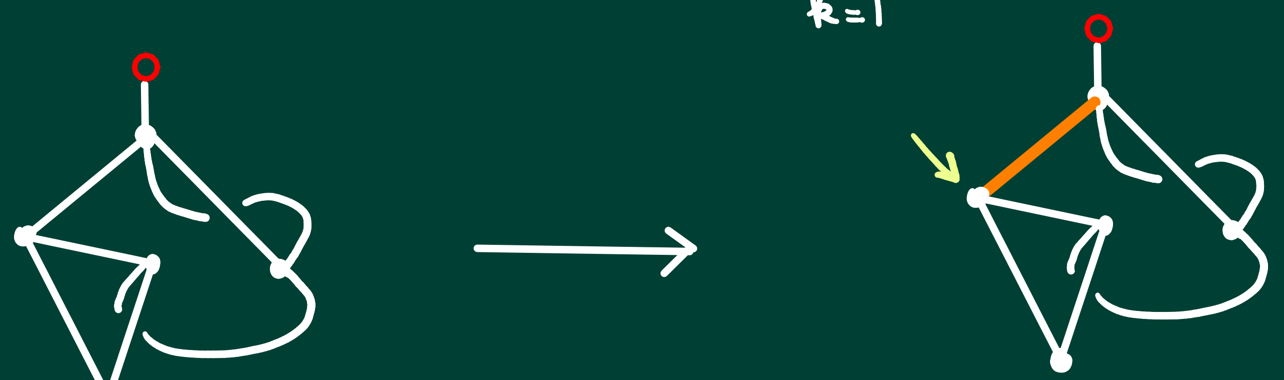

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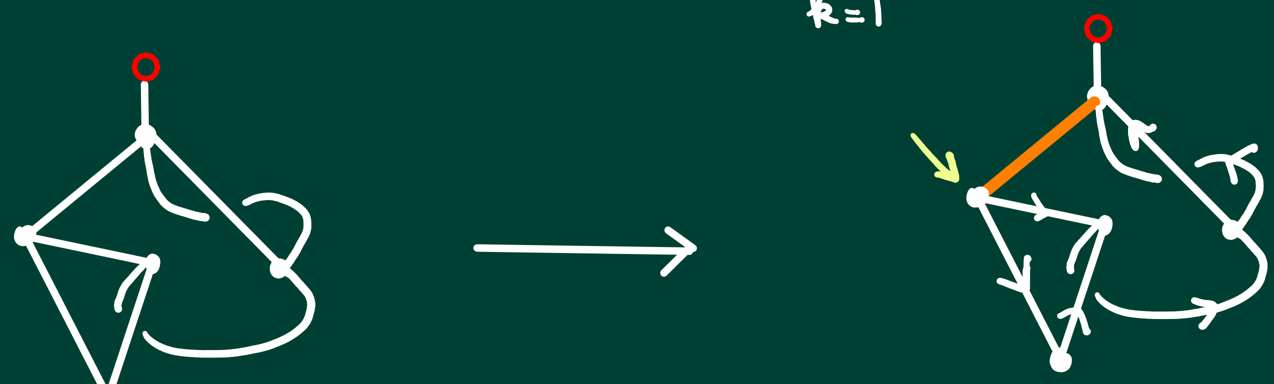

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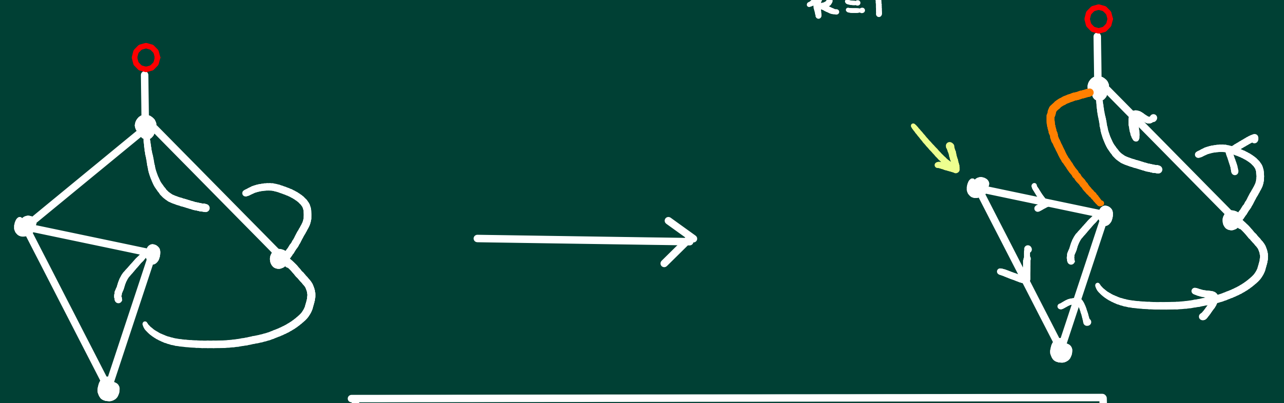

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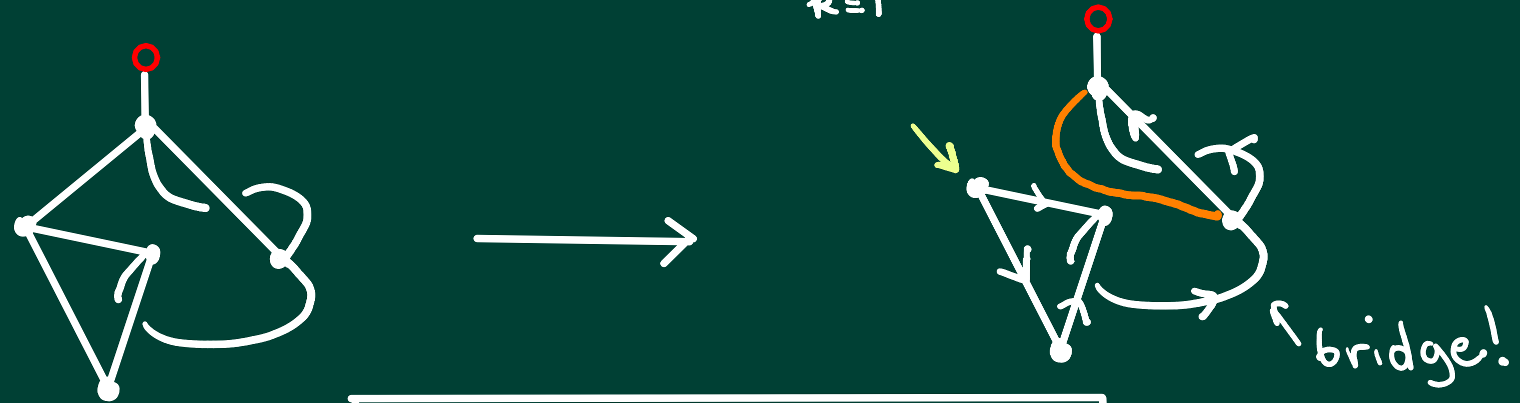

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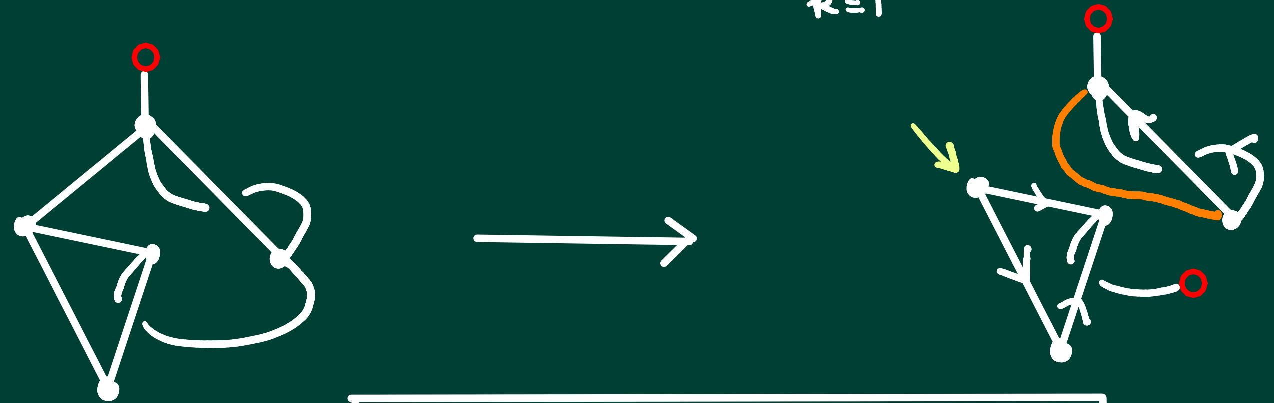

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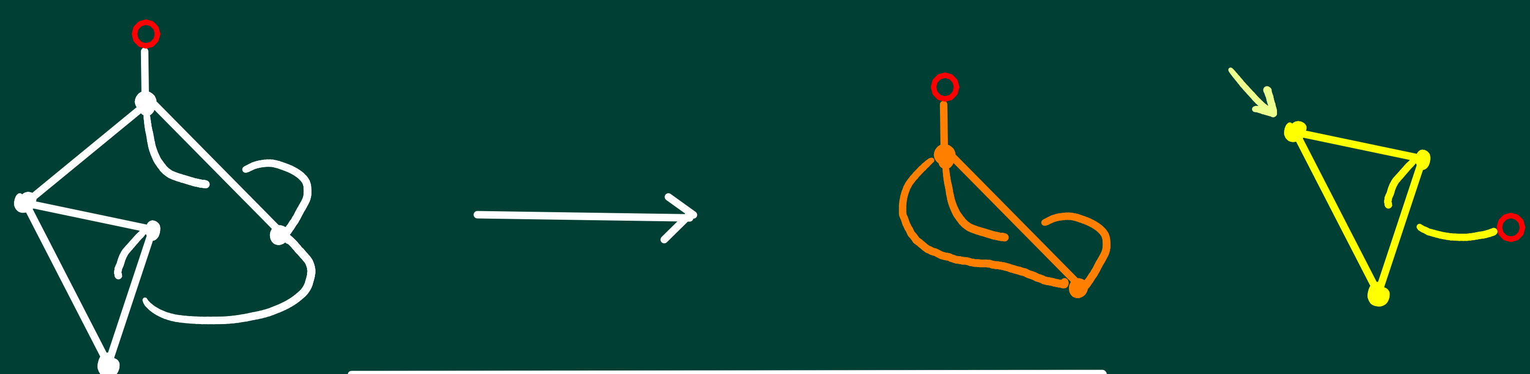

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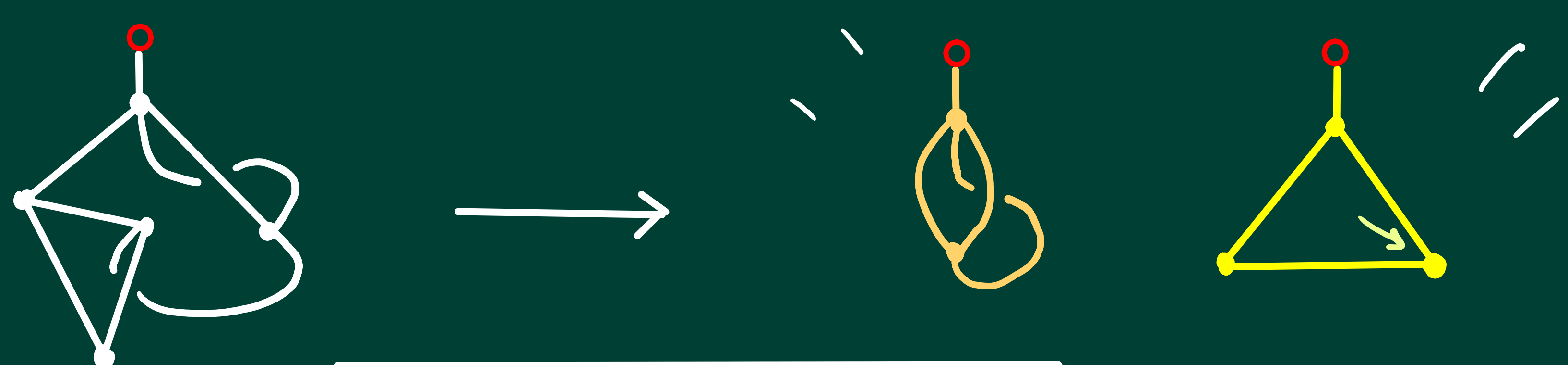

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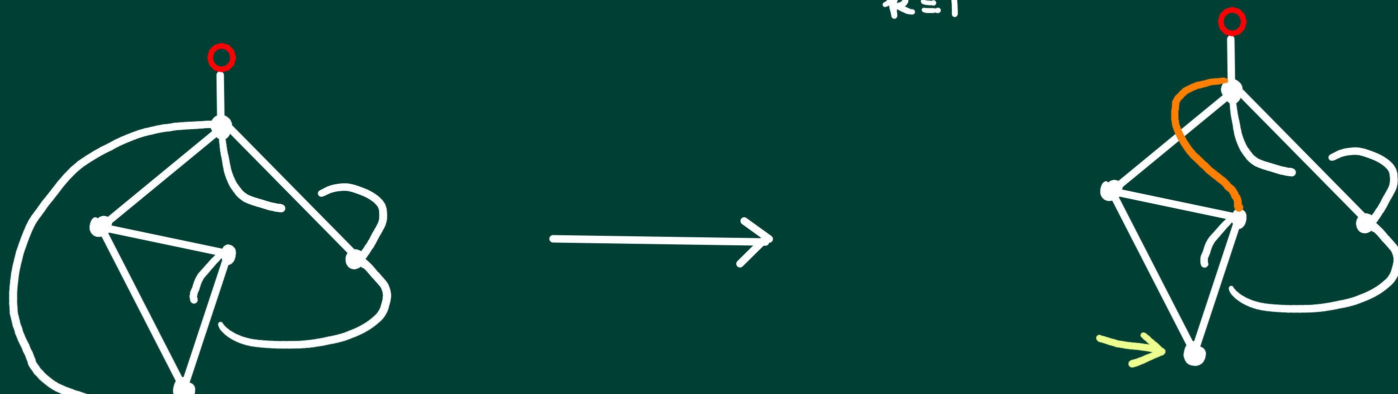

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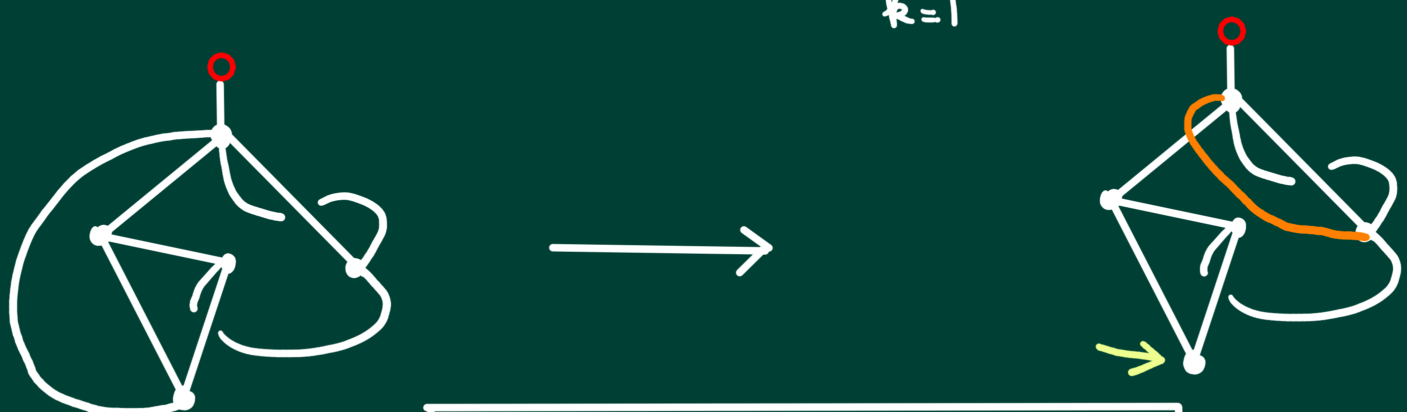

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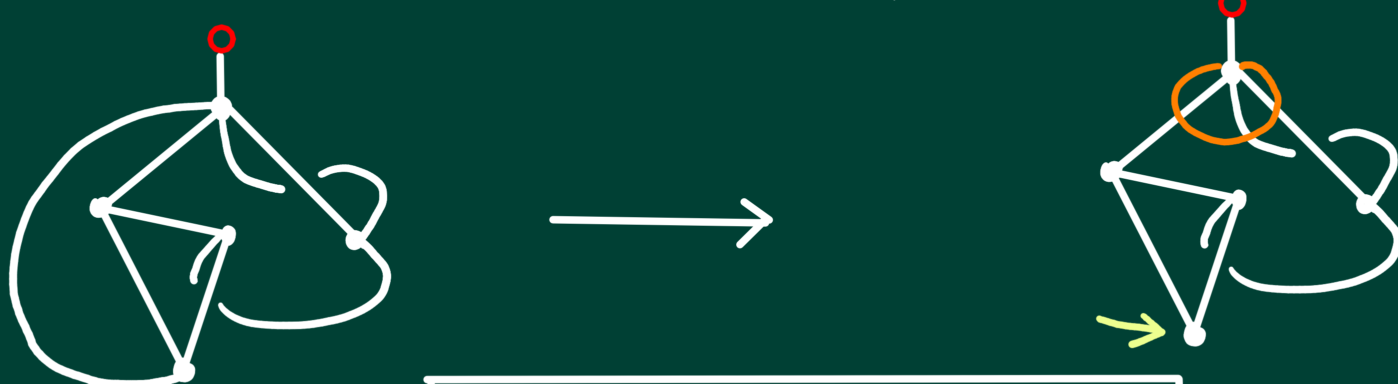

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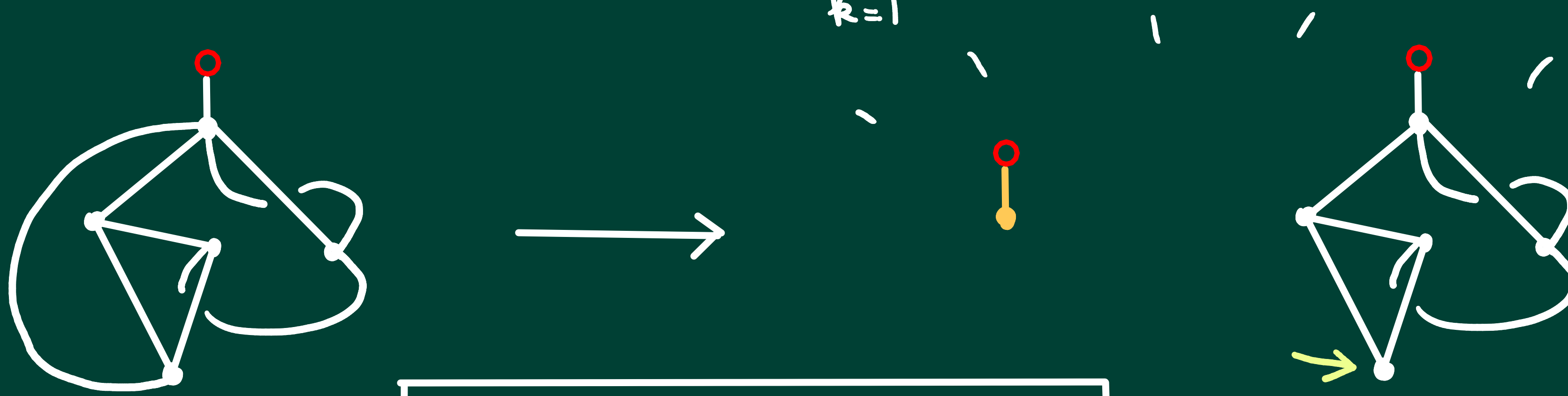

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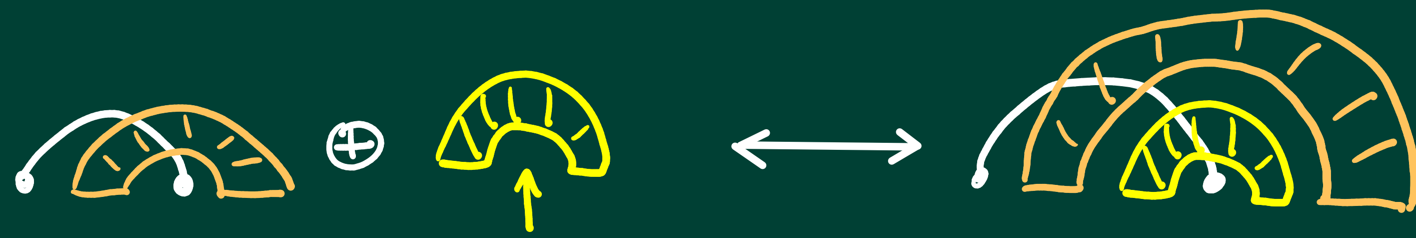

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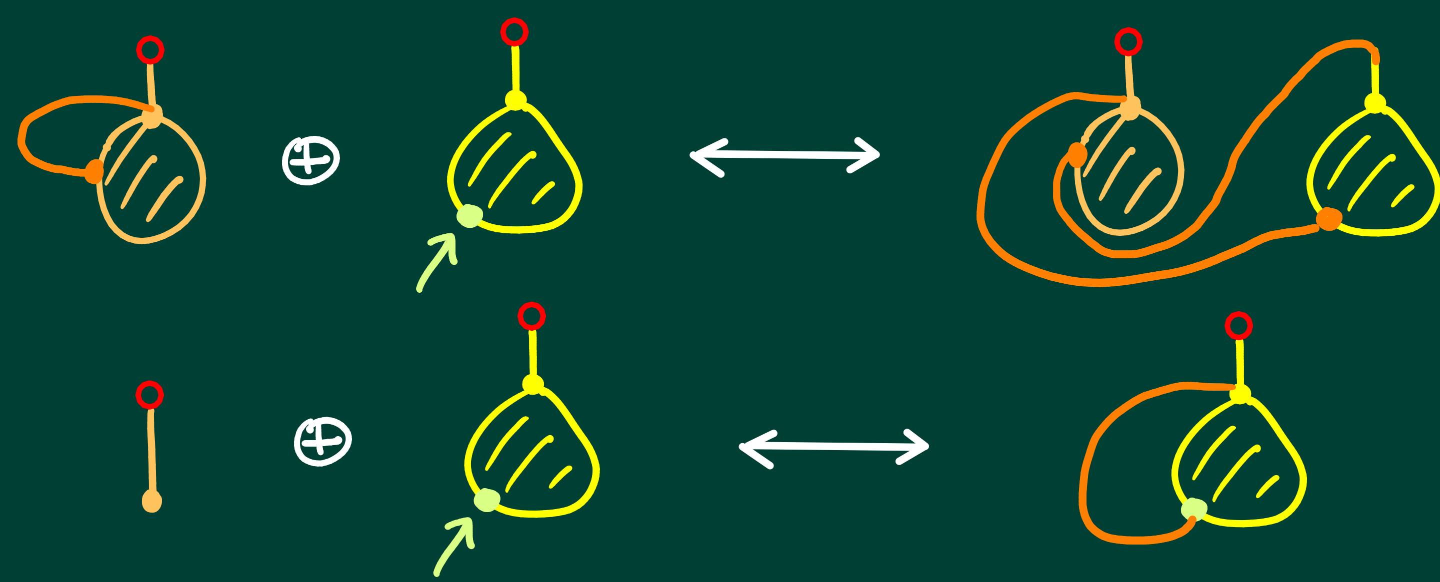

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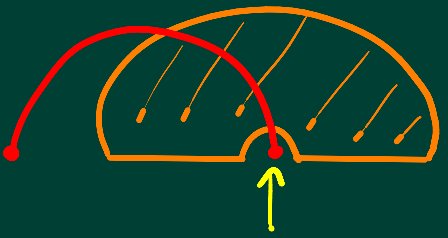
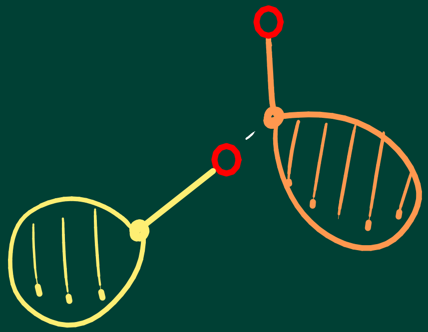
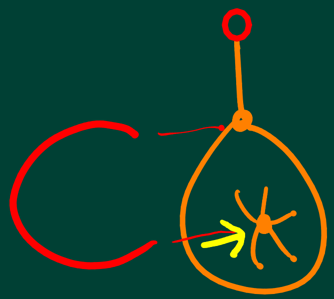
indecomposable diagram =  or  or 

Decomposition of maps: [Arquès - Béraud]

map =  or  or 

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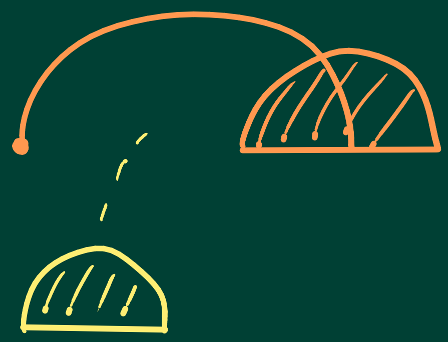
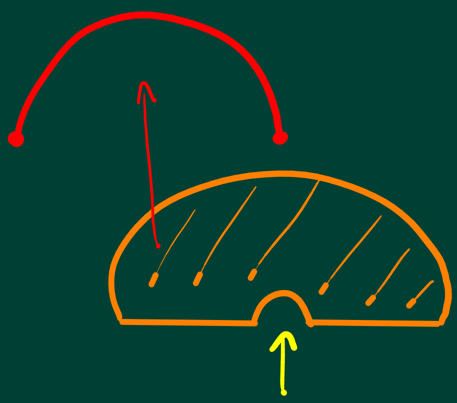
indecomposable diagram =  or  or 

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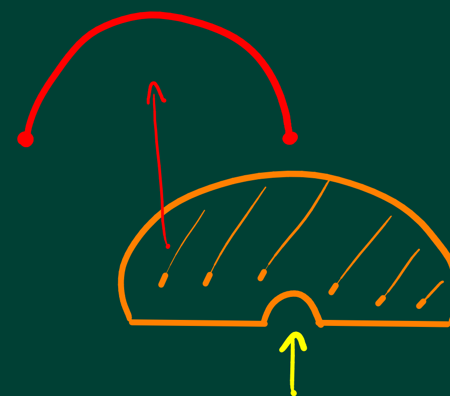
indecomposable diagram =  or  or 

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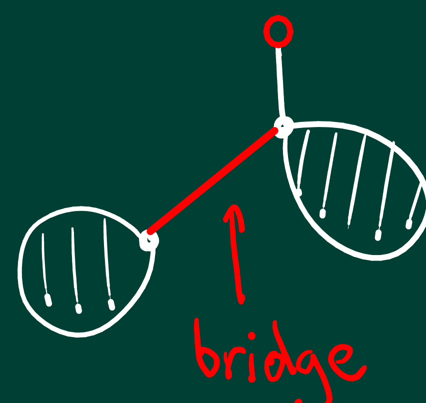
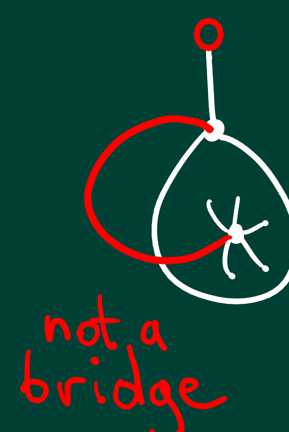
map =  or 

or 

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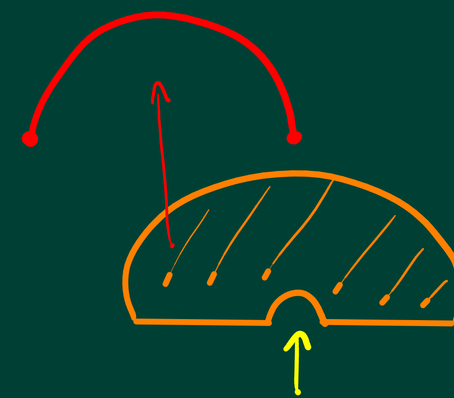
indecomposable diagram =  or  or 

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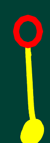
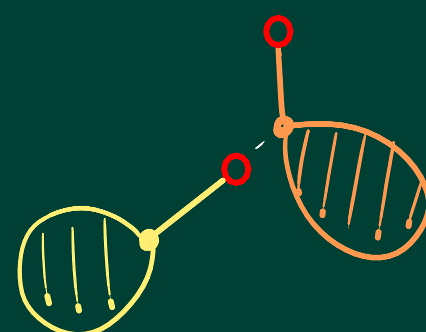
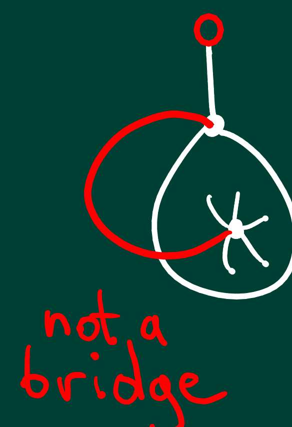
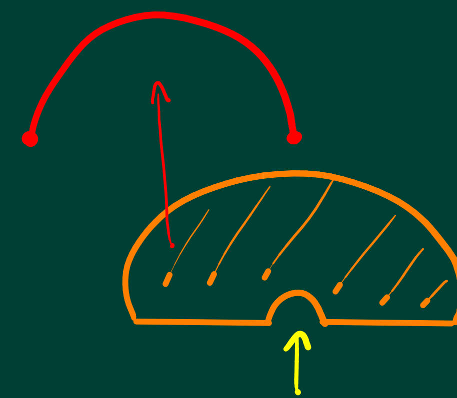
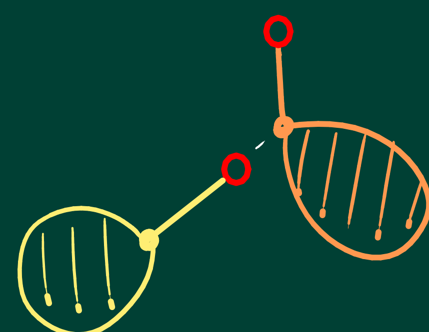
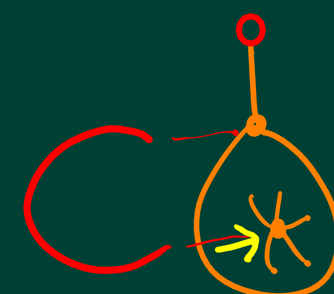
map =  or  or 

BETWEEN INDECOMPOSABLE DIAGRAMS AND MAPS

Decomposition of indecomposable diagrams:

indecomposable diagram =  or  or 

Decomposition of maps: [Arquès - Béraud]

map =  or  or 

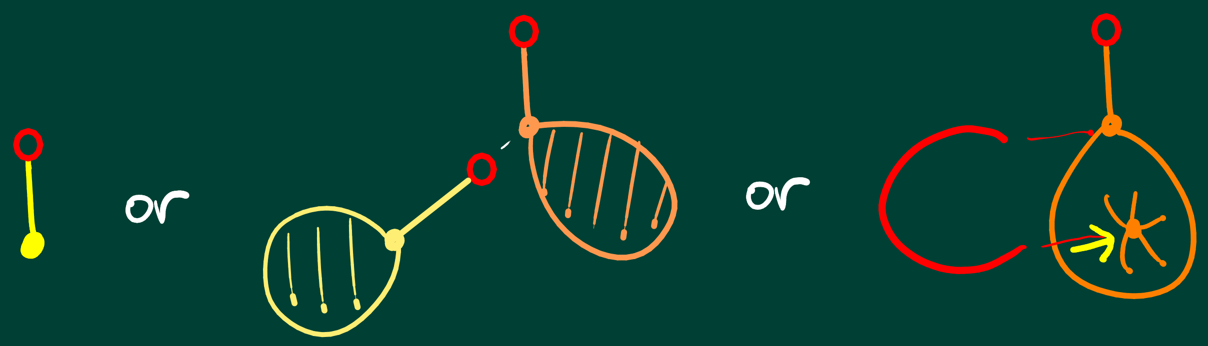
BETWEEN INDECOMPOSABLE DIAGRAMS AND MAPS

Decomposition of indecomposable diagrams:

indecomposable diagram =

Decomposition of maps: [Arquès - Béraud]

map =



Same decomposition = same numbers!

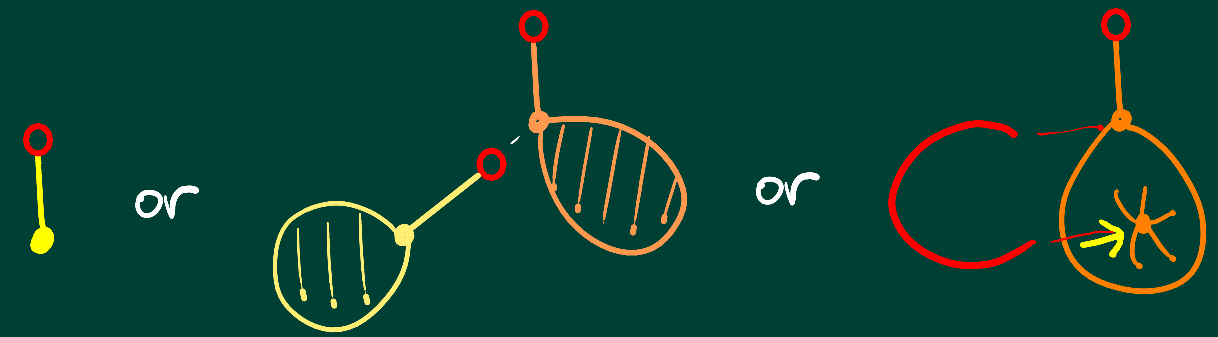
BETWEEN INDECOMPOSABLE DIAGRAMS AND MAPS

Decomposition of indecomposable diagrams:

indecomposable diagram =

Decomposition of maps: [Arquès - Béraud]

map =



Same decomposition = same numbers! But where is the bijection?

THE BIJECTION

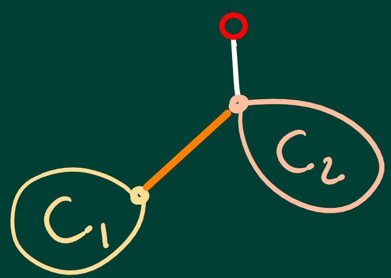
$$\text{maps} \xmapsto{\ \ \phi\ \ } \begin{array}{c}\text{(indecomposable)}\\ \text{diagrams}\end{array}$$

THE BIJECTION

maps $\underset{\phi}{\longmapsto}$ (indecomposable) diagrams

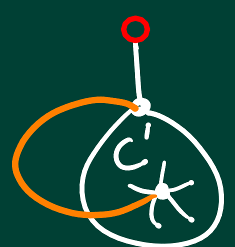
$\longrightarrow$

bridge



$\longrightarrow$

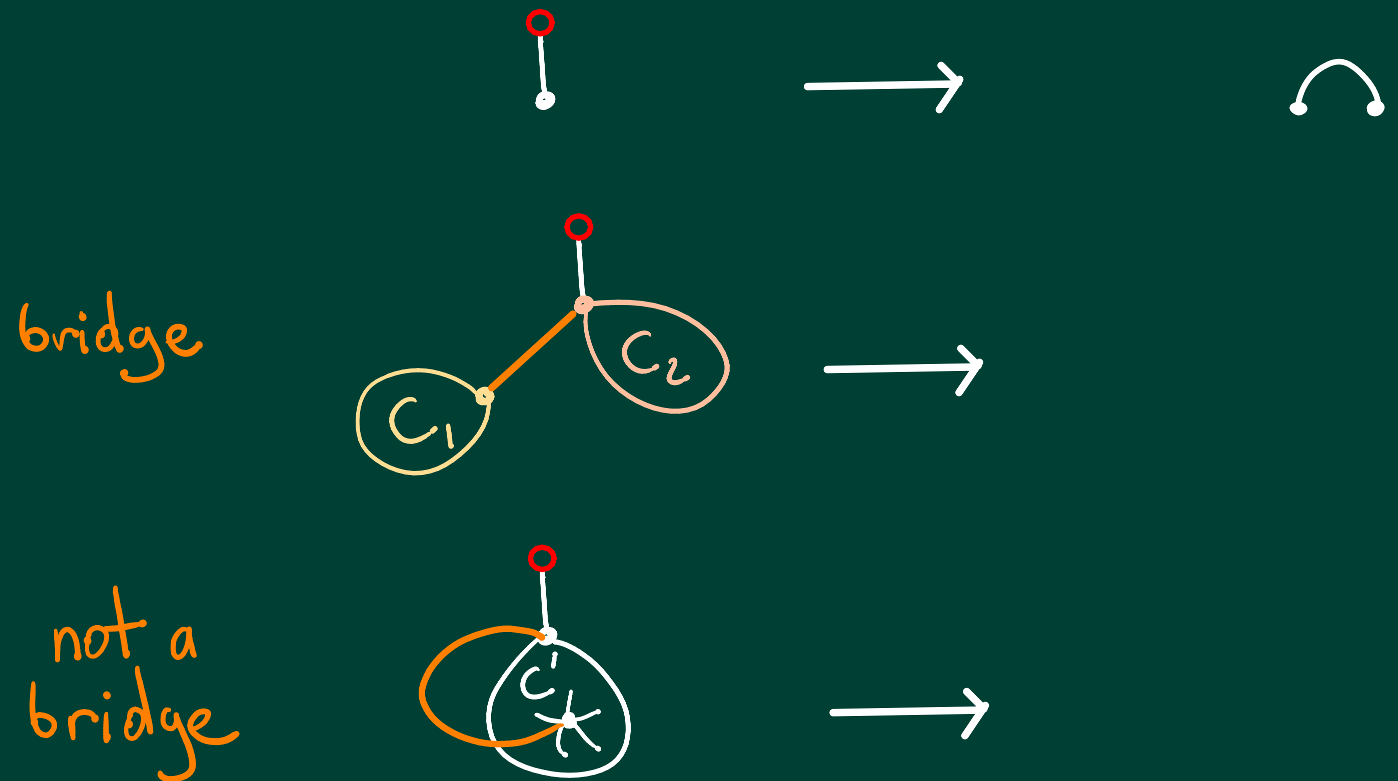
not a bridge



$\longrightarrow$

THE BIJECTION

maps $\overset{\phi}{\longmapsto}$ (indecomposable) diagrams

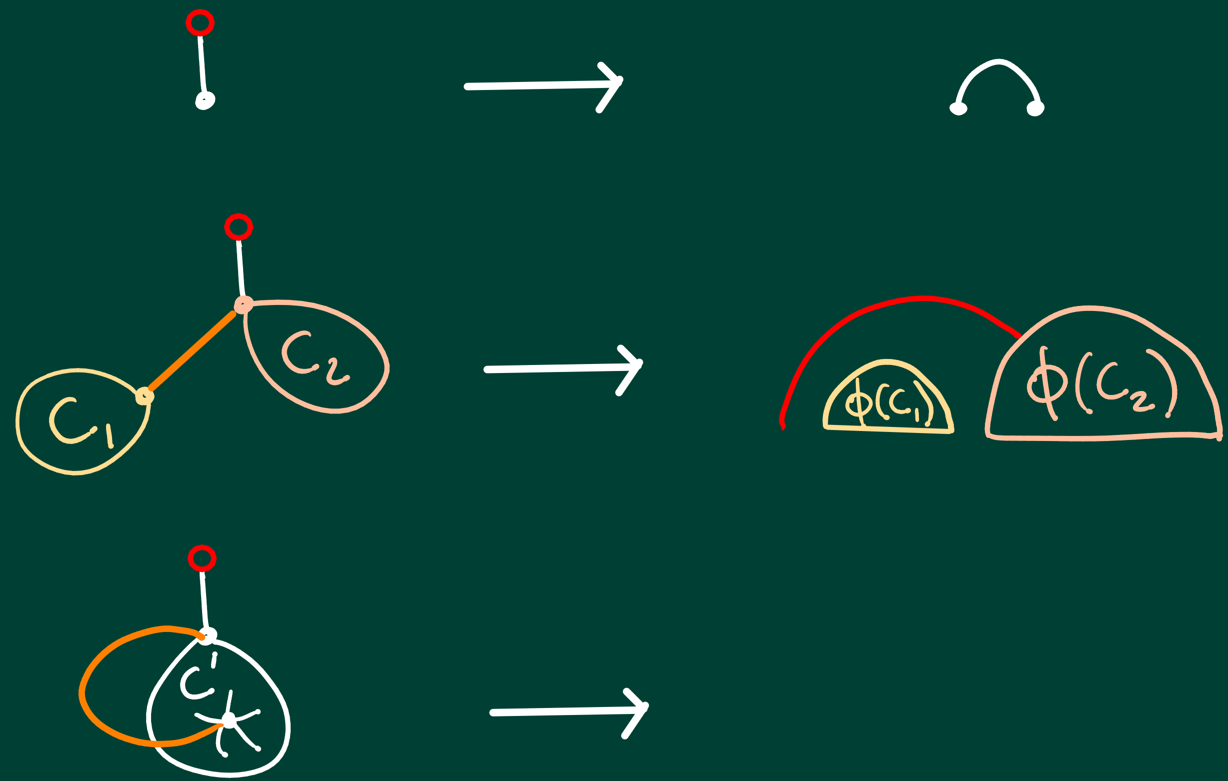


THE BIJECTION

$$\text{maps} \xmapsto{\ \phi\ } \text{(indecomposable) diagrams}$$

bridge

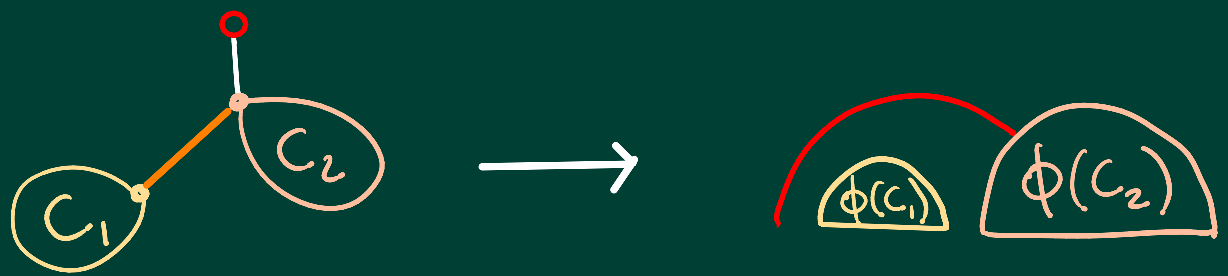
not a bridge



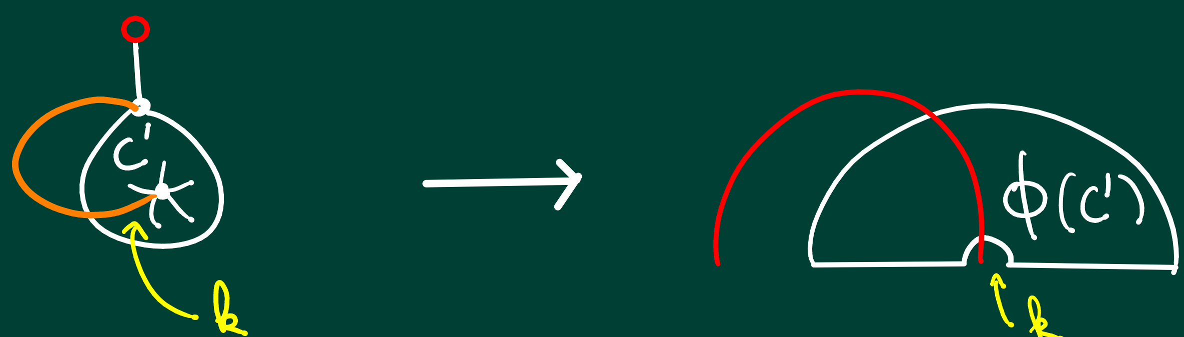
THE BIJECTION

maps $\underset{\phi}{\longmapsto}$ (indecomposable) diagrams

bridge



not a bridge



We label the corners from 1 to n

SOME PROPERTIES OF THE BIJECTION

[C. Yeats Zeilberger]

<u>maps</u> $\longleftrightarrow$ <u>indecomposable diagrams</u>

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<u>maps</u>	$\longleftrightarrow$	<u>indecomposable diagrams</u>
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leaves	$\longleftrightarrow$	isolated chords

SOME PROPERTIES OF THE BIJECTION

[C. Yeats Zeilberger]

maps ⟷ indecomposable diagrams

bridgeless maps ⟷ connected diagrams

leaves ⟷ isolated chords

planar maps ⟷ indecomposable diagrams avoiding 

SOME PROPERTIES OF THE BIJECTION

<u>maps</u>	$\xleftrightarrow{\text{[C. Yeats Zeilberger]}}$	<u>indecomposable diagrams</u>
bridgeless maps	$\longleftrightarrow$	connected diagrams
leaves	$\longleftrightarrow$	isolated chords
planar maps	$\longleftrightarrow$	indecomposable diagrams avoiding 
? ? ?	$\longleftrightarrow$	terminal chords

SOME PROPERTIES OF THE BIJECTION

[C. Yeats Zeilberger]

<u>maps</u> ⟷ <u>indecomposable diagrams</u>

bridgeless maps ⟷ connected diagrams

leaves ⟷ isolated chords

planar maps ⟷ indecomposable diagrams

avoiding 

vertices! ⟷ terminal chords

APPLICATION TO PERTURBATIVE QFT

Theorem [Marie, Yeats] [Hihn, Yeats]

The Dyson-Schwinger equation

$$G(x,L) = 1 - \sum_{k\geq 1} x^k G(x,\partial_{-p})^{1-sk} (e^{-Lp}-1) F_k(p)$$

has for solution

$$G(x,L) = 1 - \sum_{\substack{C \text{ decorated} \\ \text{connected chord diagram} \\ \text{such that } t_1 < t_2 < \dots < t_k \\ \text{are the positions of the terminal chords}}} w(C) \left(\sum_{i=1}^{t_1} f_{d(t_1), t_1 - i} \frac{(-L)^i}{i!} \right) \times \prod_{c \text{ non terminal}} f_{d(c),0} \times \prod_{i=1}^{k-1} f_{d(t_i), t_i - t_{i-1}} \, x^{\|C\|}$$

where $F_k(p) = f_{k,0}\, p^{-1} + f_{k,1} + f_{k,2}\, p + f_{k,3}\, p^2 + \dots$ = regularized Feynman integral of the primitive graphs of size k

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$$G(x,L) = 1 - \sum_{\substack{C \text{ decorated} \\ \text{bridgeless maps} \\ \text{such that } t_1<t_2<\ldots<t_k \\ \text{are the positions of the vertices}}} w(C)\left(\sum_{i=1}^{t_1} f_{d(t_1),t_1-i}\frac{(-L)^i}{i!}\right)\times\prod_{c \text{ non terminal}} f_{d(c),0}\times\prod_{i=1}^{k-1} f_{d(t_i),t_i-t_{i-1}}\, x^{\|C\|}$$

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APPLICATION TO PERTURBATIVE QFT

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Some physical equation

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has for solution

$$\sum \quad \text{BAD BAD FORMULA}$$

bridgeless maps

vertices

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"New" proof: Now the recurrence can be explained combinatorially.

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"New" proof: Now the recurrence can be explained combinatorially.

magic = science?

APPLICATION TO ASYMPTOTICS

<u>Ex</u> under the uniform distribution:

<u>Theorem</u> [Stein-Everett]

> A diagram is connected with proba $\xrightarrow[n\to+\infty]{} \frac{1}{e}$

can be (almost) straightforwardly translated by

<u>Theorem</u>

> A map is bridgeless with proba $\xrightarrow[n\to+\infty]{} \frac{1}{e}$

APPLICATION TO ASYMPTOTICS

Ex 2 under the uniform distribution:

Theorem

A random map with n edges has $\sim \ln(n)$ vertices.

can be (almost) straightforwardly translated by

Theorem

A random connected diagram with n chords has $\sim \ln(n)$ terminal chords.

APPLICATION TO LAMBDA - CALCULUS

ANOTHER STORY...

Ils vécurent heureux et eurent beaucoup de papiers...

THE END