The Tutte Polynomial and Planar Maps

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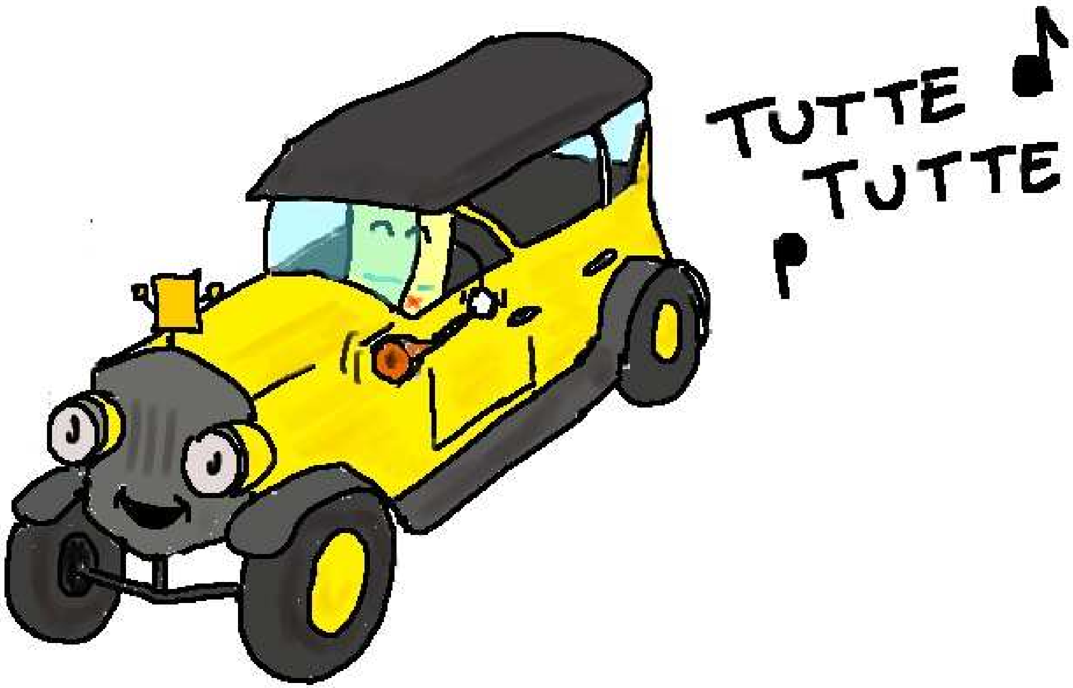


THE TUTTE POLYNOMIAL

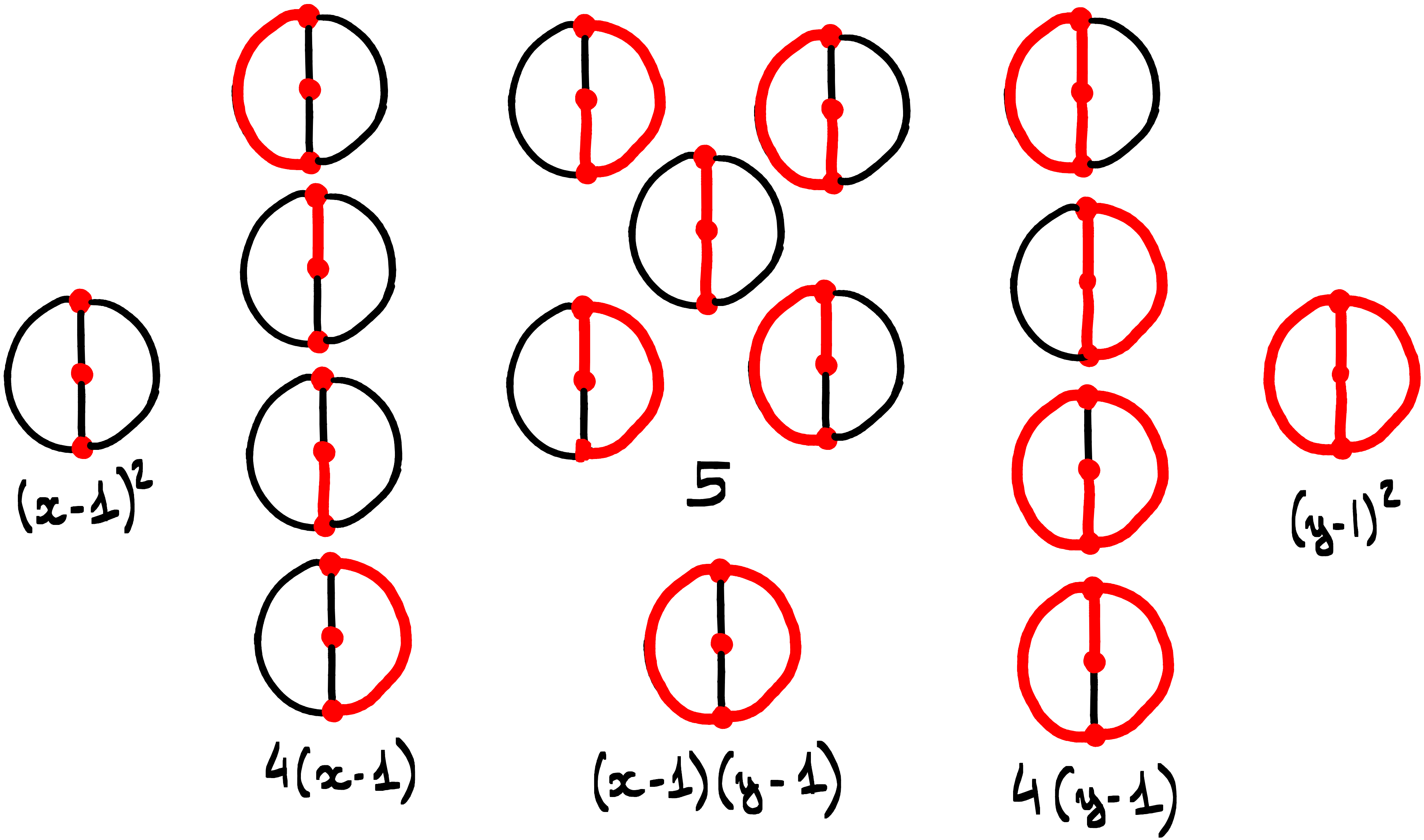
The Tutte polynomial of a connected graph G =

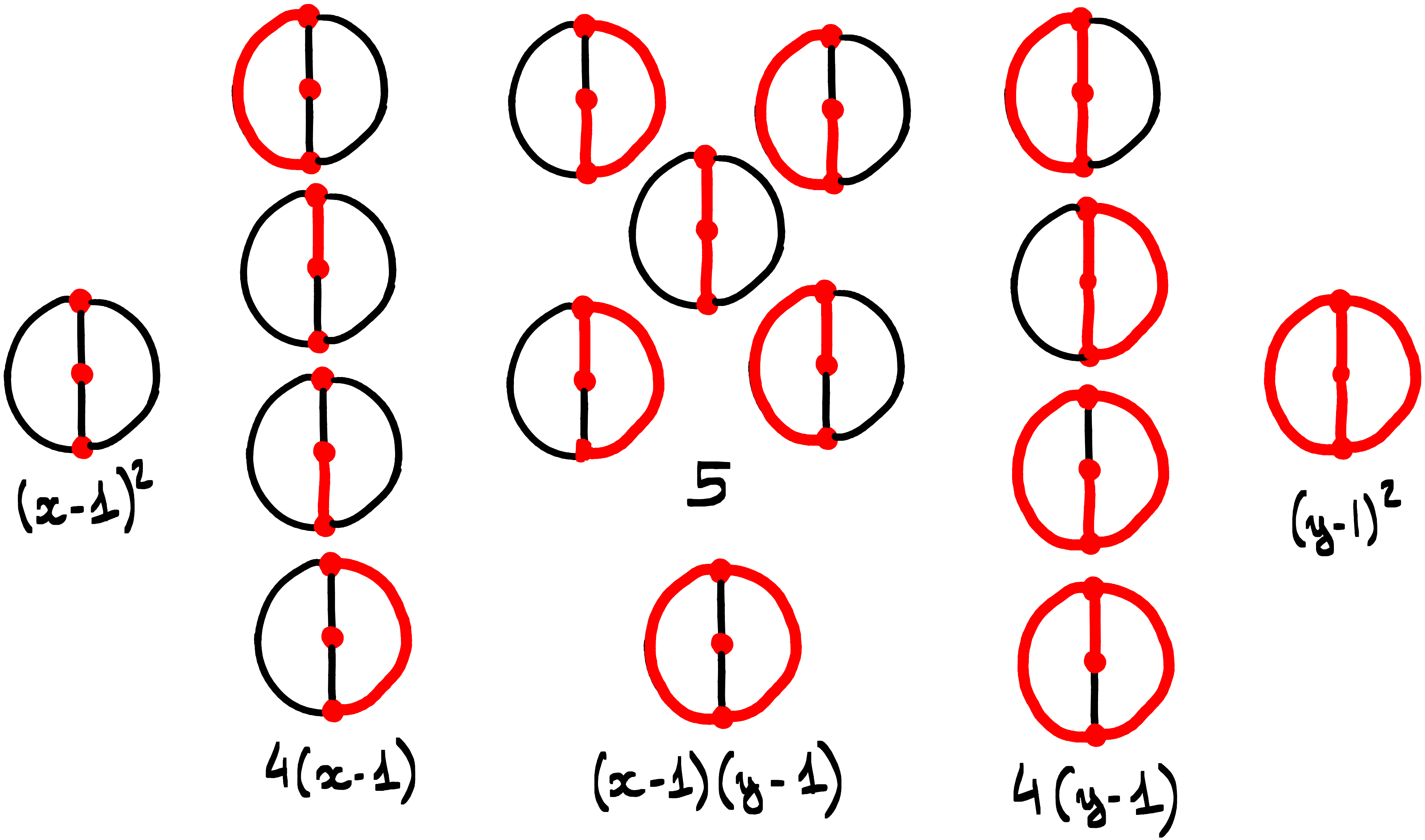
$$T_G(x,y) = \sum_{S \text{ subgraph of } G} (x-1)^{cc(S)-1} (y-1)^{cycl(S)}$$

$cc(S)$ = number of connected components of S.
$cycl(S)$ = cyclomatic number of S
= minimal number of edges we need to remove from S to obtain an acyclic graph.

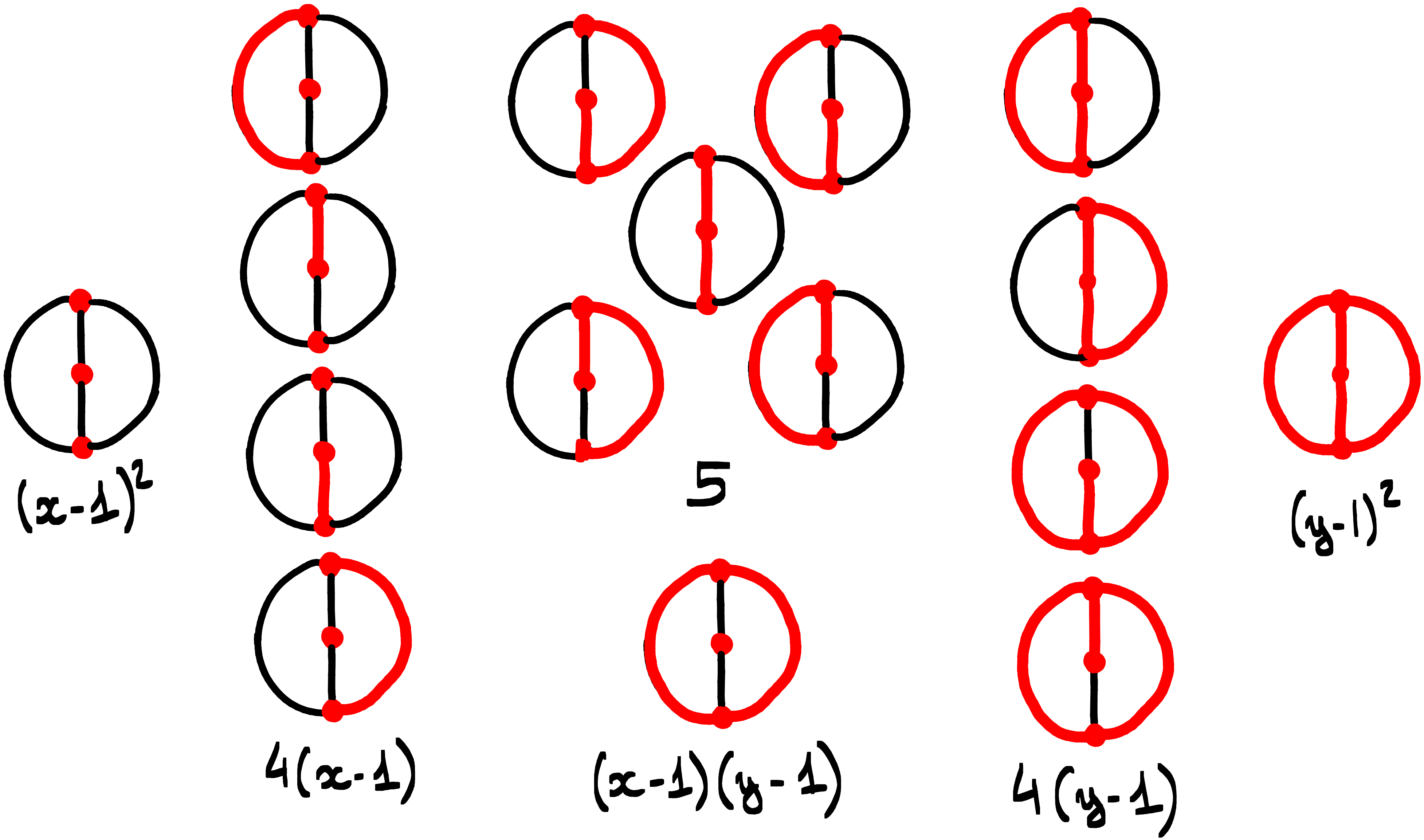
Prop:

$$T_G(x,y) \in \mathbb{N}[x,y]$$


$(x-1)^2$
$4(x-1)$
5
$(x-1)(y-1)$
$4(y-1)$
$(y-1)^2$



$$T_G(x,y) = (x-1)^2 + 4(x-1) + (x-1)(y-1) + 5 + 4(y-1) + (y-1)^2$$



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$$= x^2 + x + xy + y + y^2.$$

INTEREST

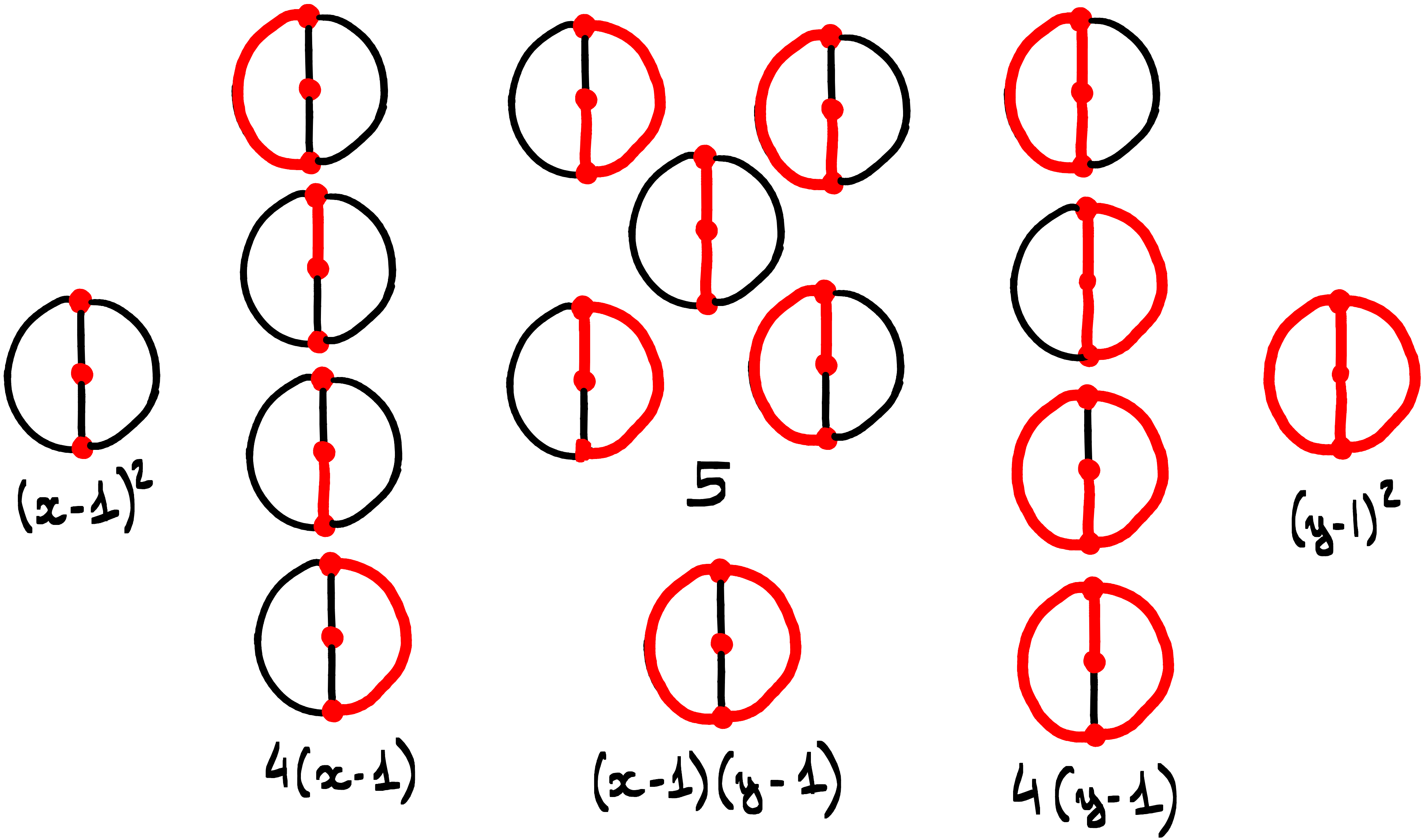
→ Numerous interesting specializations

→ Closely related to the Potts model.
(statistical physics)

→ ...

Spanning forests in planar maps

with Mireille Bousquet-Mélou



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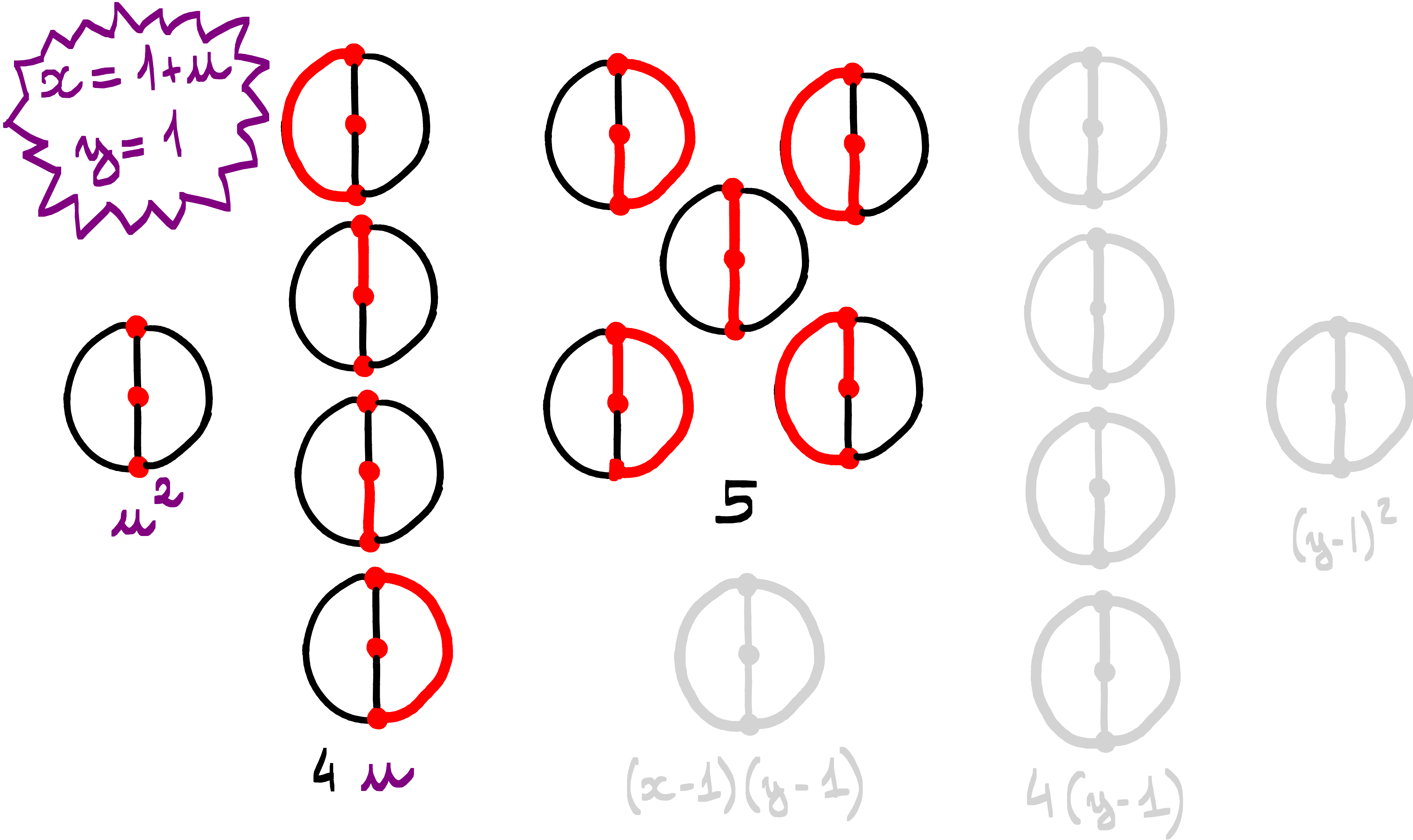
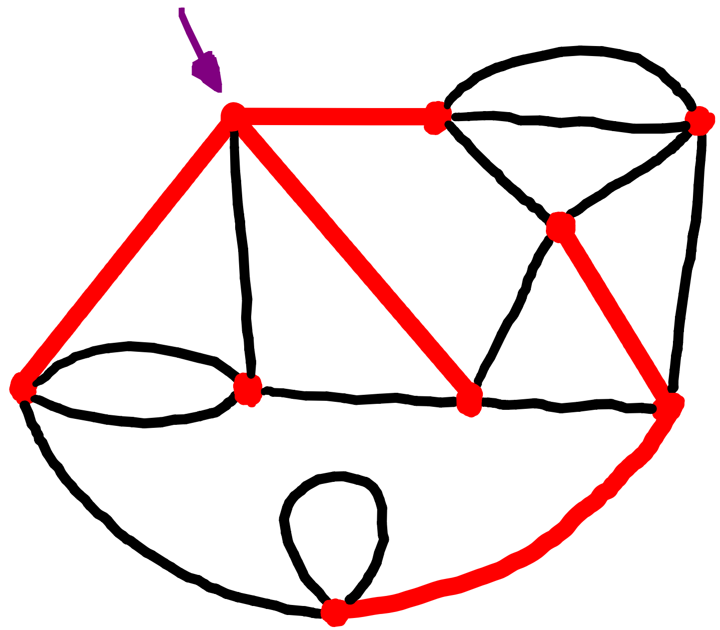


$u\,T_G(1+u, 1) =$ number of spanning forests of G with a weight u per connected component

FORESTED MAPS : DEFINITION

Forested map (M,F) =
Rooted plane map M
with a spanning forest F.



$$F(z,u) = \sum_{\substack{(M,F)\ \text{4-valent}^* \\ \text{forested map}}} z^{\#\text{faces}}\, u^{\#\text{components}-1}$$

$$= \sum_{\substack{M\ \text{4-valent}^* \\ \text{rooted map}}} T_M(1+u,1)\, z^{\#\text{faces}}$$

*: or cubic or Eulerian or...

A DIFFERENTIAL EQUATION FOR F

Th: F is D-algebraic,

i.e. F is solution of some polynomial differential equation.

Remarks:

- The equation is BIG.
- F is not D-finite.

(i.e. F is not solution of any linear differential equation.)

ASYMPTOTIC BEHAVIOUR

Fix u as a real number. $\qquad f_n(u) = [z^n]\, F(z,u)$

$-1 \leq u < 0$	$u = 0$	$0 < u$
$f_n(u) \sim \dfrac{c_u\, \rho_u^{-n}}{n^3 \ln^2 n}$	$f_n(u) \sim \dfrac{c_u\, \rho_u^{-n}}{n^3}$	$f_n(u) \sim \dfrac{c_u\, \rho_u^{-n}}{n^{\underbrace{5/2}_{\text{standard}}}}$
New "Universality class" for maps	maps with a spanning tree [Mullin]	

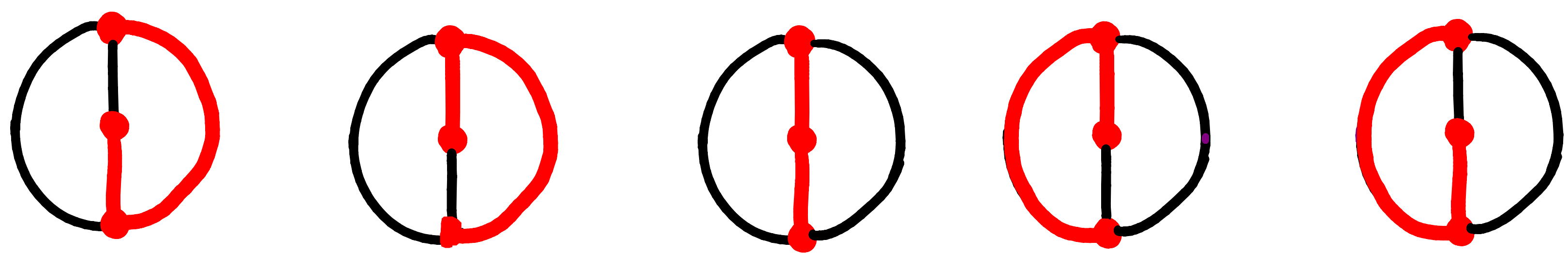
A General Framework for Activities

PRINCIPLE

Map each spanning tree onto a set of edges, called "active" edges such that

$$T_G(x,y) = \sum_{T \text{ spanning tree of } G} x^{i(T)}\, y^{e(T)}$$

where $i(T)$ = number of active edges inside T
$e(T)$ = number of active edges outside T.



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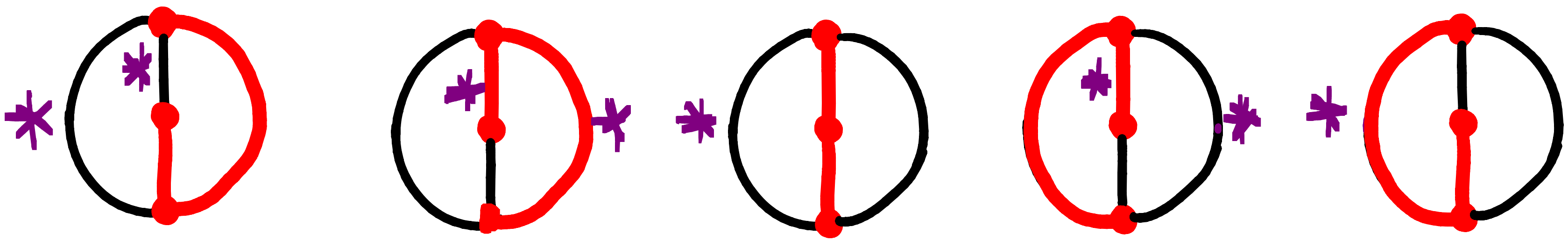

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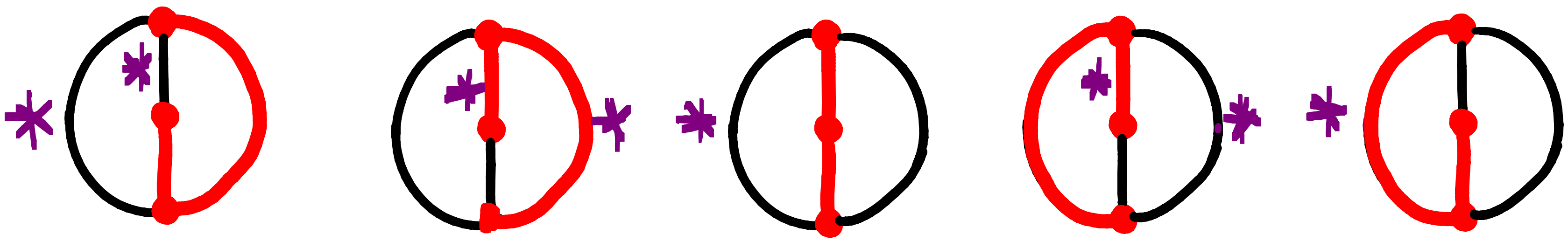
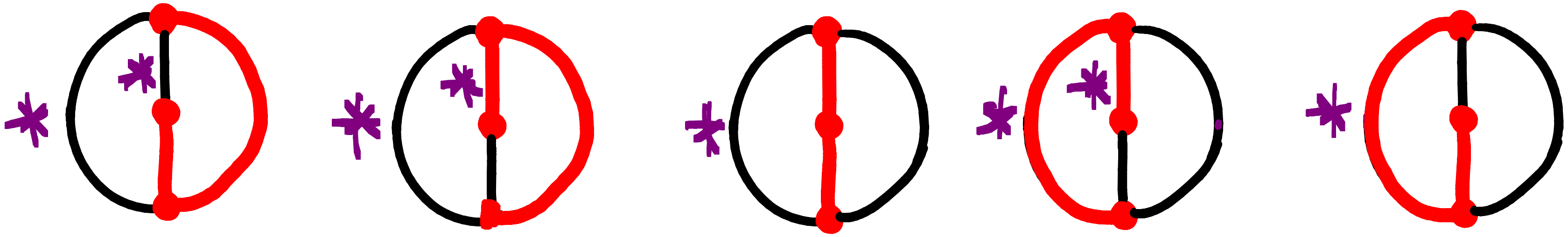


y^2 x^2 y xy x

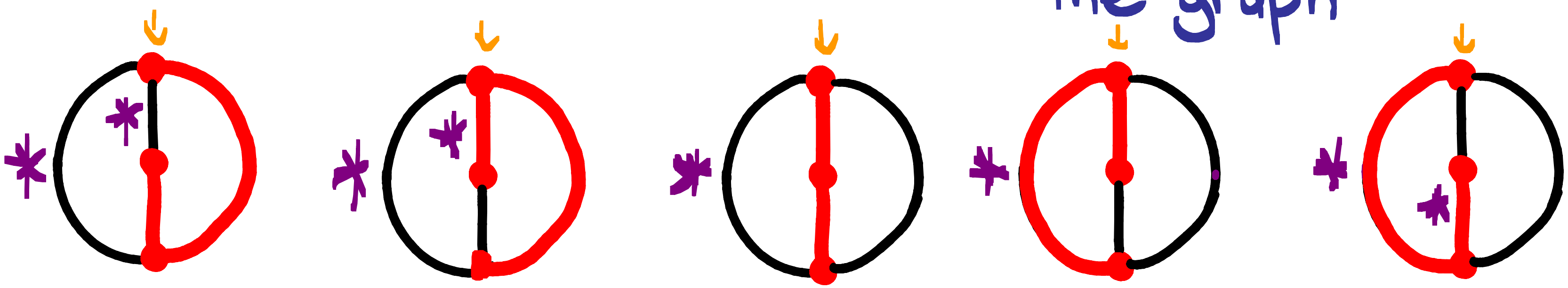
Activities

Some known activities :

- Tutte's activity   requires a linear ordering on the edge set.



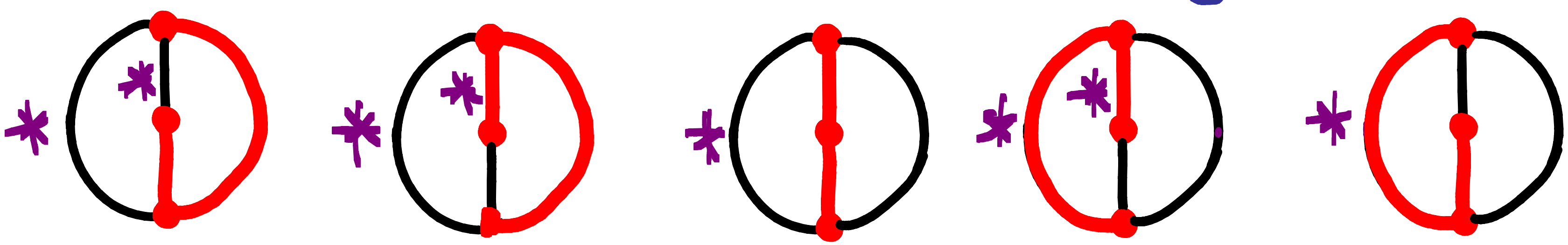
- Bernardi's activity   requires an embedding of the graph



ACTIVITIES

We define a meta-activity called Δ-activity (which requires an object named "decision tree") that gathers all the known activities as:

- Tutte's activity requires a linear ordering on the edge set.



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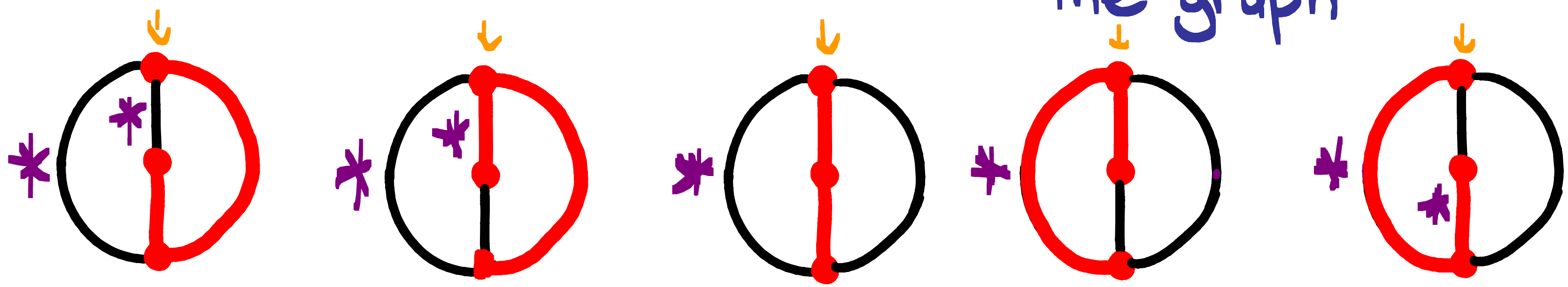

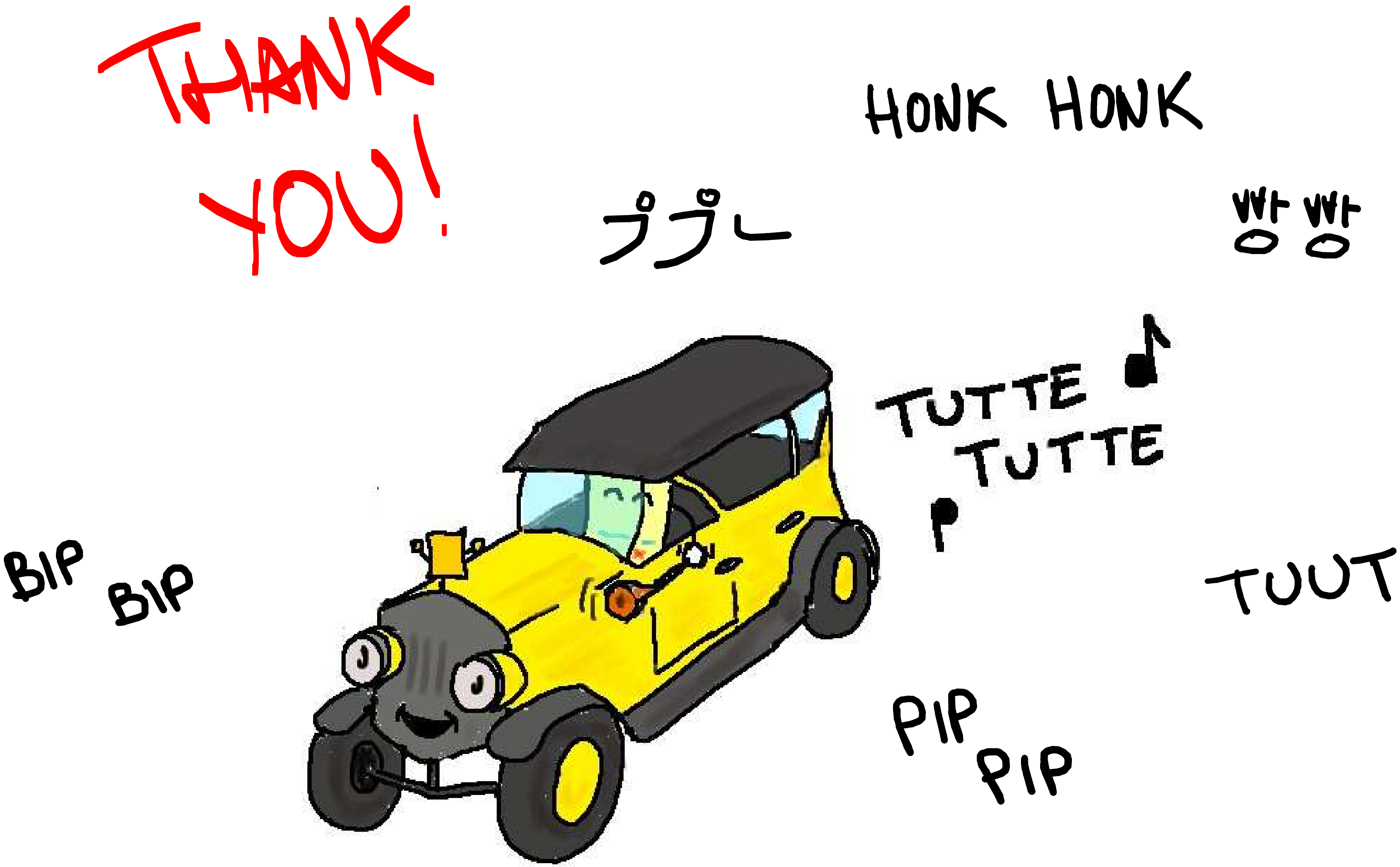
THANK YOU!
HONK HONK
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빵빵
TUTTE TUTTE
BIP BIP
TUUT
PIP PIP