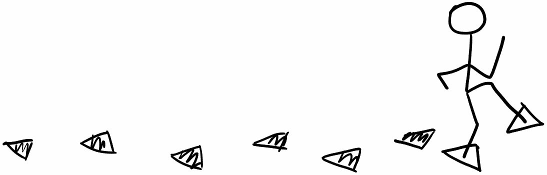
SOLVING MORTIMER & PRELLBERG'S CONJECTURE:

BIJECTION(S) BETWEEN MOTZKIN PATHS AND TRIANGULAR WALKS



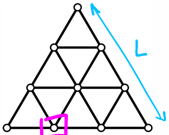
Julien COURTIEL (GREYC, Univ. Caen Normandie, France)

in collaboration with Andrew ELVEY-PRICE (Tours, France)
and Irène MARCOVICI (Nancy, France)

WALKING IN TRIANGLATTAN

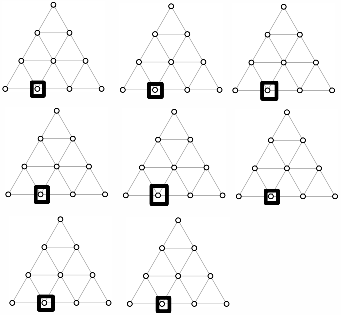
PART 1

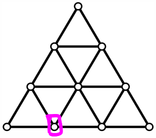
TWO WAYS OF WALKING IN A TRIANGLE



TRIANGLATTAN

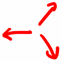
DIRECTIONS:

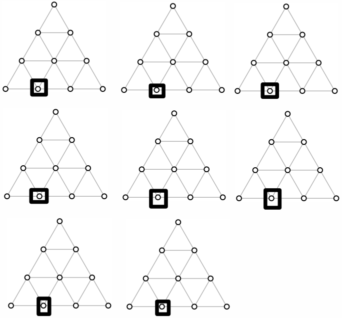




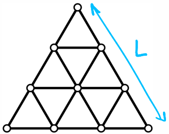
NATTALGNAIRT

DIRECTIONS:





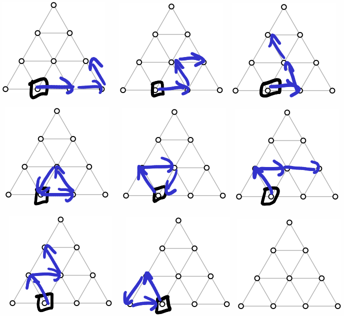
TWO WAYS OF WALKING IN A TRIANGLE

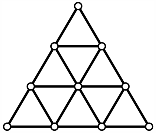


TRIANGLATTAN

DIRECTIONS:

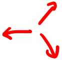
= forward paths



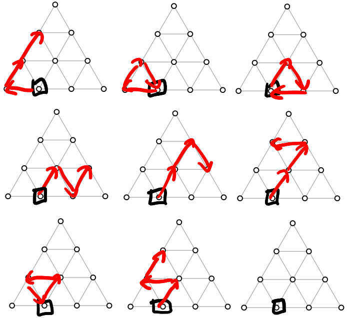


NATTALGNAIRT

DIRECTIONS:



= backward paths



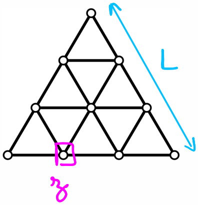
Theorem [Courtiel Elvey-Price Marcovici]

number of forward paths of length n starting from z

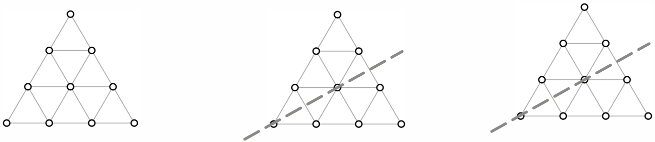
$=$

number of backward paths of length n starting from z

(for any z, n and L)

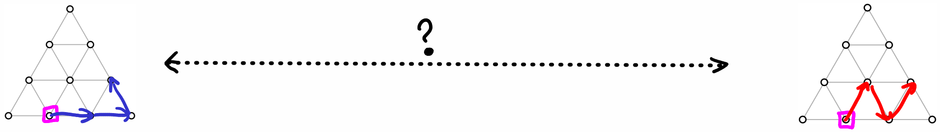


Isn't it obvious?



Caption: If two points are outlined in blue/red then the numbers of forward/paths with fixed length starting from these points are equal.

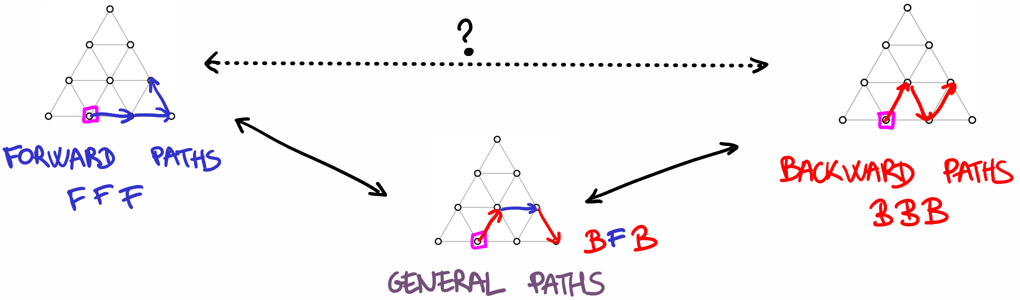
TRANSITIONAL OBJECTS



FORWARD PATHS

BACKWARD PATHS

TRANSITIONAL OBJECTS



Definitions

General path = path in Trianglattan using as steps:

$\rightarrow, \nwarrow, \swarrow, \leftarrow, \searrow, \nearrow.$

Direction vector of a general path of length n

$$= \text{word} \in \{F, B\}^n \text{ such that the } i^{th} \text{ letter } = \begin{cases} F & \text{if } i^{th} \text{ step} \in \{\nwarrow, \swarrow, \rightarrow\} \\ B & \text{if } i^{th} \text{ step} \in \{\leftarrow, \nearrow, \searrow\} \end{cases}$$

THE BIJECTION

REWRITTING RULES

LAST-STEP FLIPS

$$\Delta_1 \longleftrightarrow \overline{\Delta_3}$$

$$\Delta_2 \longleftrightarrow \overline{\Delta_1}$$

$$\Delta_3 \longleftrightarrow \overline{\Delta_2}$$

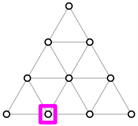
SWAPPING FLIPS

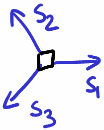
$$\Delta_i \, \overline{\Delta_j} \underset{i \neq j}{\longleftrightarrow} \overline{\Delta_j} \, \Delta_i$$

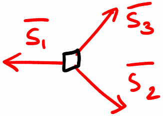
$$\Delta_1 \, \overline{\Delta_1} \longleftrightarrow \overline{\Delta_3} \, \Delta_3$$

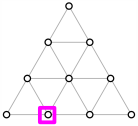
$$\Delta_2 \, \overline{\Delta_2} \longleftrightarrow \overline{\Delta_1} \, \Delta_1$$

$$\Delta_3 \, \overline{\Delta_3} \longleftrightarrow \overline{\Delta_2} \, \Delta_2$$



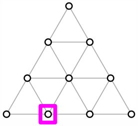


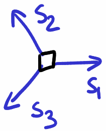


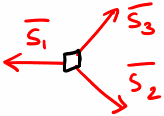


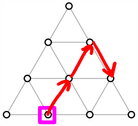
THE BIJECTION

REWRITTING RULES		
LAST-STEP FLIPS		
Δ_1	$\longleftrightarrow$	$\overline{\Delta_3}$
Δ_2	$\longleftrightarrow$	$\overline{\Delta_1}$
Δ_3	$\longleftrightarrow$	$\overline{\Delta_2}$
SWAPPING FLIPS		
$\Delta_i\,\overline{\Delta_j}$	$\longleftrightarrow$ $i \neq j$	$\overline{\Delta_j}\,\Delta_i$
$\Delta_1\,\overline{\Delta_1}$	$\longleftrightarrow$	$\overline{\Delta_3}\,\Delta_3$
$\Delta_2\,\overline{\Delta_2}$	$\longleftrightarrow$	$\overline{\Delta_1}\,\Delta_1$
$\Delta_3\,\overline{\Delta_3}$	$\longleftrightarrow$	$\overline{\Delta_2}\,\Delta_2$

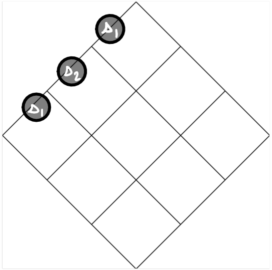




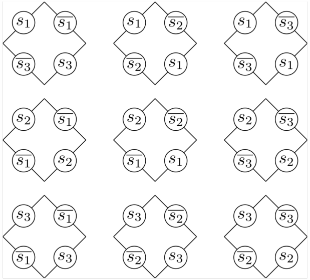




WHY IS THERE UNIQUENESS?



SET OF NINE TILES



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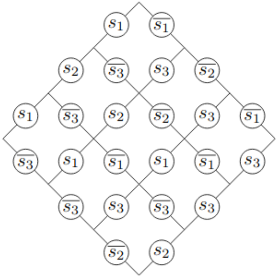
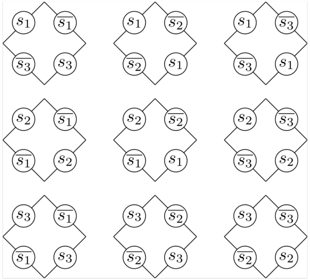


SET OF NINE TILES

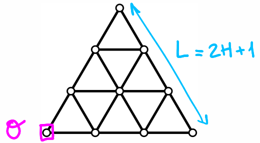


ANSWERING MORTIMER & PRELLBERG'S QUESTION

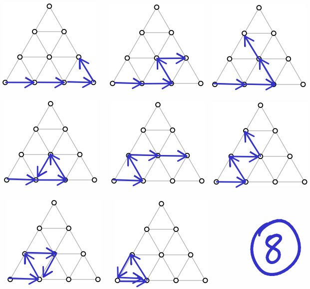
PART 2

A NEW FAMILY

FORWARD PATHS



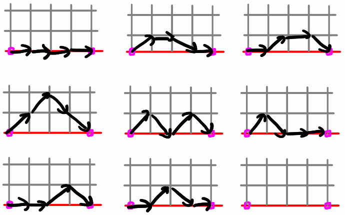
forward path starting from origin
(= bottom-left corner)



8

MOTZKIN PATHS

Motzkin path = path
using - increasing steps ↗
- horizontal steps →
- decreasing steps ↘
starting at height = 0
staying at height ≥ 0
ending at height = 0



MORTIMER & PRELLBERG'S QUESTION

Theorem [Courtiel Elvey-Price Marcovici]

number of forward paths of length n in a triangle of size $2H+1$

= number of Motzkin paths of length n with height $\leq H$

(for any n and H) And there is a bijection.

MORTIMER & PRELLBERG'S QUESTION

Theorem [Courtiel Elvey-Price Marcovici]

number of forward paths of length n in a triangle of size $2H+1$
= number of Motzkin paths of length n with height $\leq H$
(for any n and H) And there is a bijection.

Actual Mortimer and Prellberg's conjecture

Is there a bijection explaining:

number of general paths of length n in a triangle of size $2H+1$
= number of 2-coloured Motzkin paths of length n with height $\leq H$?

MORTIMER & PRELLBERG'S QUESTION

Theorem [Courtiel Elvey-Price Marcovici]

number of forward paths of length n in a triangle of size $2H+1$
= number of Motzkin paths of length n with height $\leq H$
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Actual Mortimer and Prellberg's conjecture

Is there a bijection explaining:

number of general paths of length n in a triangle of size $2H+1$
= number of 2-coloured Motzkin paths of length n with height $\leq H$?

It's a corollary of our two theorems!

general paths $\xleftrightarrow{\text{bij}}$ forward paths + direction vector

FBBF

MORTIMER & PRELLBERG'S QUESTION

Theorem [Courtiel Elvey-Price Marcovici]

number of forward/general paths of length n in a triangle of size $2H+1$
= number of 1-coloured/2-coloured Motzkin paths of length n with height $\leq H$
(for any n and H) And there is a bijection.

MORTIMER & PRELLBERG'S QUESTION

Theorem [Courtiel Elvey-Price Marcovici]

(*) number of forward/general paths of length n in a triangle of size $2H+1$
= number of 1-coloured/2-coloured Motzkin paths of length n with height $\leq H$
(for any n and H) And there is a bijection.

WHAT WAS KNOWN?

→ (*) was proved by [Mortimer Prellberg 2014] thanks to the Kernel method with 2 catalytic variables.

→ the existence of a bijection was an open question.

→ When $H = +\infty$, many bijections exist:

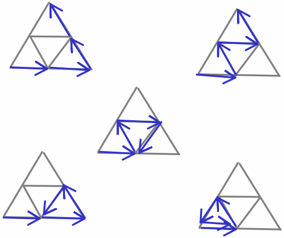
- Between forward paths & Motzkin paths [Gouyou-Beauchamps 89, Eu 10, Chyzak Yeats 20, ...]
- Between general paths & 2-coloured Motzkin paths [Yeats 14]

EVEN CASE ?

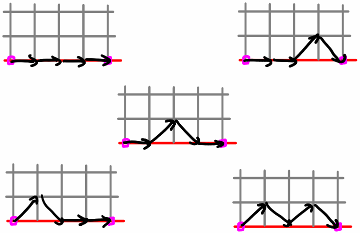
Theorem [Courtiel Elvey-Price Marcovici]

number of forward paths of length n in a triangle of size $2H$
= number of Motzkin paths of length n with height $\leq H$
with no horizontal step at height $= H$
(for any n and H) also explained by a bijection.

FORWARD PATHS



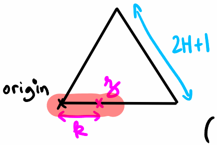
MOTZKIN PATHS



AN ELEMENTARY PROOF

Lemma

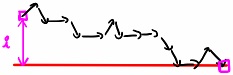
number of forward paths of length n starting at



$$= \sum_{i=0}^{k} m_n(i)$$

$(k \leq H)$

where $m_n(i)$ = number of Motzkin paths of length n starting at height $= i$ and with height $\leq H$



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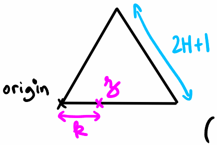

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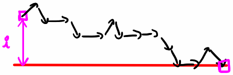
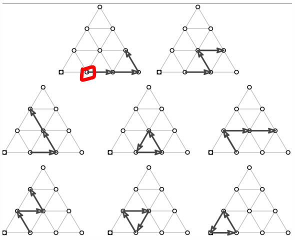
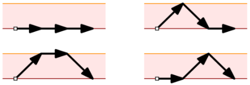


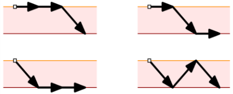
FORWARD PATHS



MOTZKIN PATHS



starting at height $= 0$

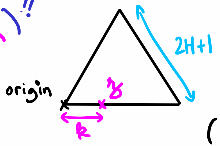


starting at height $= 1$

AN ELEMENTARY PROOF

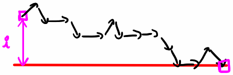
Lemma

number of forward paths of length n starting at $f_n(k)$!!



$$= \sum_{i=0}^{k} m_n(i)$$

where $m_n(i)$ = number of Motzkin paths of length n starting at height $= i$ and with height $\leq H$



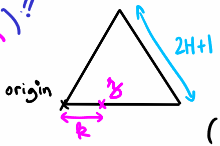
$(k \leq H)$

(Half of a) proof:

AN ELEMENTARY PROOF

Lemma

number of forward paths of length n starting at $f_n(k)$!!



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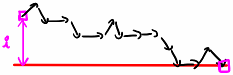


(Half of a) proof:

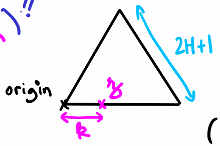
We want to show $f_n(k) - f_n(k-1) = m_n(k)$

(In this talk, we assume $0 \ll k \ll H$)

AN ELEMENTARY PROOF

Lemma

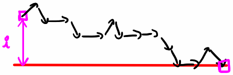
number of forward paths of length n starting at $f_n(k)$!!



$(k \leq H)$

$$= \sum_{i=0}^{k} m_n(i) \quad \text{where}$$

$m_n(i)$ = number of Motzkin paths of length n starting at height = i and with height $\leq H$

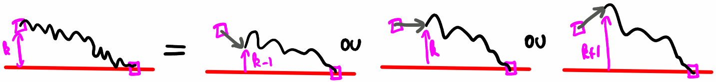


(Half of a) proof:

We want to show $f_n(k) - f_n(k-1) = m_n(k)$

(In this talk, we assume $0 \ll k \ll H$)

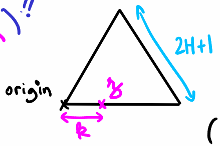
We have $m_n(k) = m_{n-1}(k-1) + m_{n-1}(k) + m_{n-1}(k+1)$



AN ELEMENTARY PROOF

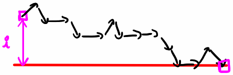
Lemma

number of forward paths of length n starting at $f_n(k)$!!



$$= \sum_{i=0}^{k} m_n(i)$$

where $m_n(i)$ = number of Motzkin paths of length n starting at height $= i$ and with height $\leq H$



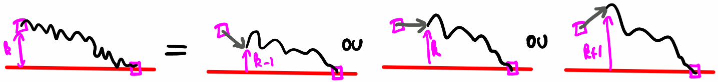
$(k \leq H)$

(Half of a) proof:

We want to show $f_n(k) - f_n(k-1) = m_n(k)$

(In this talk, we assume $0 \ll k \ll H$)

We have $m_n(k) = m_{n-1}(k-1) + m_{n-1}(k) + m_{n-1}(k+1)$



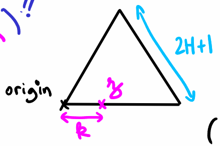
Let us prove that $f_n(k) - f_n(k-1)$ satisfies the same recurrence.

AN ELEMENTARY PROOF

Lemma

number of forward paths of length n starting at γ

$f_n(k)$!!



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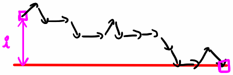
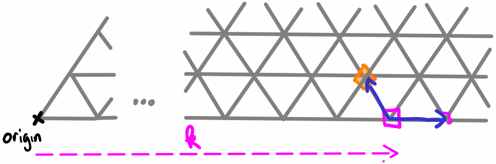


(Half of a) proof:

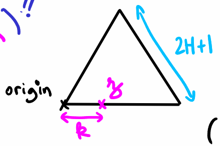


$$f_n(k) = f_{n-1}(k+1) + \diamond$$

AN ELEMENTARY PROOF

Lemma

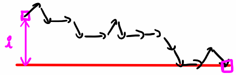
number of forward paths of length n starting at



$f_n(k)$!!

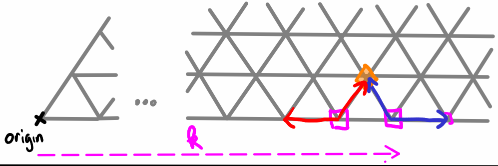
$$= \sum_{i=0}^{k} m_n(i)$$

where $m_n(i)$ = number of Motzkin paths of length n starting at height $= i$ and with height $\le H$



$(k \le H)$

(Half of a) proof:

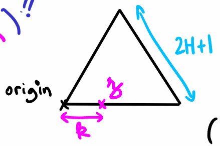


$f_n(k-1) = b_n(k-1) = b_{n-1}(k-2) + \diamond$ $= f_{n-1}(k-2) + \diamond$	$f_n(k) = f_{n-1}(k+1) + \diamond$

AN ELEMENTARY PROOF

Lemma

number of forward paths of length n starting at $f_n(k)$!!



$(k \leq H)$

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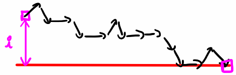
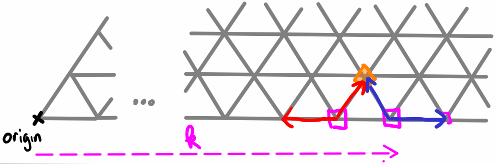


<u>(Half of a) proof:</u>



$f_n(k-1) = f_n(k-1) = f_{n-1}(k-2) + \diamond$ $= f_{n-1}(k-2) + \diamond$	$f_n(k) = f_{n-1}(k+1) + \diamond$

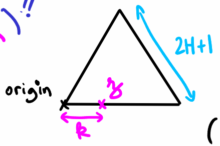
$$f_n(k) - f_n(k-1) = f_{n-1}(k+1) - f_{n-1}(k-2)$$

AN ELEMENTARY PROOF

Lemma

number of forward paths of length n starting at

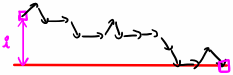
$f_n(k)$!!



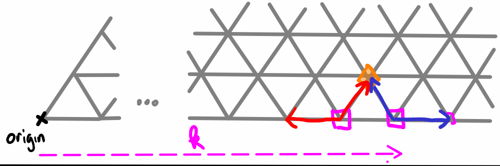
$$= \sum_{i=0}^{k} m_n(i)$$

$(k \leq H)$

where $m_n(i)$ = number of Motzkin paths of length n starting at height = i and with height $\leq H$



(Half of a) proof:



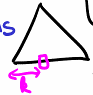
$f_n(k-1) = f_n(k-1) = f_{n-1}(k-2) + \diamond$ $= f_{n-1}(k-2) + \diamond$	$f_n(k) = f_{n-1}(k+1) + \diamond$

$$f_n(k) - f_n(k-1) = f_{n-1}(k+1) - f_{n-1}(k-2)$$

$$= \big(f_{n-1}(k+1) - f_{n-1}(k)\big) + \big(f_{n-1}(k) - f_{n-1}(k-1)\big) + \big(f_{n-1}(k-1) - f_{n-1}(k-2)\big)$$

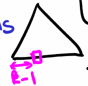
THE "HEADACHE" BIJECTION

By the previous lemma, there should be a bijection

between $\left\{ \text{forward paths starting at } \right.$ 

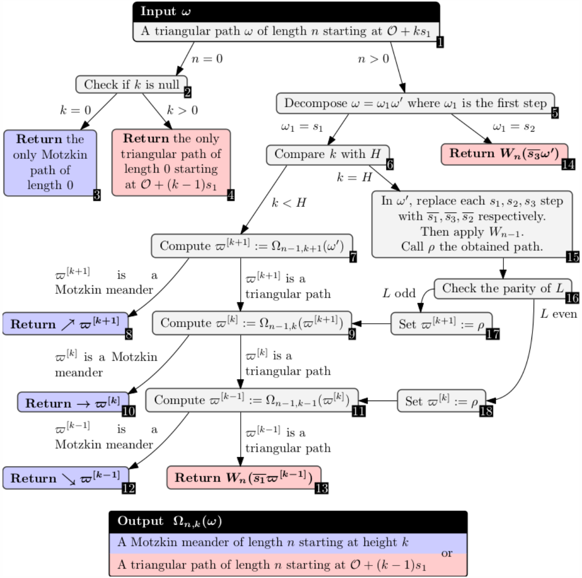
$\left. \right\}$

and

$\left\{ \text{Motzkin paths starting at height} = k \right\} \cup \left\{ \text{forward paths starting at } \right.$ 

$\left. \right\}$

THE "HEADACHE" BIJECTION



By the previous lemma, there should be a bijection

between $\left\{ \text{forward paths starting at } k \right\}$

and

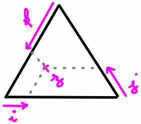
$\left\{ \text{Motzkin paths starting at height} = k \right\} \cup \left\{ \text{forward paths starting at } k-1 \right\}$

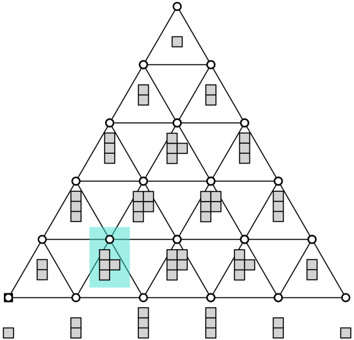
MANY OTHER BIJECTIONS

PART 3

PROFILE

coordinates of a point z $= (i, j, k)$ such that





Definition

profile of a point with coordinates $(i, j, k) =$ vector $(p_0, p_1, \dots, p_H)$

$$\frac{(1-x^{i+1})(1-x^{j+1})(1-x^{k+1})}{(1-x)^2}$$

$$= p_0 + p_1 x + \dots + p_H x^H + p_{H+1} x^{H+1} + \dots + p_{L+1} x^{L+1}$$

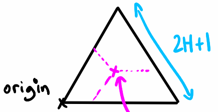
Ex: For $(1,1,3)$, $\rightarrow (1,2,1)$

$$\frac{(1-x^2)^2(1-x^4)}{(1-x)^2} = 1 + 2x + 1x^2 - x^4 - 2x^5 - x^6$$

THE GENERALISATION OF THE EARLIER LEMMA

Lemma

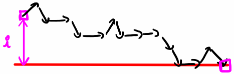
number of forward paths of length n starting at

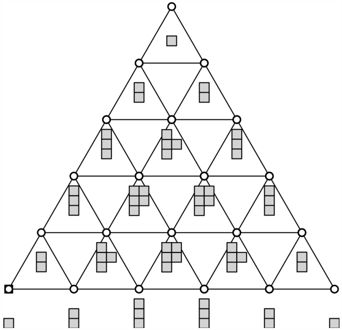


z with profile $p_0, p_1, \dots, p_H$

$$= \sum_{i=0}^{H} p_i \times m_n(i)$$

where $m_n(i)$ = number of Motzkin paths of length n starting at height $= i$ and with height $\leq H$

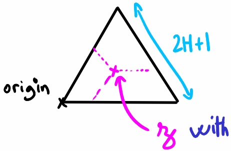




THE GENERALISATION OF THE EARLIER LEMMA

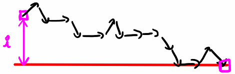
Lemma

number of forward paths of length n starting at



$$= \sum_{i=0}^{H} p_i \times m_n(i)$$

where $m_n(i)$ = number of Motzkin paths of length n starting at height $= i$ and with height $\leq H$



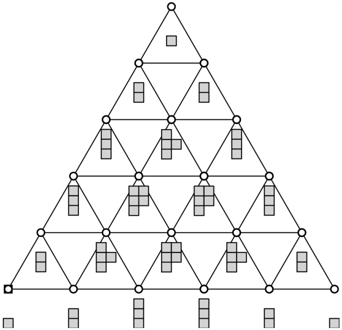
So there should be a bijection between

$\{(\text{cell of } z, \text{Motzkin path})\}$

compatible height

and

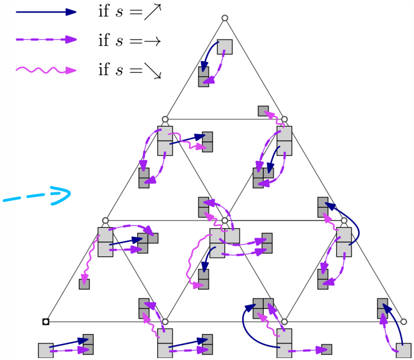
$\{\text{forward paths starting from } z\}$



SCAFFOLDING AND BIJECTION

Definition

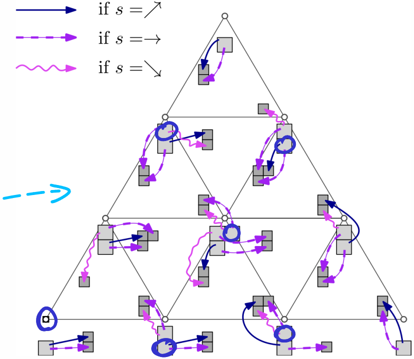
scaffolding = a riot of arrows like this



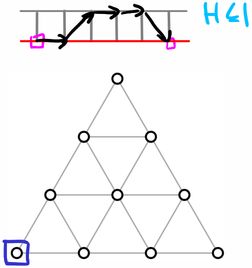
Theorem

For each scaffolding, there is a bijection between forward paths and Motzkin paths

SCAFFOLDING AND BIJECTION



Ex:



Definition

scaffolding = a riot of arrows like this

Theorem

canonical scaffolding

For each scaffolding, there is a bijection between forward paths and Motzkin paths

GENERALISATION

PART 4

TRIANGLATTAN IN HIGHER DIMENSION?

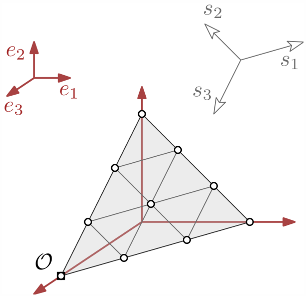
Actually,

Trianglattan of size L =

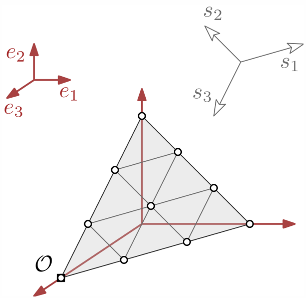
$$\{i\, e_1 + j\, e_2 + k\, e_3 : i+j+k = L\}$$

and step set:

$$\underbrace{e_1 - e_3}_{\rightarrow},\ \underbrace{e_2 - e_1}_{\nwarrow},\ \underbrace{e_3 - e_2}_{\swarrow}$$



TRIANGLATTAN IN HIGHER DIMENSION?



Actually,

Trianglattan of size L =

$$\{ i\, e_1 + j\, e_2 + k\, e_3 : i + j + k = L \}$$

and step set:

$$\underbrace{e_1 - e_3}_{\rightarrow},\ \underbrace{e_2 - e_1}_{\nwarrow},\ \underbrace{e_3 - e_2}_{\swarrow}$$

$\Downarrow$ shift in higher dimension

Hyperattan of dimension d =

$$\{ i_1 e_1 + i_2 e_2 + \dots + i_d e_d : \sum_{k=1}^{d} i_k = L \}$$

Step set: $e_1 - e_d,\ e_2 - e_1,\ \dots,\ e_d - e_{d-1}$

Extension of the previous theorems?

- Symmetry between forward paths & backward paths?
- Bijection with other family of paths?

TRIANGLATTAN IN HIGHER DIMENSION?

Actually,

Trianglattan of size L =

$$\{ i\, e_1 + j\, e_2 + k\, e_3 \ : \ i+j+k = L \}$$

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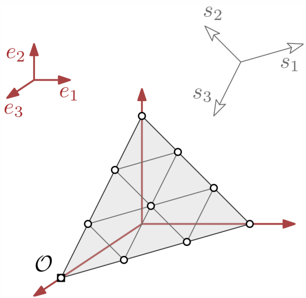


$\Downarrow$ shift in higher dimension

Hyperattan of dimension d =

$$\left\{ i_1 e_1 + i_2 e_2 + \dots + i_d e_d \ : \ \sum_{k=1}^{d} i_k = L \right\}$$

Step set: $e_1 - e_d,\ e_2 - e_1,\ \dots,\ e_d - e_{d-1}$

Extension of the previous theorems?

- Symmetry between forward paths & backward paths? Yes, EZ
- Bijection with other family of paths?

TRIANGLATTAN IN HIGHER DIMENSION?

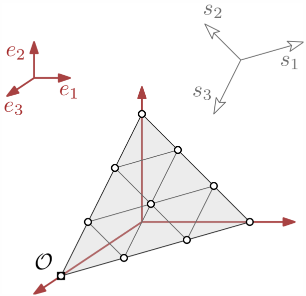
Actually,

Trianglattan of size L =

$\{ i\, e_1 + j\, e_2 + k\, e_3 : i + j + k = L \}$

and step set:

$\underbrace{e_1 - e_3}_{\rightarrow}, \underbrace{e_2 - e_1}_{\nwarrow}, \underbrace{e_3 - e_2}_{\swarrow}$



$\Downarrow$ shift in higher dimension

Hyperattan of dimension d =

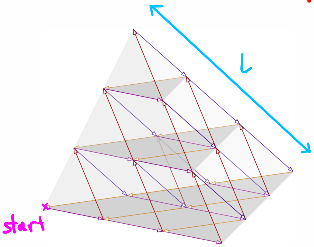
$\{ i_1 e_1 + i_2 e_2 + \dots + i_d e_d : \sum_{k=1}^{d} i_k = L \}$

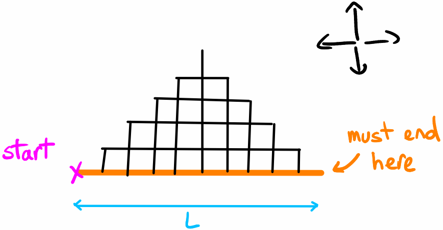
Step set: $e_1 - e_d, e_2 - e_1, \dots, e_d - e_{d-1}$

<u>Extension of the previous theorems?</u>

- Symmetry between forward paths & backward paths? Yes, EZ
- Bijection with other family of paths? Only dimension 4...

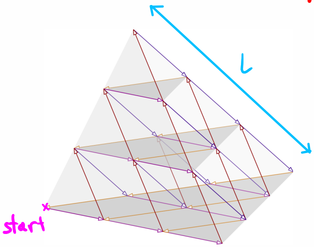
A NEW BIJECTION

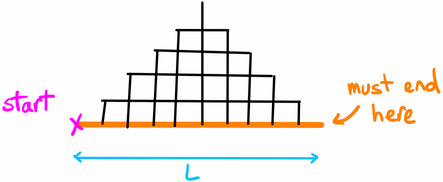




{"pyramidal" walks} $\xleftrightarrow{\text{bijection}}$ {"waffle" walks}

A NEW BIJECTION

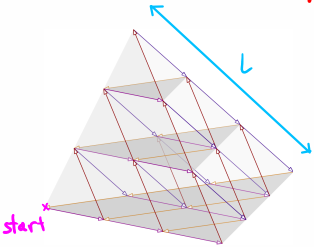


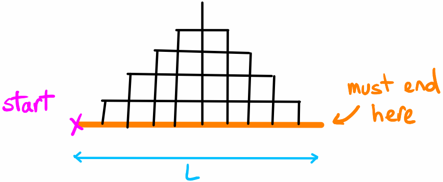


{"pyramidal" walks} $\xleftrightarrow{\text{bijection}}$ {"waffle" walks}

The counting of these walks was an open question from [Mortimer Prellberg]

A NEW BIJECTION





{"pyramidal" walks} $\longleftrightarrow$ bijection $\longleftrightarrow$ {"waffle" walks}

The counting of these walks was an open question from [Mortimer Prellberg]

Theorem The generating function is

$$P(t) = \frac{1}{(L+4)^2} \sum_{\substack{1 \le j < k \le L+3 \\ 2 \nmid j,k}}^{L+4} \frac{(\alpha^k + \alpha^{-k} - \alpha^j - \alpha^{-j})^2 (2 + \alpha^j + \alpha^{-j})(2 + \alpha^{-k} + \alpha^k)}{1 - (\alpha^j + \alpha^{-j} + \alpha^k + \alpha^{-k})t}$$

where $\alpha = e^{\frac{i\pi}{L+4}}$