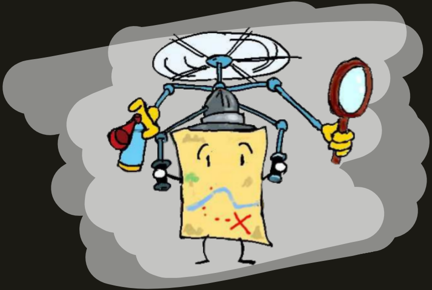
ASYMPTOTIC DISTRIBUTION OF PARAMETERS IN RANDOM MAPS

Julien COURTIEL (AMACC, Caen, France)



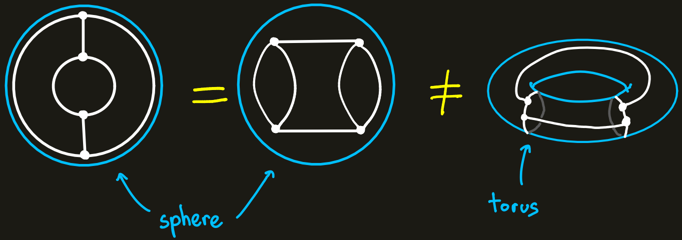
Algebraic Combinatorics (and enumeration?) Seminar, Waterloo

Co-authors: Olivier BODINI (Paris 13), Sergey DOVGAL (Paris 13), Hsien-Kuei HWANG (Taïwan)

DEFINITION

combinatorial map = cellular embedding of a connected graph onto an oriented surface.

Examples:



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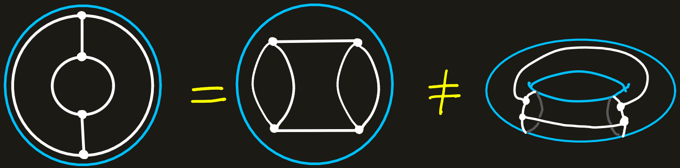

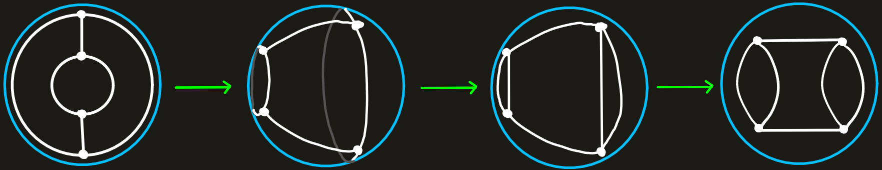

DEFINITION

combinatorial map = *cellular* embedding of a connected graph onto an oriented surface.

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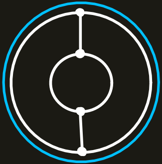
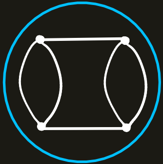
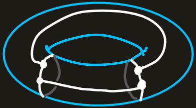
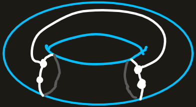
 =  ≠ 

Counter-example:

 not cellular because  is not a disk

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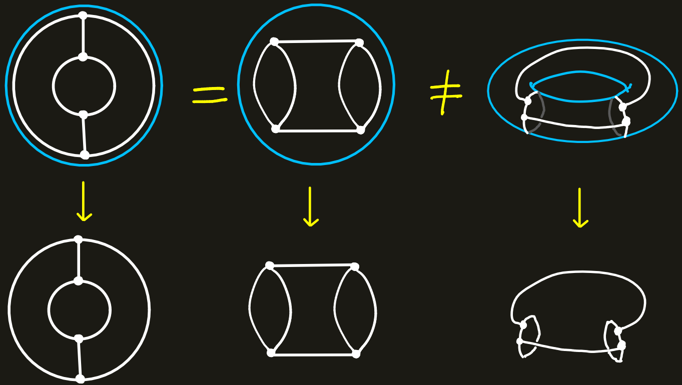

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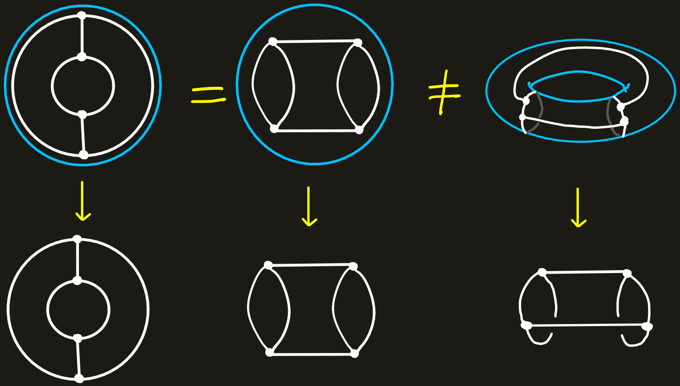

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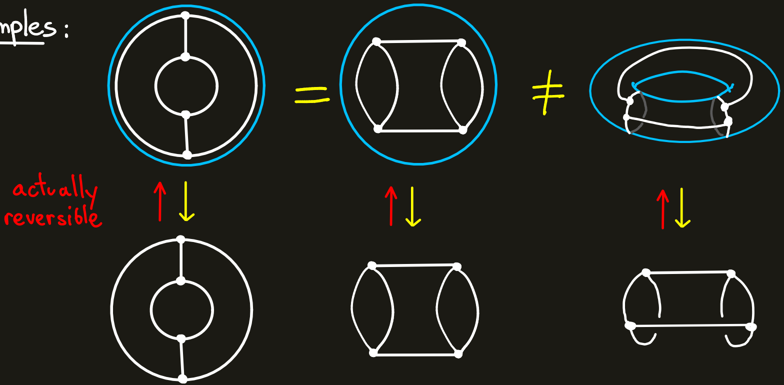


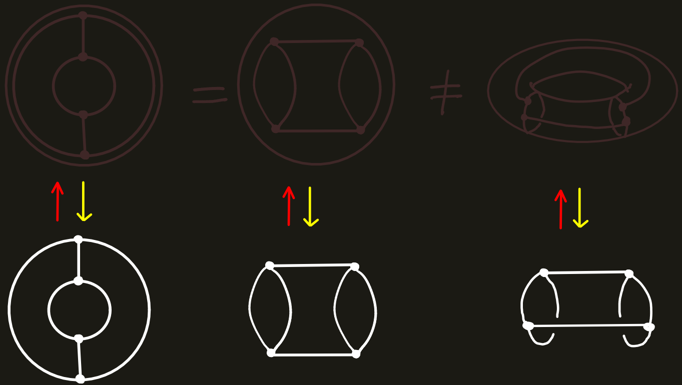
DEFINITION

FORGET IT!

combinatorial map = cellular embedding of a connected graph onto an oriented surface.

Examples:

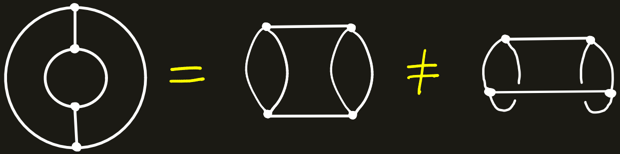
actually reversible



DEFINITION

combinatorial map = connected graph where we have cyclically ordered the half-edges around each vertex.

<u>Examples</u>:



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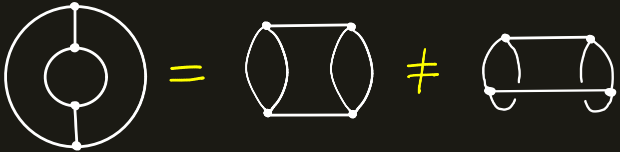
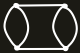
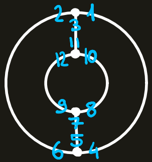
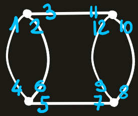


Why is  the same as  ?





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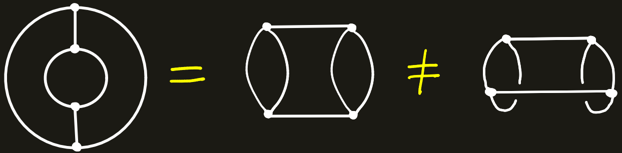
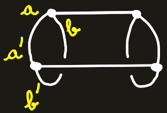


Why is  different from  ?



Absent pattern in  :

$a \leftrightarrow a'$ $\quad$ $a \curvearrowright b$

$b \leftrightarrow b'$ $\quad$ $a' \curvearrowright b'$

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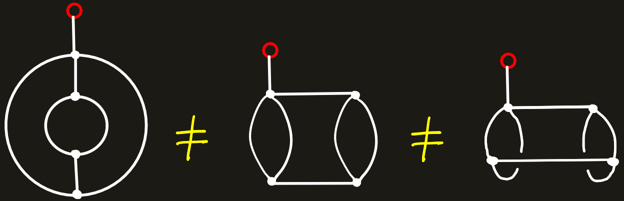
Examples:



We root every map on a leaf.

DEFINITION

combinatorial map = connected graph where we have cyclically ordered the half-edges around each vertex.

1 edge

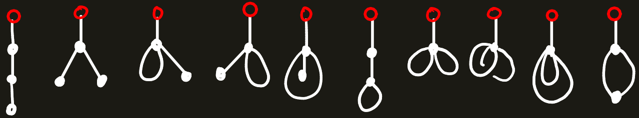
1

2 edges

2

3 edges

10



RECURRENCE FORMULA

c_n = number of combinatorial maps with n edges

Recurrence formula : [Arquès Béraud]

$$c_1 = 1 \qquad c_n = \sum_{k=1}^{n-1} c_k c_{n-k} + (2n-3)\, c_{n-1}$$

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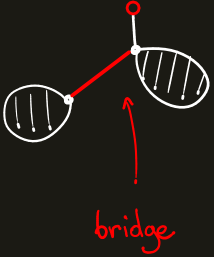
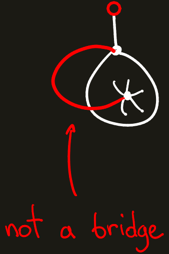
map =  or 

or 

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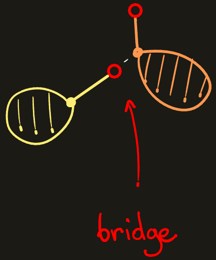
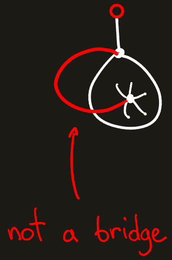
map =  or 

or 

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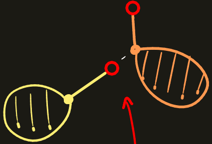
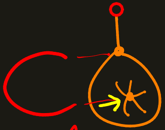
map =  or  or 

bridge

not a bridge

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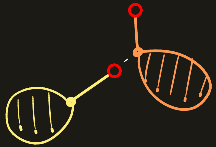
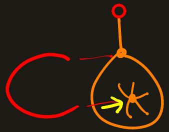
map =  or  or 

Generating function : $C(z) = \sum_{n \geq 0} c_n z^n$

$$C(z) = z + C(z)^2 + z\left(2z\frac{\partial C(z)}{\partial z} - C(z)\right)$$

WHY COUNTING MAPS WITH NO CONSIDERATION FOR GENUS?

→ Good framework to study parametric Riccati equations.

→ Connections with other combinatorial families.

- indecomposable chord diagrams (link with the Quantum Fields Theory)
- lambda-terms
- Schur functions

WHY COUNTING MAPS WITH NO CONSIDERATION FOR GENUS?

Part 2

→ Good framework to study parametric Riccati equations.

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- indecomposable chord diagrams
  (link with the Quantum Fields Theory)
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Part 1

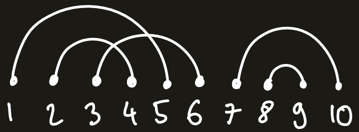
PART I

Connections with other combinatorial families

CHORD DIAGRAMS

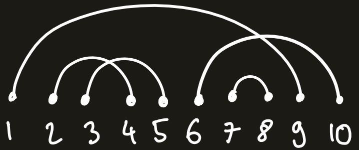
diagram of n chords

= matching of the set $\{1, \ldots, 2n\}$



indecomposable diagram

= diagram that is <u>not</u> the concatenation of two diagrams.



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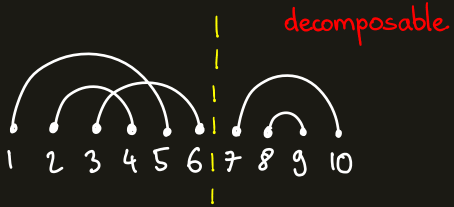

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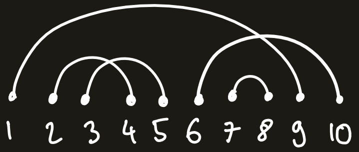
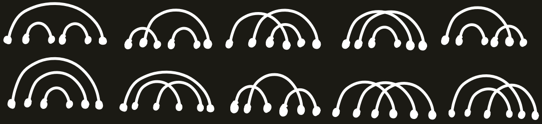


CHORD DIAGRAMS

1 chord  1

2 chords 2

3 chords 10



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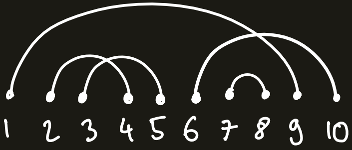

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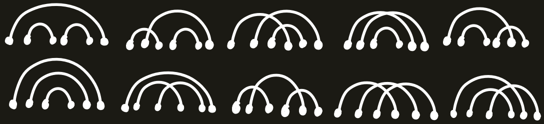

Proposition [Cvitanović Lautrup Pearson, Ossana de Mendez Rosenstiehl, Cori]

number of combinatorial maps with n edges
=
number of indecomposable diagrams with n chords

RECURRENCE FORMULA : THE COMEBACK

c_n = number of indecomposable diagrams with n chords

Recurrence formula :

$$c_1 = 1 \qquad c_n = \sum_{k=1}^{n-1} c_k c_{n-k} + (2n-3)\, c_{n-1}$$

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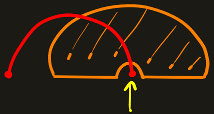
indecomposable diagram =  ou  ou 

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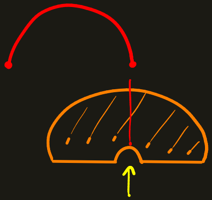
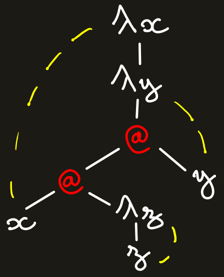
indecomposable diagram =  ou  ou 

LINEAR LAMBDA-TERMS



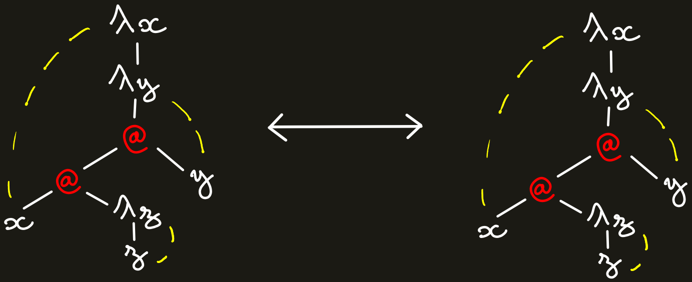
$\lambda x.\lambda y((x\ \ \lambda z.z)\ y)$

linear lambda-term = Motzkin tree where each leaf is bound by a unary vertex and each vertex binds exactly one leaf.

Theorem [Bodini Gardy Gittenberg Jacquot]

linear lambda-terms $\longleftrightarrow$ trivalent maps

LINEAR LAMBDA-TERMS



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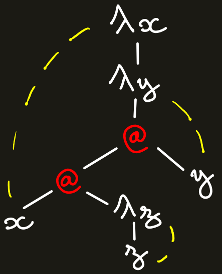
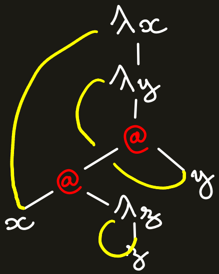


$\longleftrightarrow$



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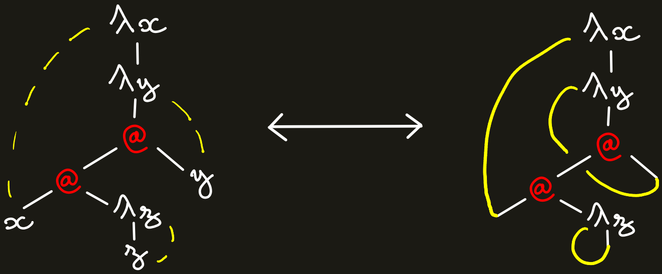

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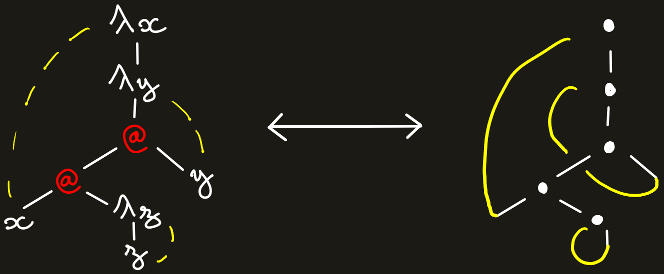

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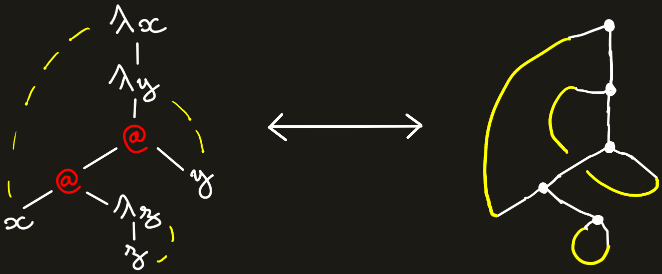

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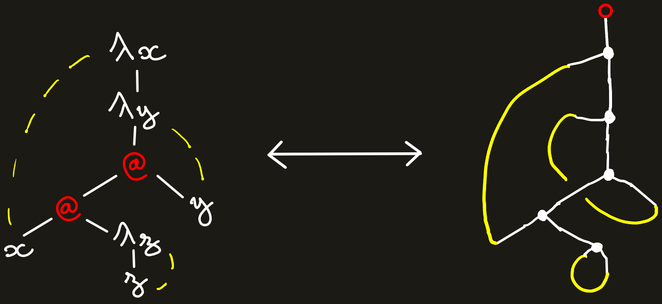

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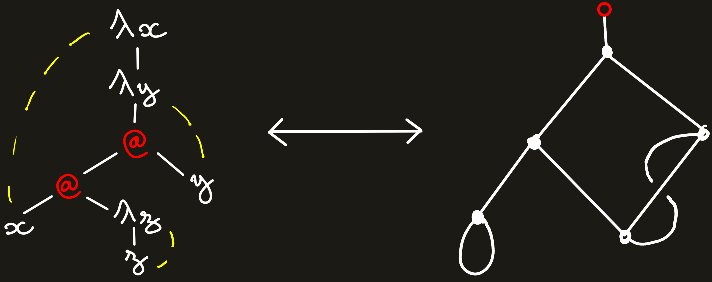

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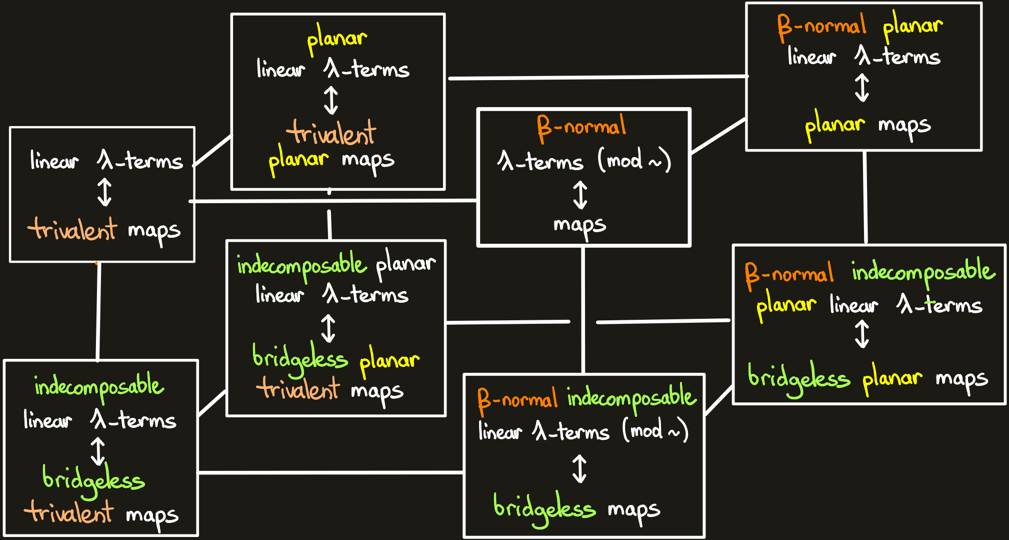
NOAM ZEILBERGER'S CUBE



PART II

Asymptotic analysis of statistics on maps

ASYMPTOTIC NUMBER OF MAPS

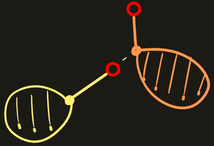
c_n = number of combinatorial maps with n edges

Recurrence formula :

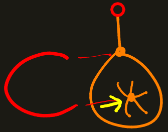
$$c_1 = 1 \qquad\qquad c_n = \sum_{k=1}^{n-1} c_k c_{n-k} + (2n-3)\, c_{n-1}$$

map =

or



or



Question 0 : Asymptotic estimate of c_n ?

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Idea: (Formally) solve it !

ASYMPTOTIC NUMBER OF MAPS

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Riccati :(

linear :)

$$C(z) = z + 2z^2\frac{\phi'(z)}{\phi(z)}$$

MAGIC TRICK!

$$2z^2\phi''(z) + (5z-1)\phi'(z) + \phi(z) = 0$$

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$$2z^2\phi''(z) + (5z-1)\phi'(z) + \phi(z) = 0$$

<u>Solution</u> : $\phi(z) = \sum_{n \geq 0} (2n-1)!!\, z^n$

$(2n-1)!! = (2n-1) \times (2n-3) \times \ldots \times 1$

ASYMPTOTIC NUMBER OF MAPS

$$C(z) = z + 2z^2 \frac{\phi'(z)}{\phi(z)}$$

$$2z^2 \phi''(z) + (5z-1)\phi'(z) + \phi(z) = 0$$

<u>Solution</u>: $\phi(z) = \sum_{n \geq 0} (2n-1)!! \, z^n$

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ASYMPTOTIC NUMBER OF MAPS

$$C(z) = z + 2z^2 \frac{\phi'(z)}{\phi(z)} \iff c_{n+1} = 2n\,\phi_n - \sum_{k=1}^{n-1} c_n \phi_{n-k}$$

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By some bootstrapping, $\boxed{c_n \sim \phi_n \left(2n - 1 - \frac{3}{2} n^{-1} - \frac{19}{4} n^{-2} + O(n^{-3})\right)}$

$$2z^2 \phi''(z) + (5z-1)\,\phi'(z) + \phi(z) = 0$$

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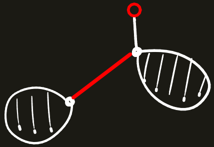
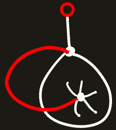
NUMBER OF VERTICES

$C(z)$ = generating function of maps where z counts the edges

Equation

$$C = z + C^2 + 2z^2 \frac{\partial C}{\partial z} - zC$$

map =  or  or 

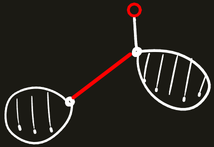
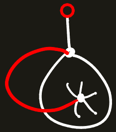
Question 1: behaviour of the number of vertices ?

NUMBER OF VERTICES

$C(z,u) =$ generating function of maps where z counts the edges and u counts the vertices

Equation

$$C = zu + C^2 + 2z^2\frac{\partial C}{\partial z} - zC$$

map $=$  or  or 

Question 1: behaviour of the number of vertices ?

NUMBER OF VERTICES

$$C = z\,u + C^2 + 2z^2\frac{\partial C}{\partial z} - z\,C$$

NUMBER OF VERTICES

$$C = z u + C^2 + 2 z^2 \frac{\partial C}{\partial z} - z C$$

MAGIC TRICK!

$$C(z,u) = z u + 2 z^2 \frac{\phi'(z,u)}{\phi(z,u)}$$

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$$C(z,u) = zu + 2z^2\frac{\phi'(z,u)}{\phi(z,u)}$$

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<u>Solution</u>: $\phi(z,u) = 1 + \frac{u(u+1)}{2}z + \frac{u(u+1)(u+2)(u+3)}{2^2\times 2!}z^2 + \dots + \frac{u(u+1)\dots(u+2n-1)}{2^n\times n!}z^n + \dots$

NUMBER OF VERTICES

Fact: $\phi(z,u)$ behaves like $C(z,u)$

Theorem:

For the uniform distribution of combinatorial maps,

$$\text{Number of vertices} \xrightarrow[\text{law}]{} \text{Gaussian law} \quad \begin{array}{l} \text{mean} \sim \ln(n) + \gamma + \dots \\ \text{variance} \sim \ln(n) + \gamma - \frac{\pi^2}{12} + \dots \end{array}$$

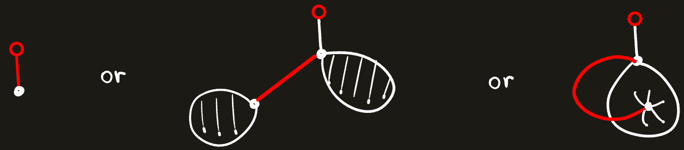
$$\phi(z,u) = 1 + \frac{u(u+1)}{2} z + \frac{u(u+1)(u+2)(u+3)}{2^2 \times 2!} z^2 + \dots + \frac{u(u+1)\dots(u+2n-1)}{2^n \times n!} z^n + \dots$$

NUMBER OF EDGES INCIDENT TO THE ROOT

$C(z,u)$ = generating function of maps where z counts the edges and u counts the number of edges incident to the root vertex.

Equation:

$$C(z,u) = zu + u\,C(z,u)\,C(z,1) + u\left(2z^2\frac{\partial C}{\partial z} - zC\right)$$

map = 

NUMBER OF EDGES INCIDENT TO THE ROOT

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MAGIC TRICK!

$$C(z,1) = z + 2z^2\,\frac{\phi'(z,1)}{\phi(z,1)}$$

$$2uz^2\,C'(z,u)\,\phi(z,1) + 2uz^2\,C(z,u)\,\phi'(z,1) = (1-2uz)\,C(z,u)\,\phi(z,1) - \phi(z,1)$$

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$$P(z,u) = \phi(z,u)\,\phi(z,1)$$

$$2uz^2\,P'(z,u) = (1-2uz)\,P(z,u) - \phi(z,1)$$

almost linear!

NUMBER OF EDGES INCIDENT TO THE ROOT

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Theorem :

For the uniform distribution of combinatorial maps,

$$\text{Number of edges incident to the root} \xrightarrow[\text{law}]{}$$

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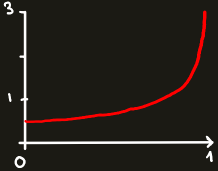
$$C(z,u) = zu + u\,C(z,u)\,C(z,1) + u\left(2z^2\frac{\partial C}{\partial z} - zC\right)$$

Theorem :

For the uniform distribution of combinatorial maps,

Number of edges incident to the root divided by n $\xrightarrow{\text{law}}$ Beta-law density $\frac{1}{2}(1-t)^{-\frac{1}{2}}$ sur $[0,1)$

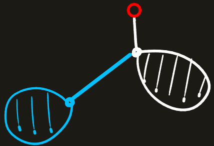
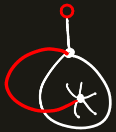


NUMBER OF COMPONENTS ATTACHED TO THE ROOT

$C(z,u)$ = generating function of maps where z counts the edges and u counts the number of connected components attached to the root vertex.

<u>Equation</u>:

$$C(z,u) = z + u\,C(z,u)\,C(z,1) + \left(2z^2\frac{\partial C}{\partial z} - zC\right)$$

map =  or  or 

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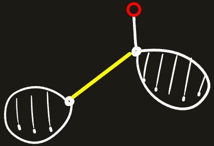
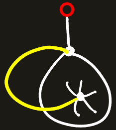
Number of connected components attached to the root vertex $\xrightarrow[\text{law}]{}$ Geometric law of parameter $1/2$.

ROOT VERTEX DEGREE

$C(z,u)$ = generating function of maps where z counts the edges and u counts the degree of the root vertex

Equation:

$$C(z,u) = zu + u\,C(z,u)\,C(z,1) + u\left(2z^2\frac{\partial C}{\partial z} - zC\right) + (u^2-u)\frac{\partial C}{\partial u}$$

map =  or  or 

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Theorem:

Degree of the root vertex $\xrightarrow[\text{law}]{}$

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Theorem:

$$\text{Degree of the root vertex divided by } n \xrightarrow[\text{law}]{} \text{uniform law on } [0,1]$$

NUMBER OF LOOPS

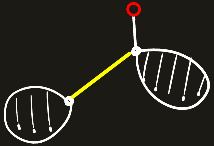
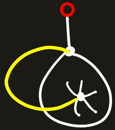
$C(z,u,l)$ = generating function of maps where z counts the edges

u counts the degree of the root vertex

and l counts the number of loops.

Equation:

$$C(z,u) = zu + u\,C(z,u)\,C(z,1) + u\left(2z^2\frac{\partial C}{\partial z} - zC\right) + (u^2 l - u)\frac{\partial C}{\partial u}$$

map =  or  or 

NUMBER OF LOOPS

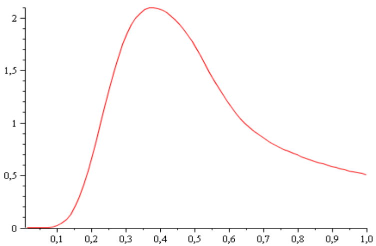
$C(z,u,\ell)$ = generating function of maps where z counts the edges
u counts the degree of the root vertex
and ℓ counts the number of loops.

Equation:

$$C(z,u) = zu + u\,C(z,u)\,C(z,1) + u\left(2z^2\frac{\partial C}{\partial z} - zC\right) + (u^2\ell - u)\frac{\partial C}{\partial u}$$

Theorem

number of loops divided by n $\xrightarrow{\text{law}}$



PART III

Lessons to learn

$\boxed{1}$ Know your bijections

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Why does the root vertex degree asymptotically follow an uniform law?

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Why does the root vertex degree asymptotically follow an uniform law?

indecomposable diagrams with n chords and k left-to-right maxima $\xleftrightarrow{\text{[Cori]}}$ maps with n edges and k vertices

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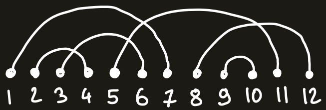

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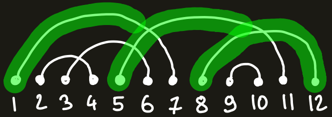

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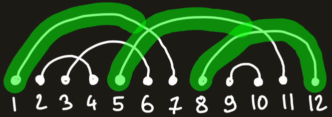

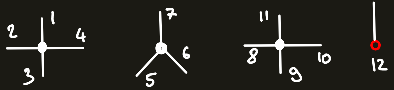

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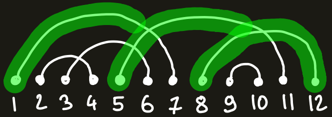

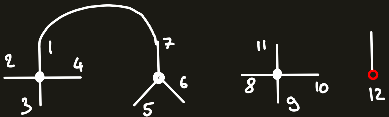

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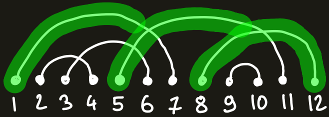

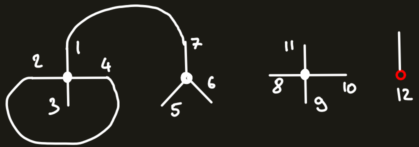

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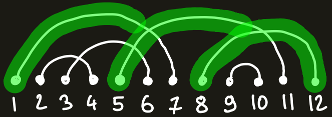

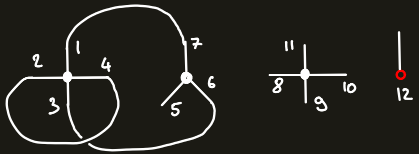

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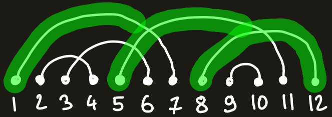

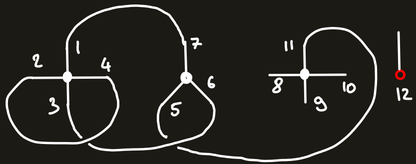

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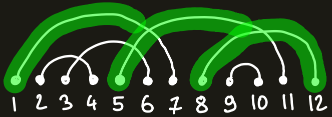

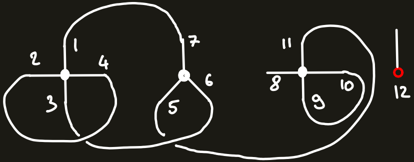

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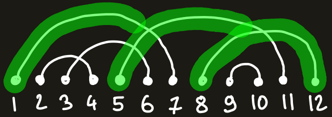

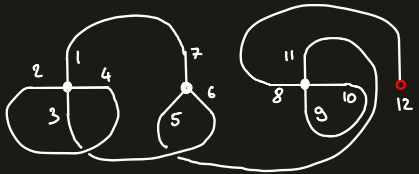

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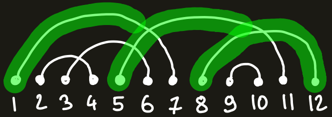

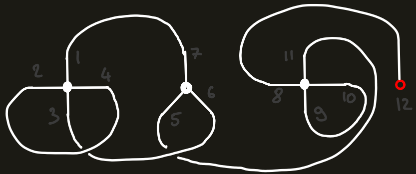

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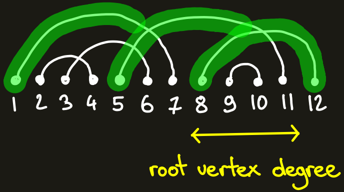

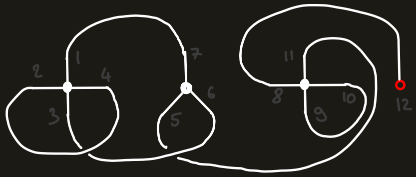

$\boxed{2}$ Think bigger

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<u>Why does the magic trick work for the number of edges incident to the root?</u>

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maps $\longleftrightarrow$ indecomposable diagrams

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Why does the magic trick work for the number of edges incident to the root?

$$\begin{tikzcd}
\text{maps} \arrow[r, leftrightarrow] \arrow[d, hook] & \text{indecomposable diagrams} \arrow[d, hook] \\
\text{sequence of maps} \arrow[r, leftrightarrow] & \text{diagrams}
\end{tikzcd}$$

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number: $(2n-1)!!$

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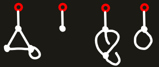
Why does the magic trick work for the number of edges incident to the root?

maps $\longleftrightarrow$ indecomposable diagrams

$\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

sequence of maps $\longleftrightarrow$ diagrams



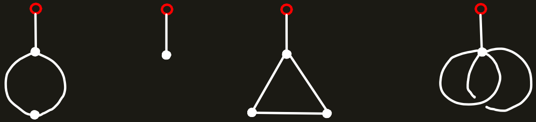
number: $(2n-1)!!$

If m_n = number of sequences $\mathrm{Map}_1, \mathrm{Map}_2, \dots, \mathrm{Map}_\ell$ with n edges,

$m_n = (2n-1) \times m_{n-1}$: explanation?

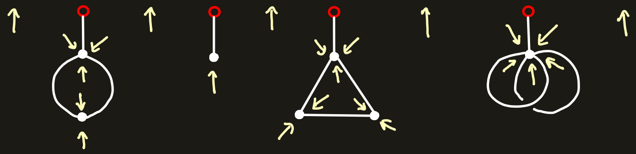
$\boxed{2}$ Think bigger

$m_n = (2n-1) \times m_{n-1}$: Why?



$\boxed{2}$ Think bigger

$m_n = (2n-1) \times m_{n-1}$: Why?

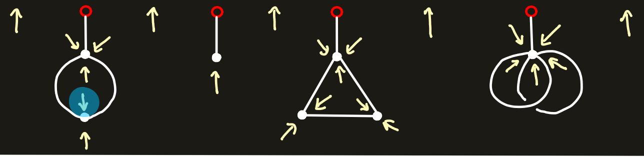


$(2n-1)$ is the number of $\uparrow$

2 Think bigger

$m_n = (2n-1) \times m_{n-1}$: Why?

Before

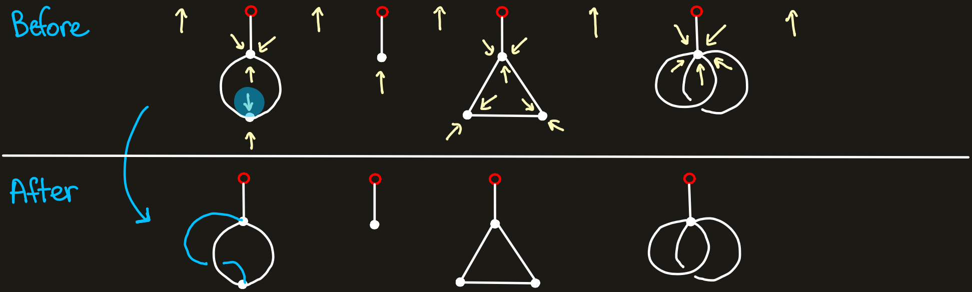


After

First case: We choose ↑ in the first map.

2 Think bigger

$m_n = (2n-1) \times m_{n-1}$: Why?

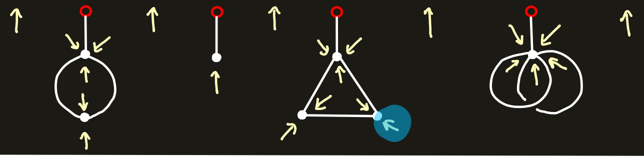


First case: We choose ↑ in the first map.

$\boxed{2}$ Think bigger

$m_n = (2n-1) \times m_{n-1}$: Why ?

Before

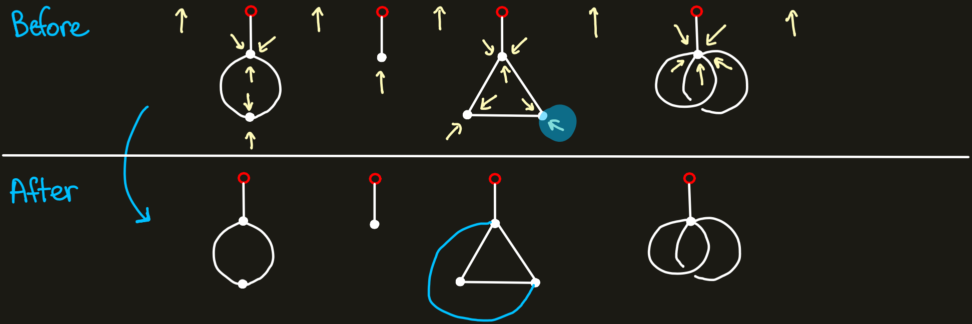


After

Real first case: We choose ↑ in a map.
ʒ

$\boxed{2}$ Think bigger

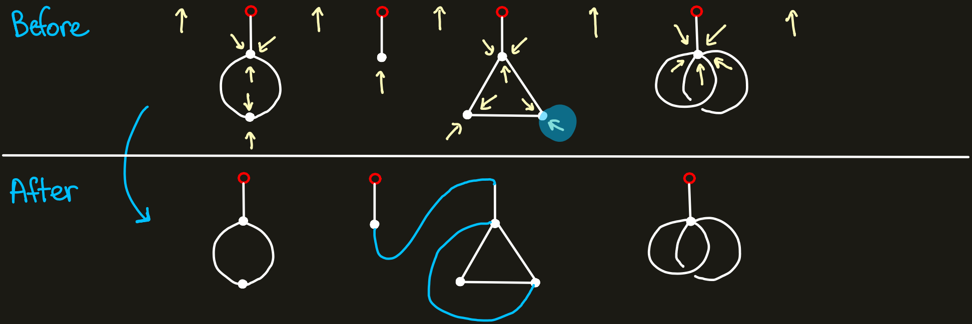
$m_n = (2n-1) \times m_{n-1}$: Why ?



Real first case: We choose $\uparrow$ in a map.
$\exists$

$\boxed{2}$ Think bigger

$m_n = (2n-1) \times m_{n-1}$: Why?



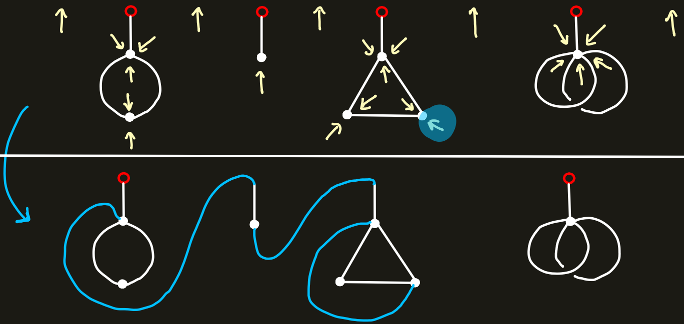
Real first case: We choose ↑ in a map.

$\boxed{2}$ Think bigger

$m_n = (2n-1) \times m_{n-1}$: Why?

Before

After

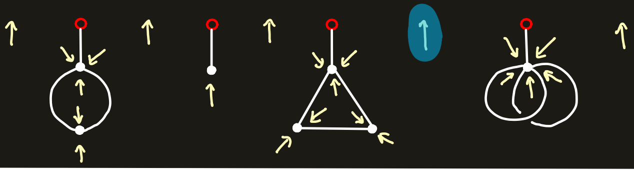


Real first case: We choose ↑ in $\underset{3}{a}$ map.

$\boxed{2}$ Think bigger

$m_n = (2n-1) \times m_{n-1}$: Why?

Before



After

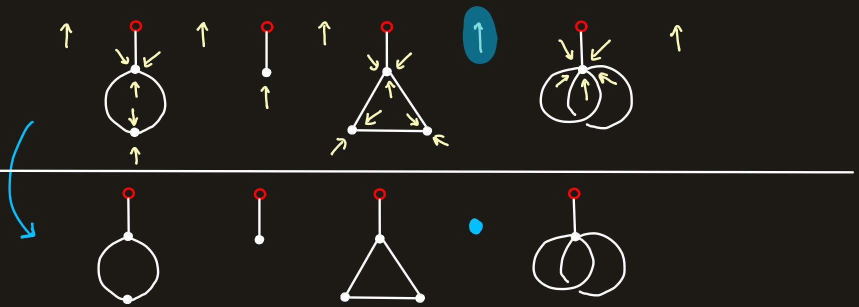
Second case: We choose ↑ in an interval but not the 1st.

$\boxed{2}$ Think bigger

$m_n = (2n-1) \times m_{n-1}$: Why?

Before

After



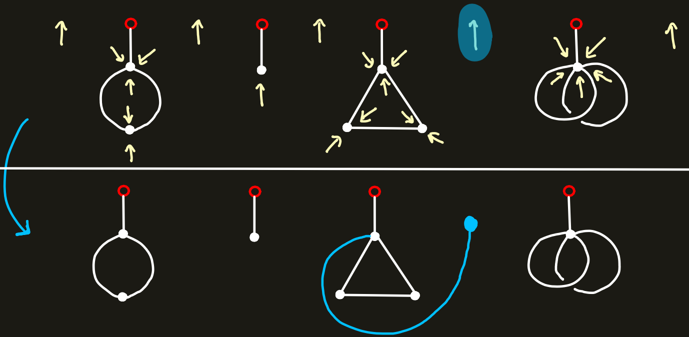
Second case: We choose ↑ in an interval but not the 1st.

$\boxed{2}$ Think bigger

$m_n = (2n-1) \times m_{n-1}$: Why?

Before

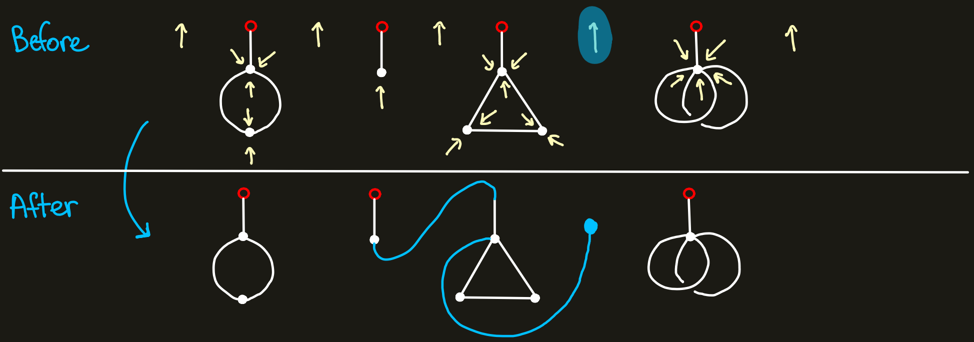
After



Second case: We choose ↑ in an interval but not the 1^{st}.

$\boxed{2}$ Think bigger

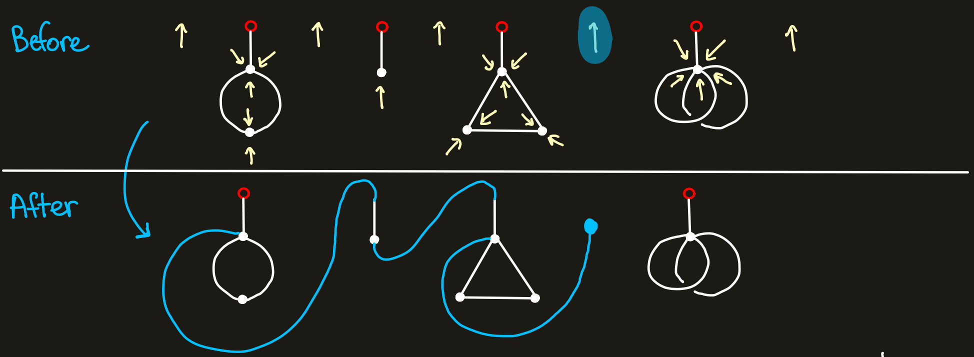
$m_n = (2n-1) \times m_{n-1}$: Why?



Second case: We choose ↑ in an interval but not the 1st.

$\boxed{2}$ Think bigger

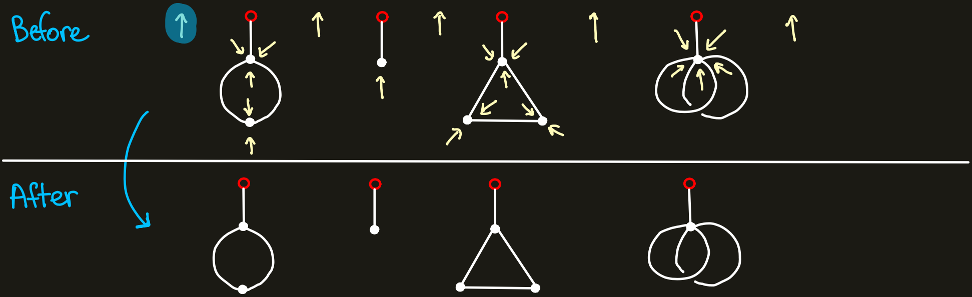
$m_n = (2n-1) \times m_{n-1}$: Why?



Second case: We choose ↑ in an interval but not the 1st.

$\boxed{2}$ Think bigger

$m_n = (2n-1) \times m_{n-1}$: Why?



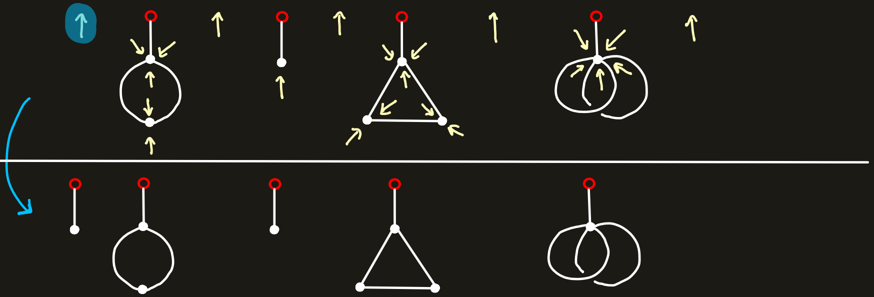
Last case: We choose ↑ in the first interval.

$\boxed{2}$ Think bigger

$m_n = (2n-1) \times m_{n-1}$: Why?

Before

After



Last case: We choose ↑ in the first interval.

$\boxed{2}$ Think bigger

$m_n = (2n-1) \times m_{n-1}$: Why?

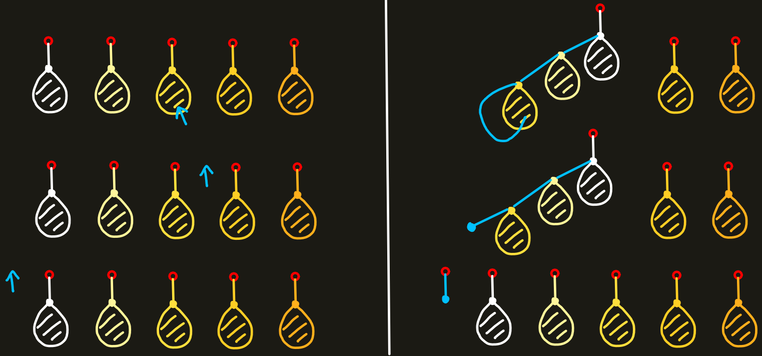
Summary:

Before | After

1st case:

2nd case:

3rd case:

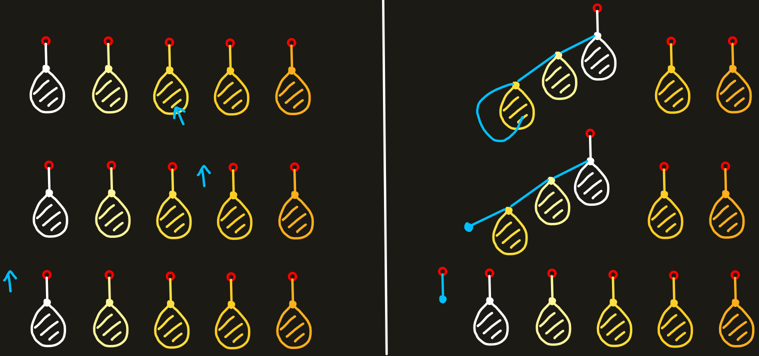


2 Think bigger

Every case except the 3rd one increases the number of edges incident to the root by 1.

Summary:

Before | After



1st case:

2nd case:

3rd case:

$\boxed{2}$ Think bigger

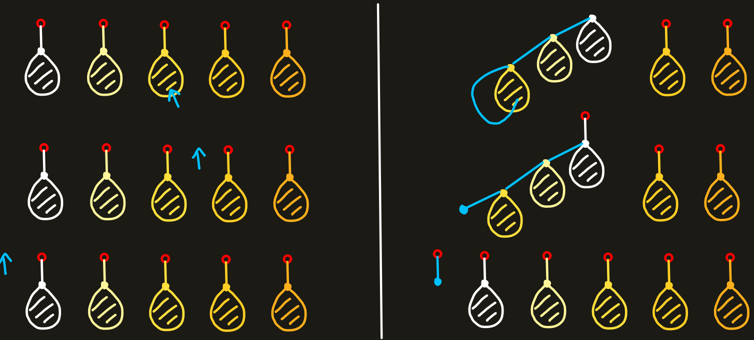
Every case except the 3rd one increases the number of edges incident to the root by 1.

In terms of GFs, it translates $(1-2uz)\,P(z,u) = 2uz^2\,P'(z,u) + \phi(z,1)$.

1st case:

2nd case:

3rd case:



$\boxed{3}$ Be humble and work, grasshopper.

→ Wide range of limits laws for combinatorial maps:
towards a taxonomy of possible laws?

→ Understand the operation $C = z + K z^2 \dfrac{\phi'}{\phi}$

→ Extension to other families of maps?
to other combinatorial families?

THANK YOU!