

Julien Courtiel (PIMS, UBC)
Karen Yeats (SFU)

Terminal \textfemale Chords in Connected Chord Diagrams



SFU, March 8

Tom Hanks
Catherine Zeta-Jones
OPERA

The Terminal

Life is waiting.

BRADLEY COOPER and JAMES RYAN from GENE TURKIN
TOM HANKS, CATHERINE ZETA-JONES, THE TERMINAL SHOT IN COLOGNE, GERMANY
DIRECTED BY STEPHEN DORFF, PRODUCED BY JONATHAN LIEBMAN AND GENE TURKIN
SCREENPLAY BY STEPHEN DORFF, JONATHAN LIEBMAN, GENE TURKIN
STORY BY STEPHEN DORFF, JONATHAN LIEBMAN, GENE TURKIN
PRODUCTION DESIGNER, CLAUDIO CAVALLI
CINEMATOGRAPHY, ROBERTSON D. HARRIS
EDITOR, CLAUDIO CAVALLI
MUSIC, STEPHEN DORFF
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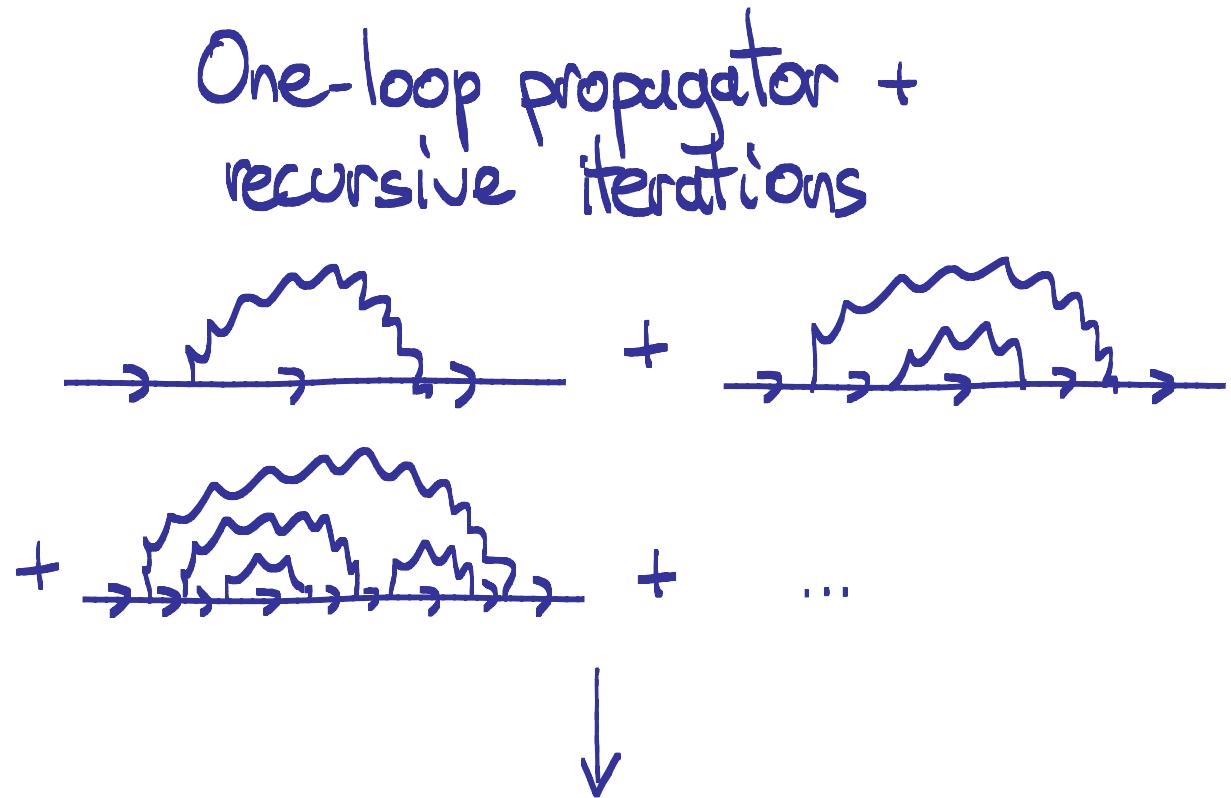


COMBINATORIAL DYSON-SCHWINGER EQUATIONS

Class of Feynman graphs

Feynman rules
↓

solution of
Dyson-Schwinger equations



$$G(x, L) = 1 - x G\left(x, \frac{\partial}{\partial(-\rho)}\right)^{-1} (e^{-L\rho} - 1) F(\rho)_{|\rho=0}$$

COMBINATORIAL DYSON-SCHWINGER EQUATIONS

Theorem

[Marie, Yeats]

The solution of the previous equation can be written under the form:

$$G(x, L) = 1 - \sum_{i \geq 1} \sum_{\substack{\text{connected} \\ \text{chord diagram} \\ \text{with terminal chords} \\ \text{in position } t_1 < t_2 < \dots < t_k \\ \text{such that } t_1 \geq i}} \frac{i^{|C|} L^{|C|-k}}{i!} f_0 f_{t_1-i} f_{t_2-t_1} f_{t_3-t_2} \dots f_{t_k-t_{k-1}}$$

where

$$\frac{f_0}{\varrho} + f_1 + f_2 \varrho + f_3 \varrho^2 + \dots = \begin{array}{l} \text{expansion of a regularized} \\ \text{Feynman integral} \end{array}$$

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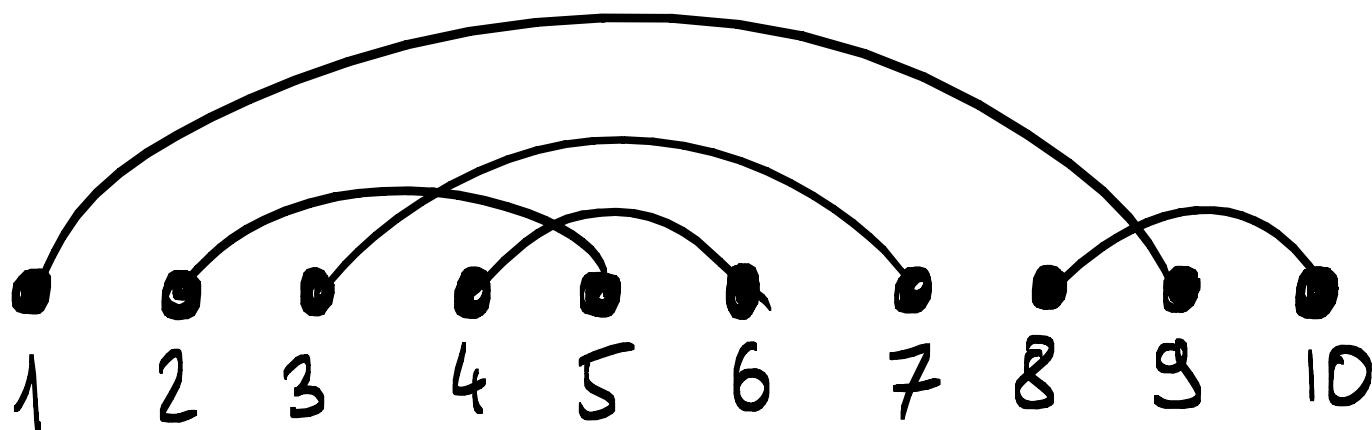
with terminal chords
in position $t_1 < t_2 < \dots < t_k$
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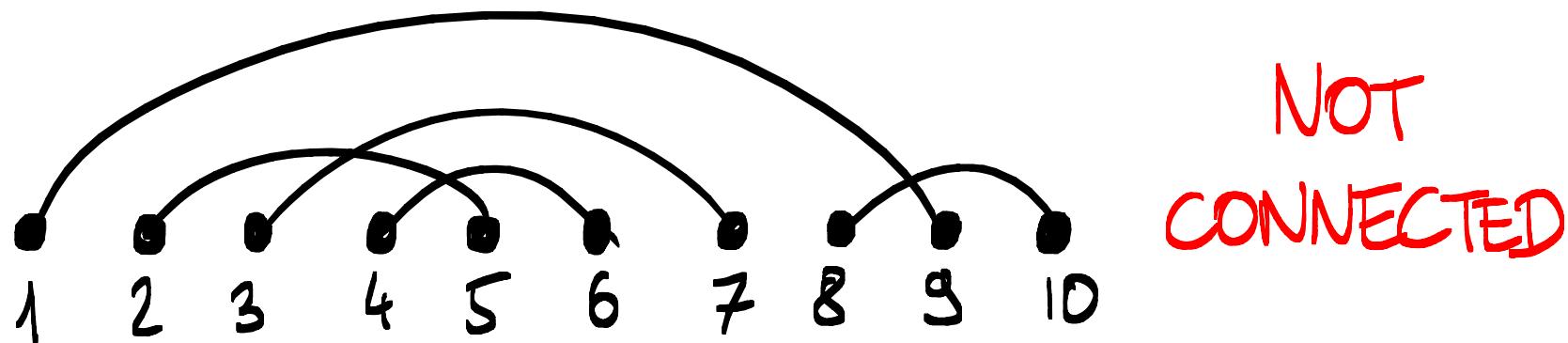
CONNECTED CHORD DIAGRAMS

Diagram of n chords = matching of $\{1, 2, \dots, 2n\}$



CONNECTED CHORD DIAGRAMS

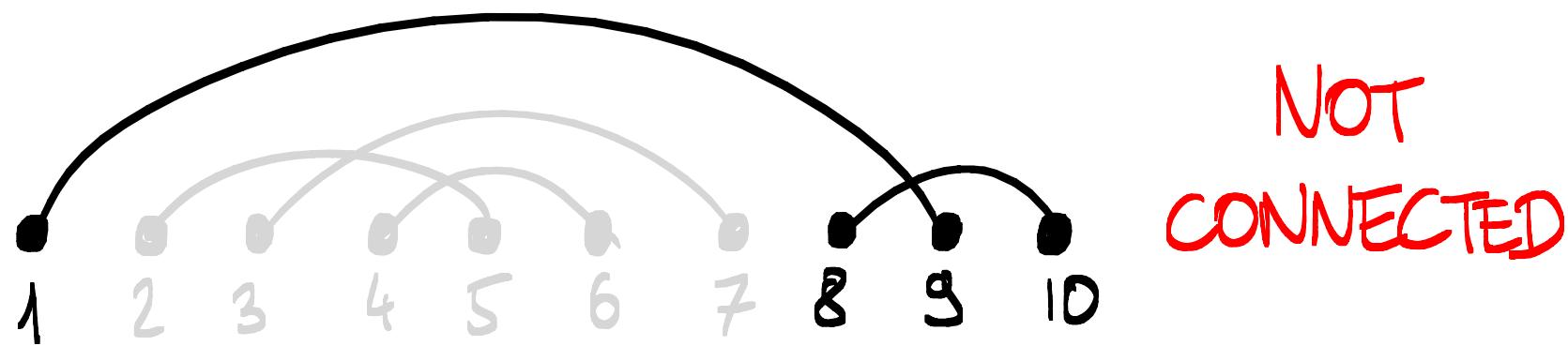
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connected diagram : its representation is
in one piece

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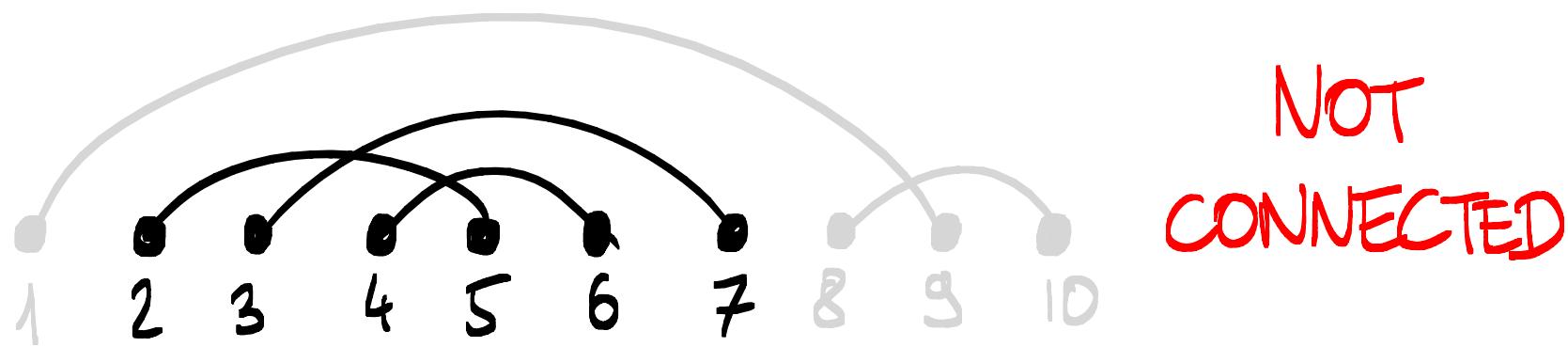
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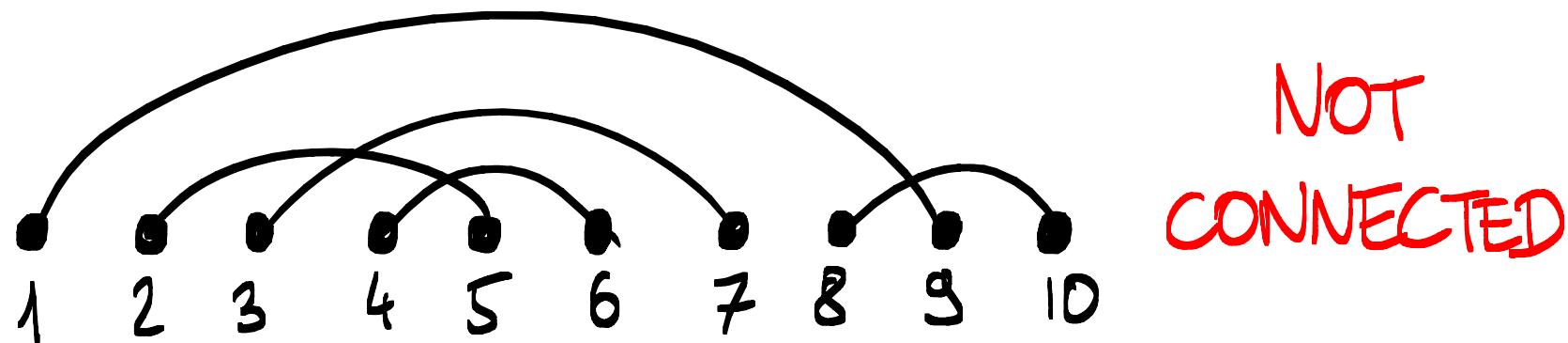
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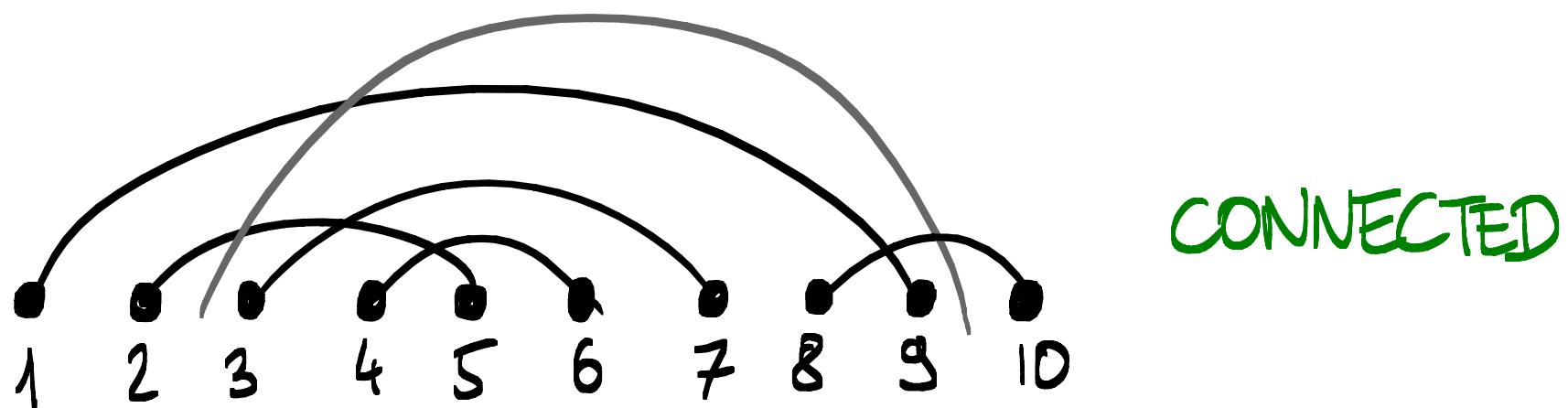
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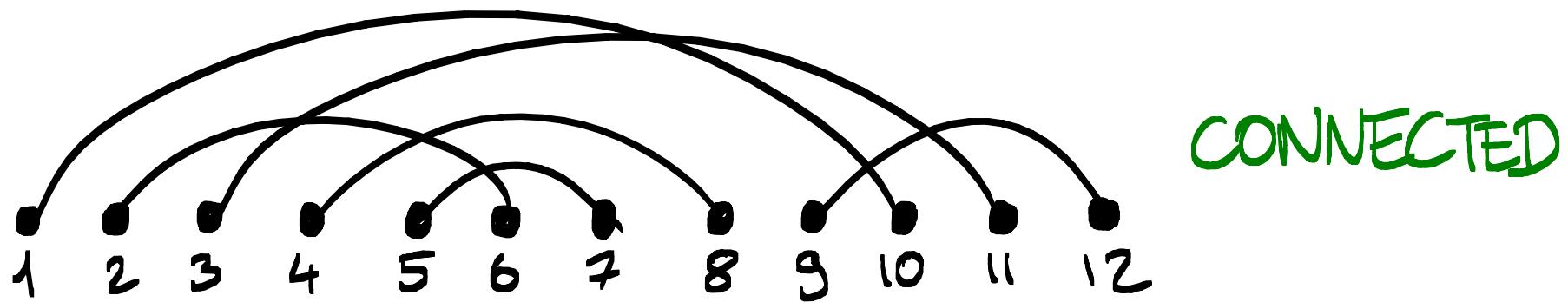
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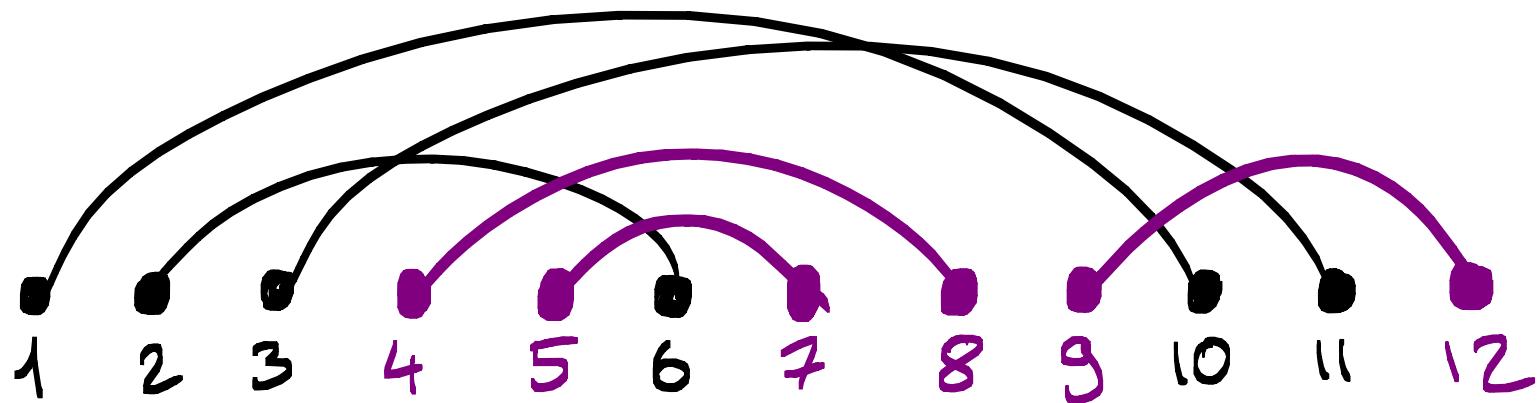
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TERMINAL CHORDS



terminal chord = chord (a, b) such that
for every chord (c, d)
that intersects it,

$$c < a < d < b.$$

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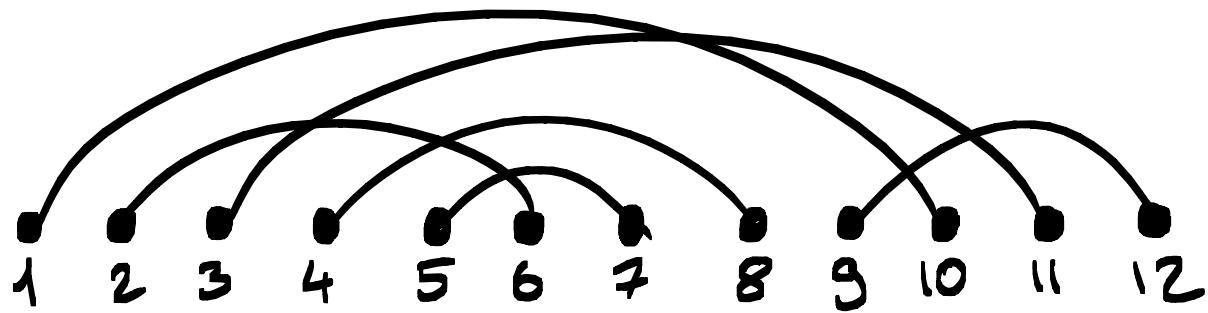
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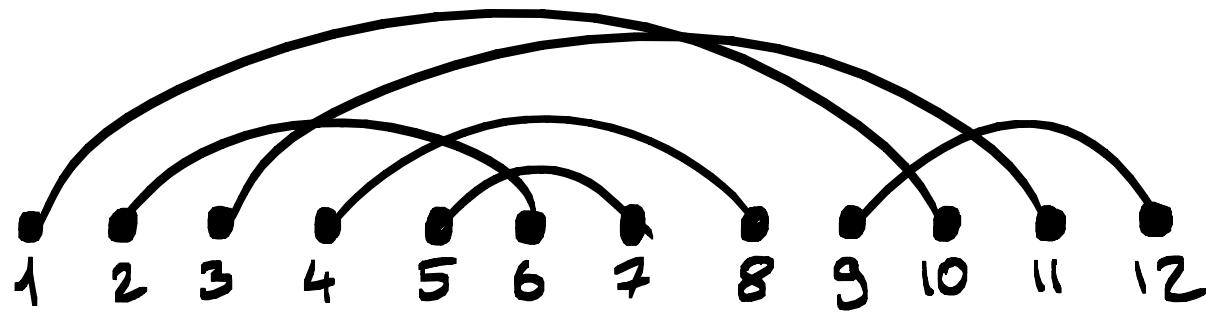
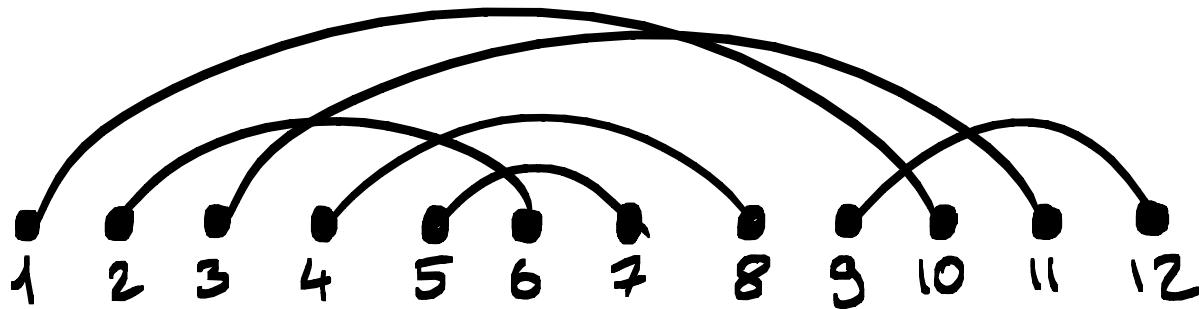
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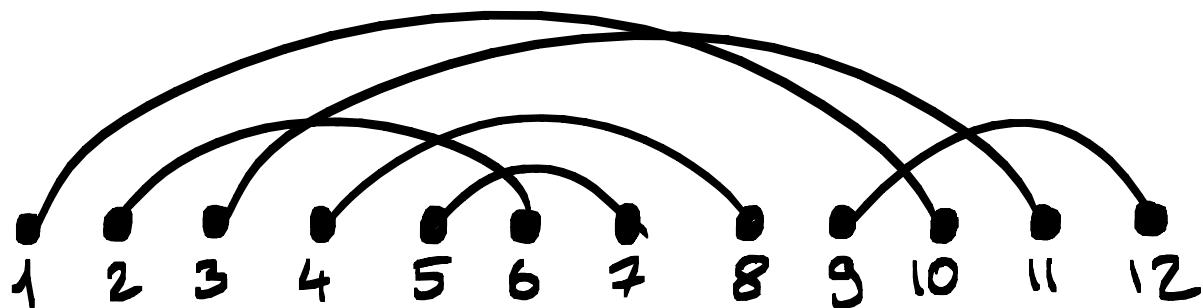
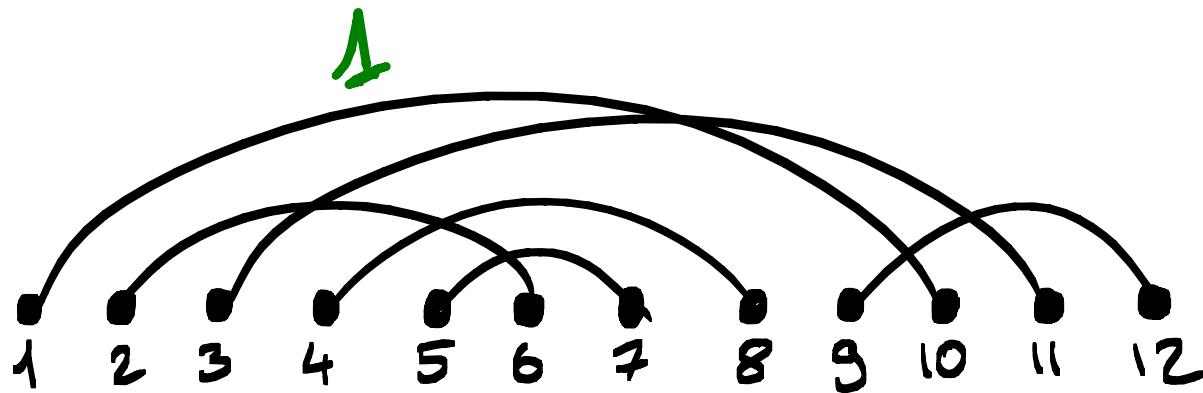
INTERSECTION ORDER



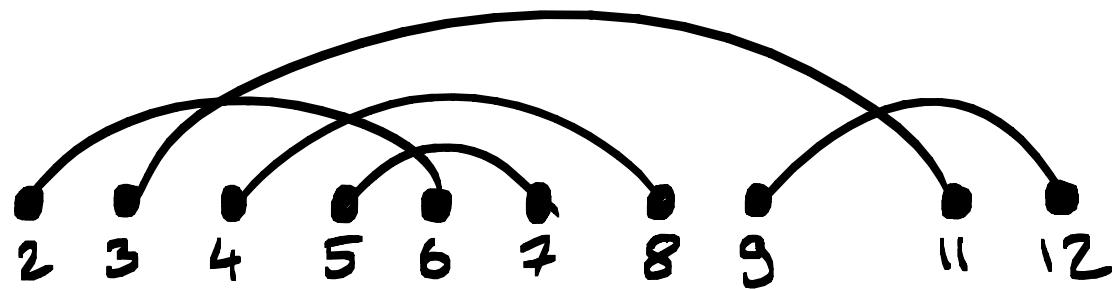
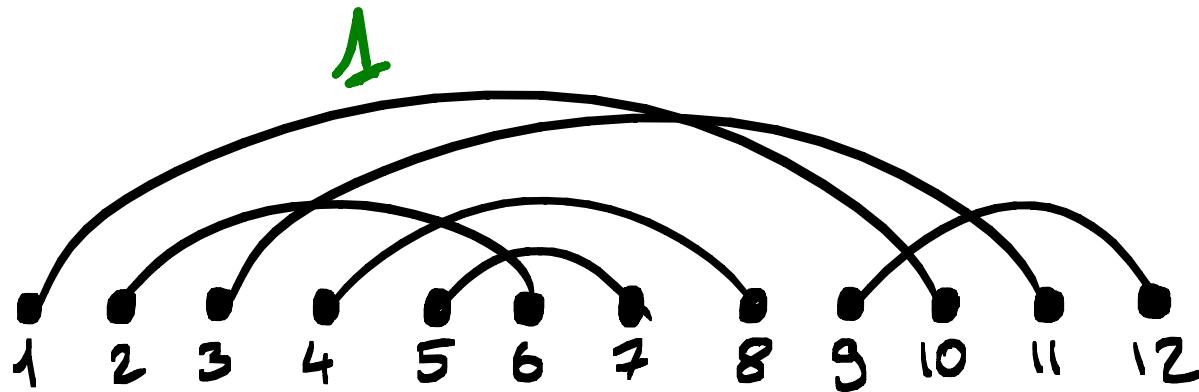
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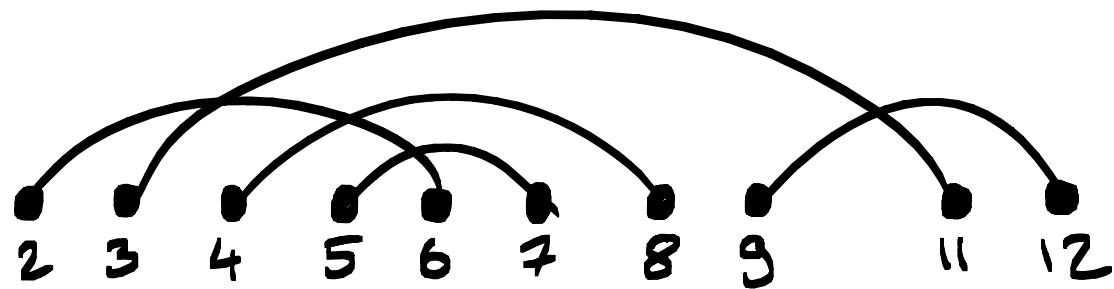
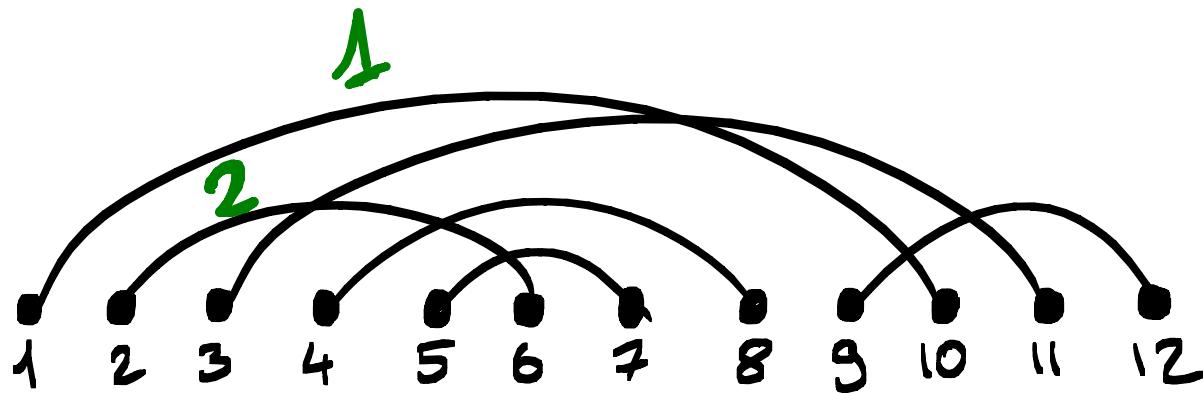
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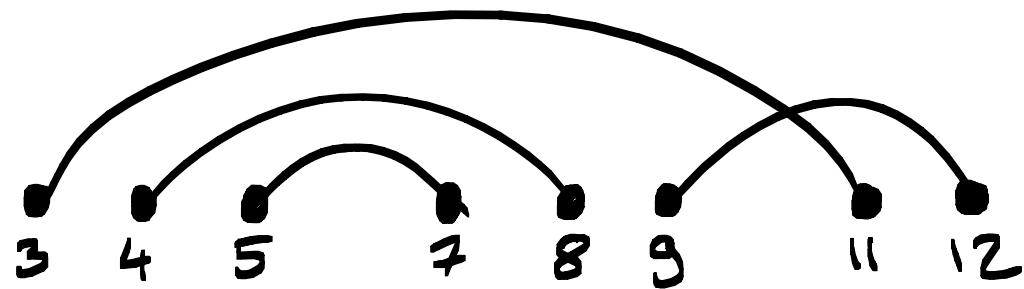
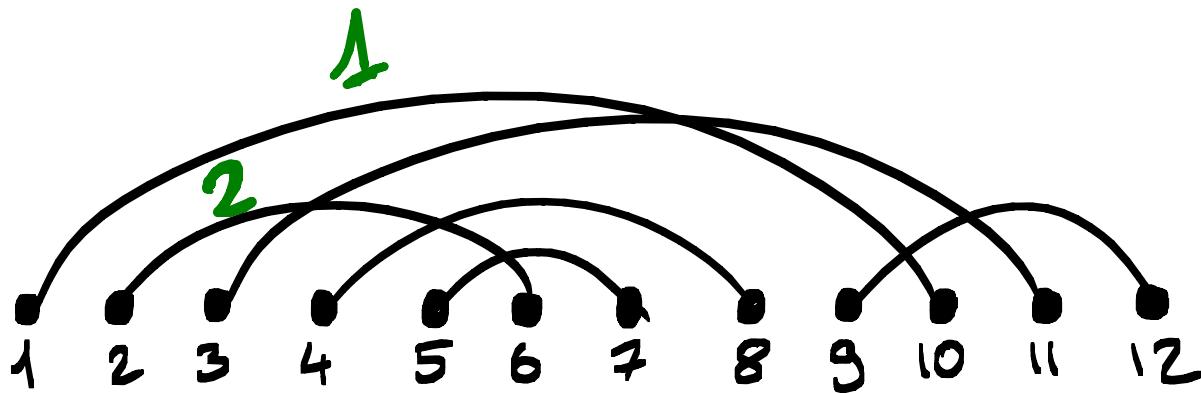
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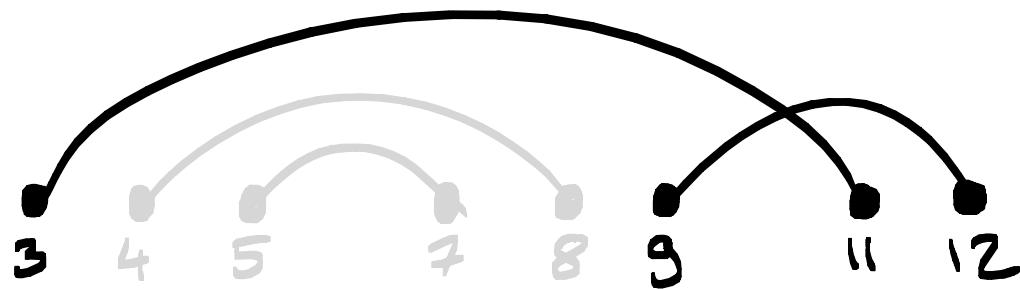
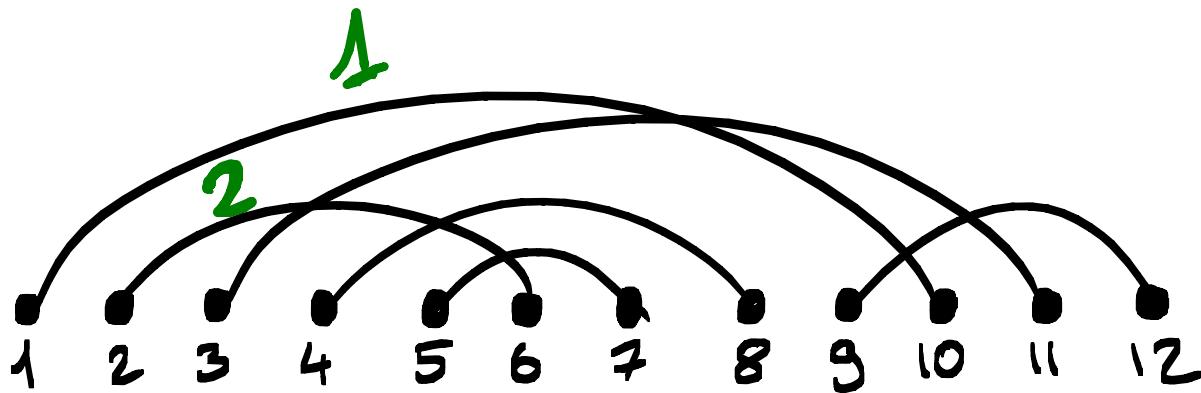
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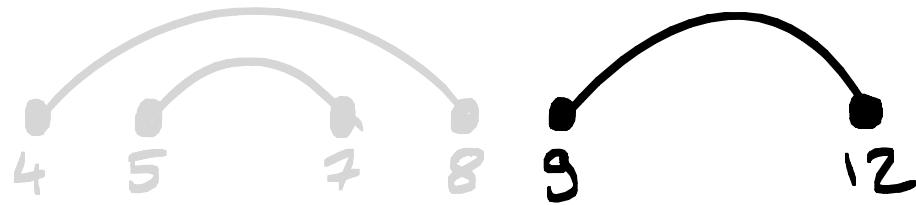
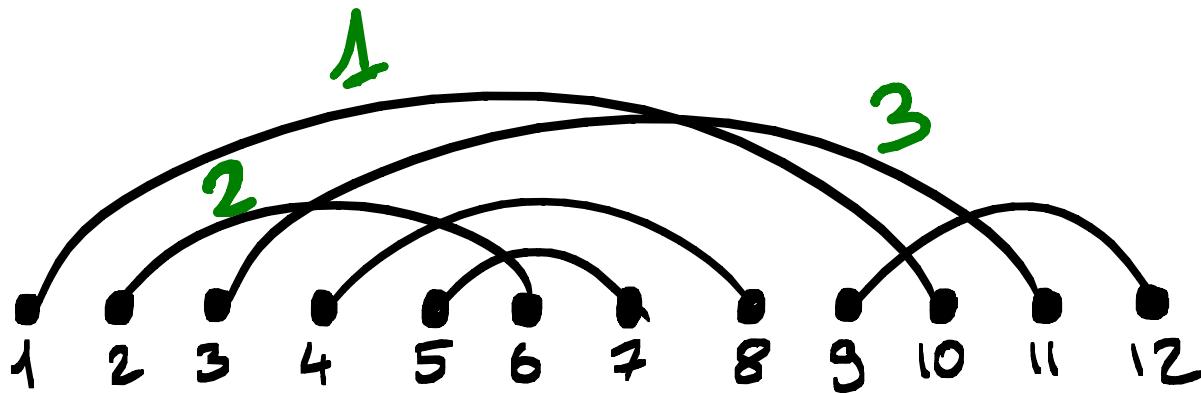
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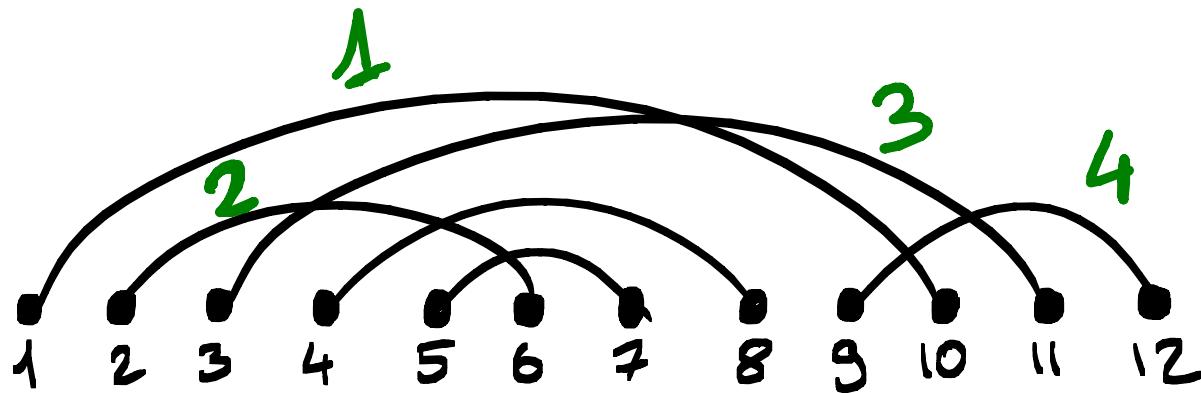
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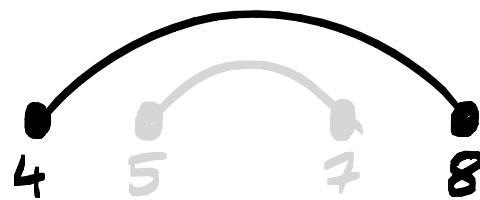
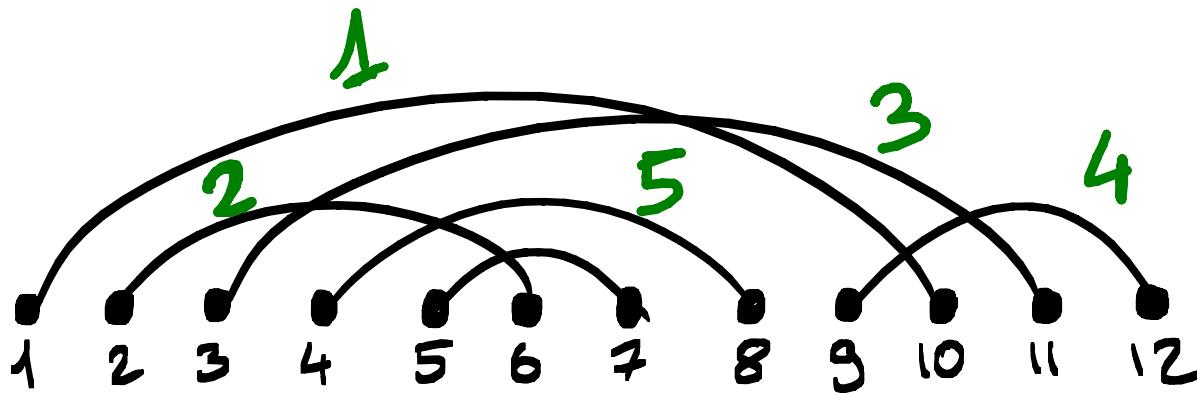
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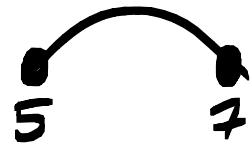
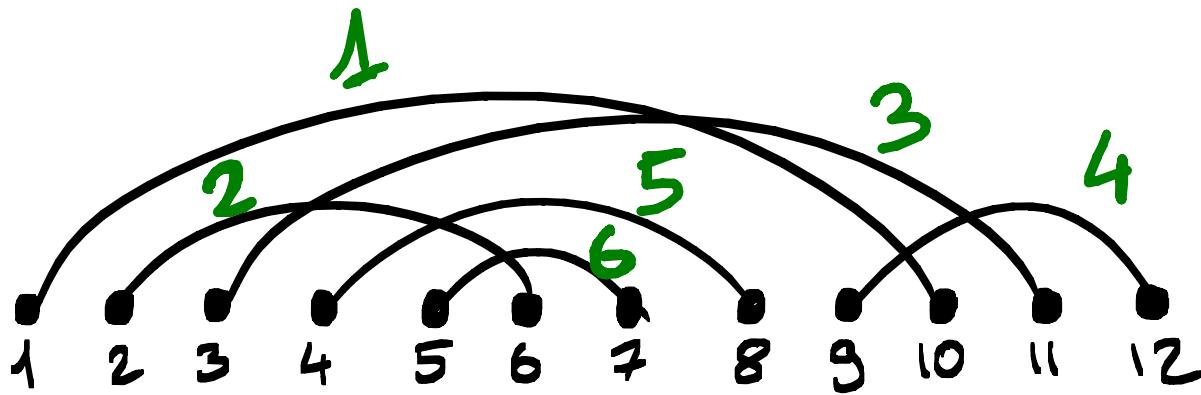
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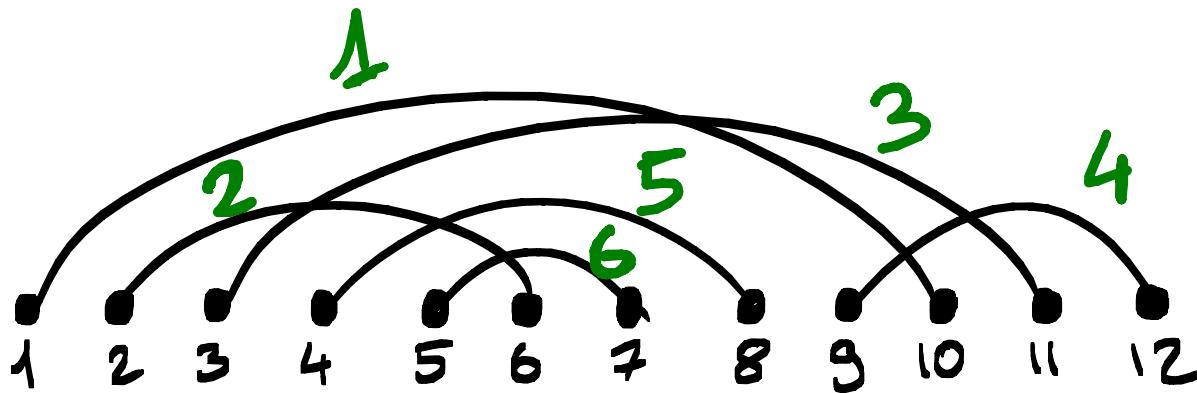
INTERSECTION ORDER



INTERSECTION ORDER



INTERSECTION ORDER



intersection order \neq left-right order

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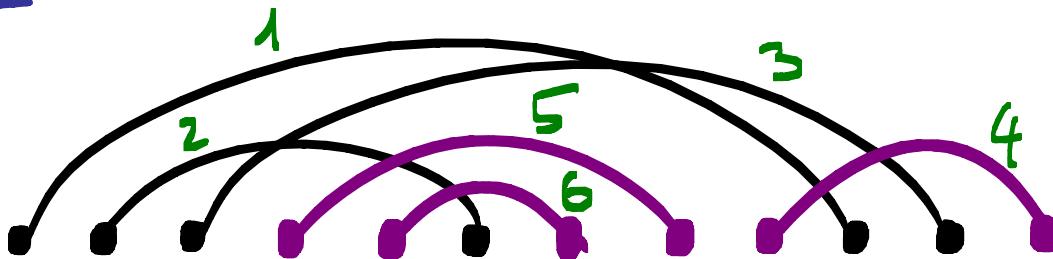
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Ex:

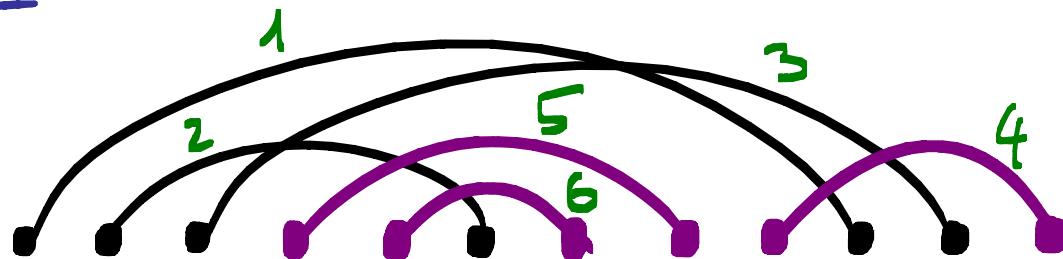


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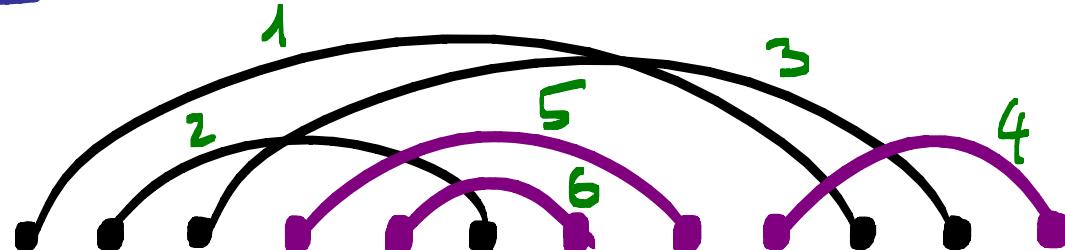
$$k=3 \quad t_1=4 \quad t_2=5 \quad t_3=6$$

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Ex: (for $i \leq 4$)



$$k=3 \quad t_1=4 \quad t_2=5 \quad t_3=6$$

$$\frac{\alpha^6}{i!} \times f_0 \times f_{4-i} \times f_1 \times f_1$$

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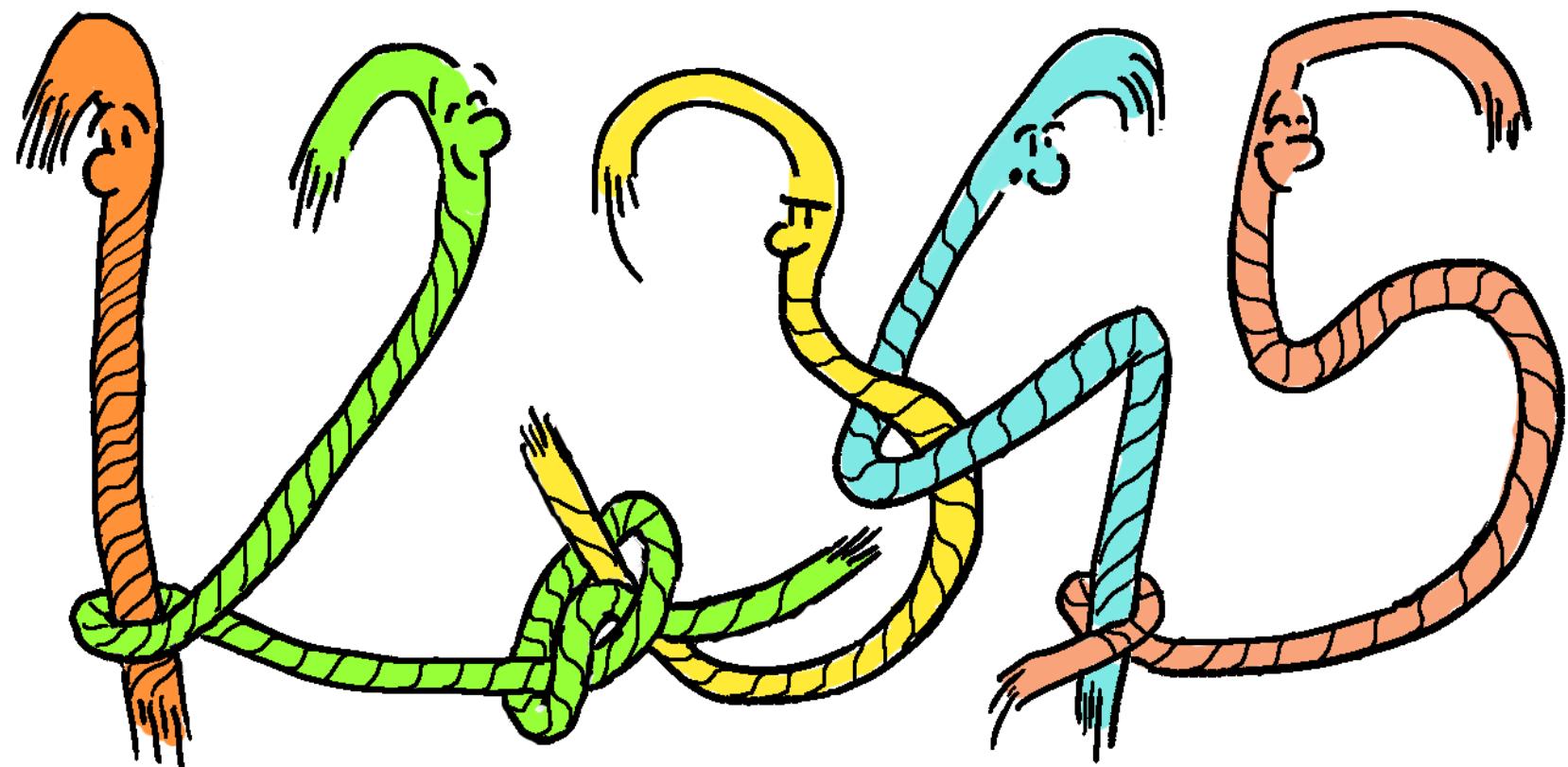
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QUESTIONS

- leading-log coefficients behaviour?
- number of terminal chords?
- position of the first terminal chord?
- number of consecutive terminal chords?

ENUMERATION OF CONNECTED CHORD DIAGRAMS



HISTORICAL BACKGROUND

About the enumeration of chord diagrams:

- Knot theory (Vassiliev invariants)
- random graph generation
- bio-informatics (RNA secondary structures)
- cumulants
- ...

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1. [Touchard, 1952] = prehistory

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2 - [Stein-Everett, 1978] = explicit formulas!

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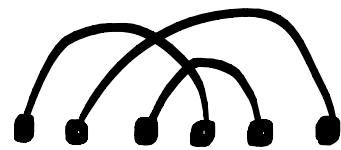
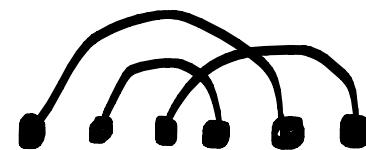
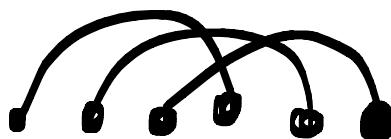
3 - [Flajolet-Noy, 2000] = analytic combinatorics!!

STEIN FORMULA

c_n = number of connected diagrams
with n chords

$$c_1 = 1 \quad c_2 = 1 \quad c_3 = 4 \quad c_4 = 27$$

For $n=3$,



TRIVIA

number of non-necessarily
connected diagrams with
 n chords = ?



number of non-necessarily
connected diagrams with $= (2n-1)!!$
 n chords

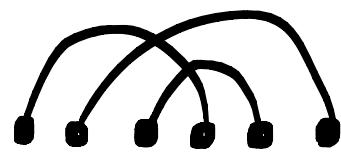
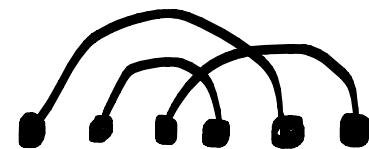
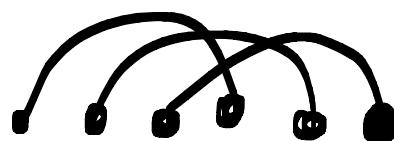
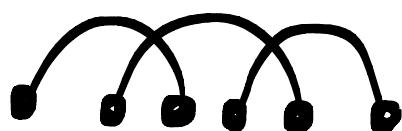
$$= (2n-1) \times (2n-3) \times \dots \times 3 \times 1$$

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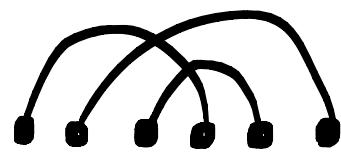
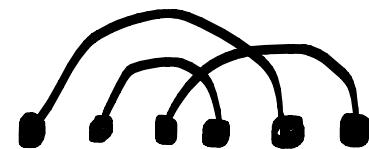


STEIN FORMULA

c_n = number of connected diagrams
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$$c_1 = 1 \quad c_2 = 1 \quad c_3 = 4 \quad c_4 = 27 \quad c_5 = 248$$

For $n=3$,

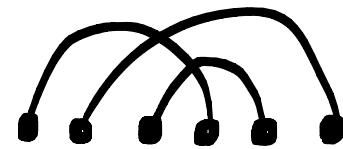
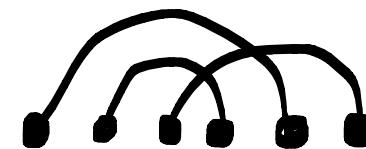
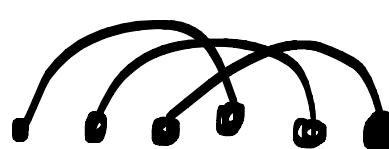
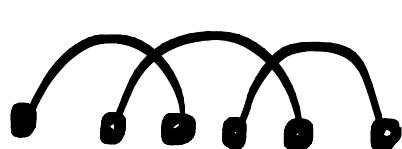


STEIN FORMULA

c_n = number of connected diagrams
with n chords

$$c_1 = 1 \quad c_2 = 1 \quad c_3 = 4 \quad c_4 = 27 \quad c_5 = 248$$

For $n=3$,



Theorem [Stein]

$$c_n = (n-1) \sum_{k=1}^{n-1} c_k \times c_{n-k}$$

STEIN FORMULA

$$\text{Theorem: } c_n = (n-1) \sum_{k=1}^{n-1} c_k \times c_{n-k}$$

STEIN FORMULA

Theorem: $c_n = \sum_{k=1}^{n-1} (2k-1) c_k \cdot c_{n-k}$

Corollary: $c_n = (n-1) \sum_{k=1}^{n-1} c_k \cdot c_{n-k}$

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Theorem: $c_n = \sum_{k=1}^{n-1} (2k-1) c_k c_{n-k}$

(var. change $k \leftarrow n-k$)

$$c_n = \sum_{k=1}^n (2n-2k-1) c_{n-k} c_k$$

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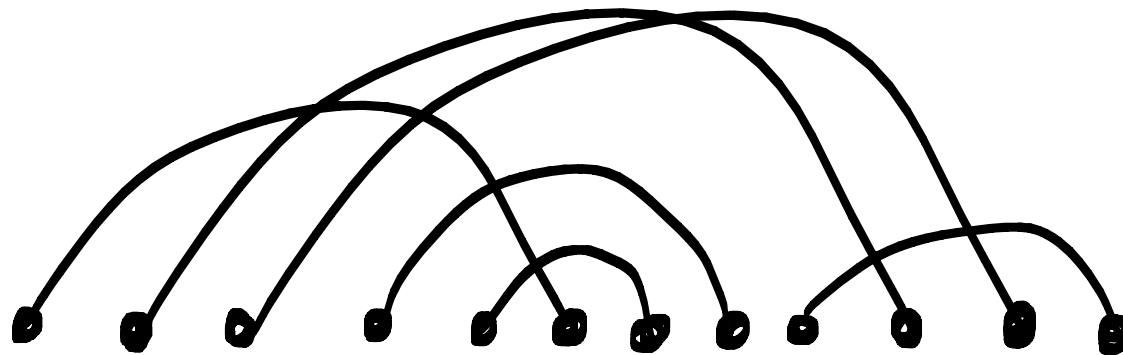
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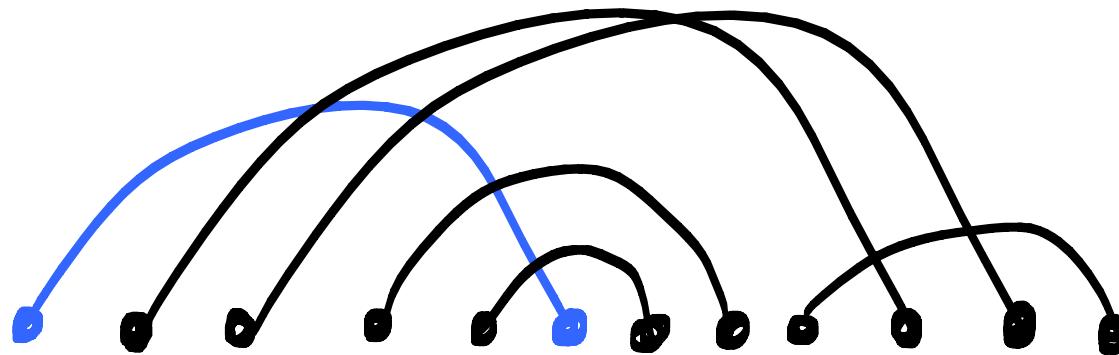
Proof:



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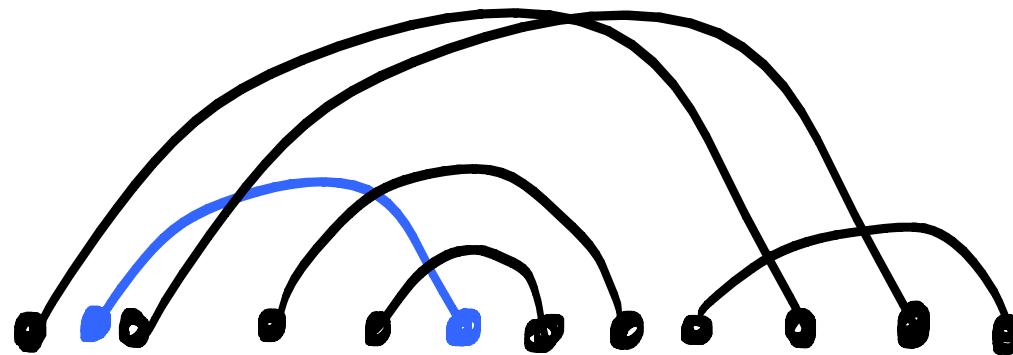
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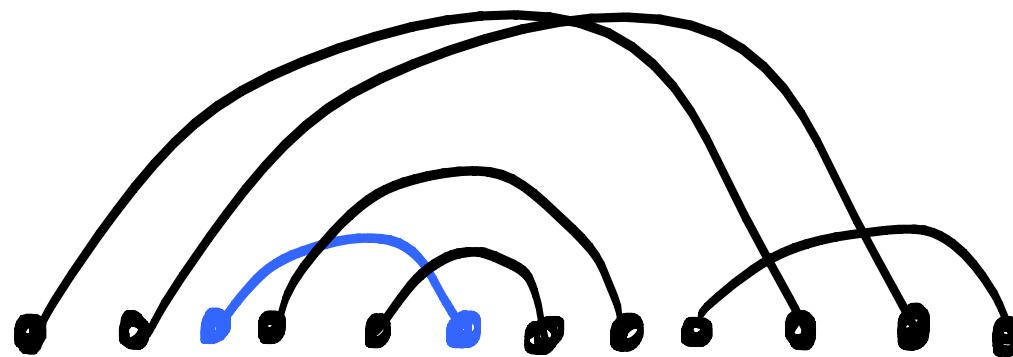
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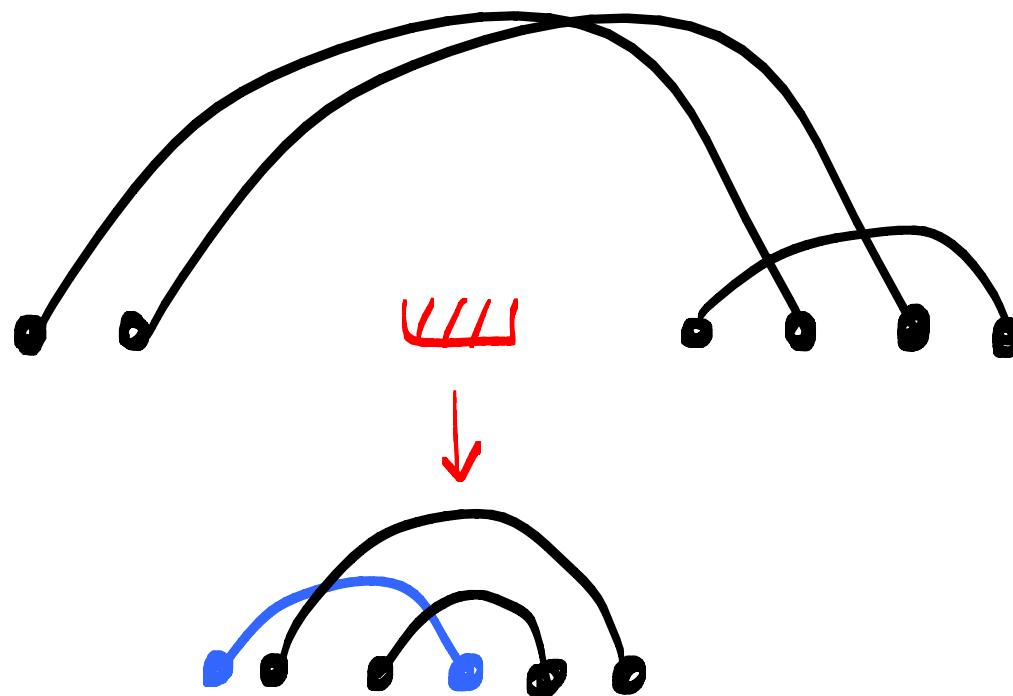
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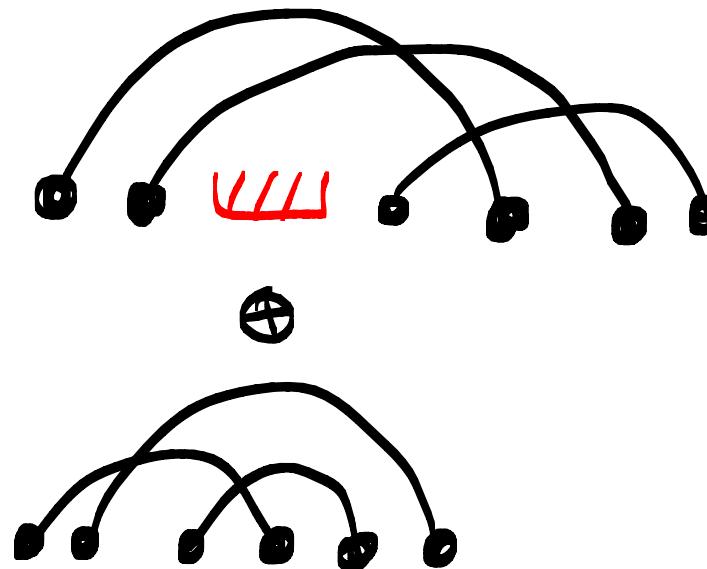
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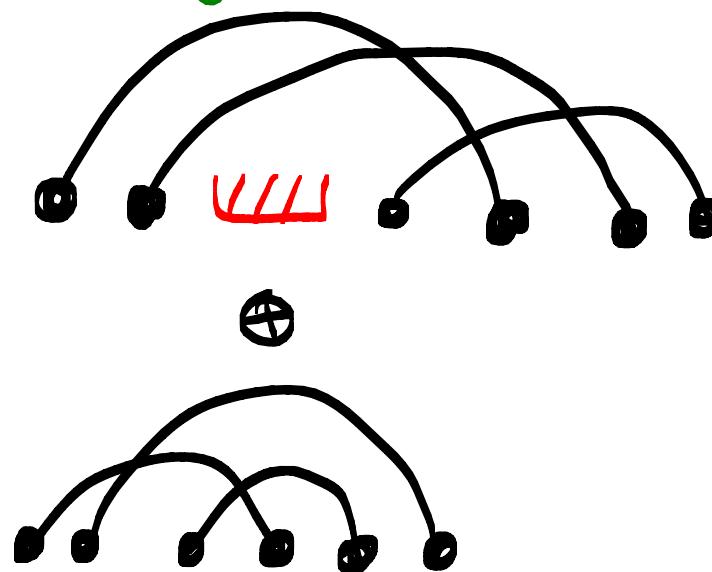
STEIN FORMULA

Theorem: $c_n = \sum_{k=1}^{n-1} (2k-1) c_k c_{n-k}$

Proof:

If k chords,
↓

then $(2k-1)$
possible
insertions



c_n VS CATALAN

CONNECTED
DIAGRAMS

$$c_n = (n-1) \sum_{k=1}^{n-1} c_k c_{n-k}$$

CATALAN

$$c_n = \sum_{k=1}^{n-1} c_k c_{n-k}$$

c_n VS CATALAN

CONNECTED
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$$c_n = (n-1) \sum_{k=1}^{n-1} c_k c_{n-k}$$

$$c_n \geq (n-1) \times c_1 \times c_{n-1}$$

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$$c_n \geq (n-1)!$$

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→ analytic

- No simple equation for the Exponential Generating Functions

ASYMPTOTIC BEHAVIOUR

[Stein-Everett]

$$c_n \sim \frac{1}{e} \times (2n-1)!!$$

Consequence : $P(\text{diagram is connected}) \rightarrow \frac{1}{e}$

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- number of connected components $\sim \text{Poisson}(1)$
- n - size of the largest component $\sim \text{Poisson}(1)$

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-

Our humble contribution : $\frac{c_{n-1}}{c_n} = \frac{1}{2n} + \frac{1}{4n^2} - \frac{1}{4n^3} + O\left(\frac{1}{n^3}\right)$

STATISTICS ON TERMINAL CHORDS

I WAS
A JOKE
IN FRENCH
BUT I DON'T
WORK ANYMORE



LEADING - LOG TERMS

$$G(x, L) = 1 - \sum_{i \geq 1} \sum_{\substack{C \text{ connected} \\ \text{chord diagram} \\ \text{with terminal chords} \\ \text{in position } t_1 < t_2 < \dots < t_k \\ \text{such that } t_1 \geq i}} \frac{i^{|C|}}{i!} \alpha^{|C|-k} f_{t_1-i} f_{t_2-t_1} f_{t_3-t_2} \dots f_{t_k-t_{k-1}}$$

LEADING - LOG TERMS

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Coefficient of $(Lx)^i x^{|C|-i}$ for i close to $|C|$?

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Coefficient of $(Lx)^i x^{|C|-i}$ for i close to $|C|$?

- $|C| = i$: leading-log expansion
- $|C| = i+1$: next-to leading-log expansion
- $|C| = i+l$: next-to ^{l} leading-log expansion

LEADING - LOG TERMS

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Coefficient of $(Lx)^i x^{|C|-i}$ for i close to $|C|$?

- $|C| = i$: leading-log expansion

$$\Leftrightarrow t_1 = |C|$$

\Leftrightarrow There is only one terminal chord.

ONLY ONE TERMINAL CHORD

number of connected diagrams with n chords
and only one terminal chord
= ?

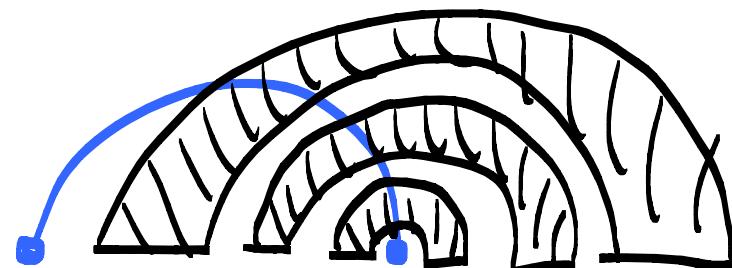
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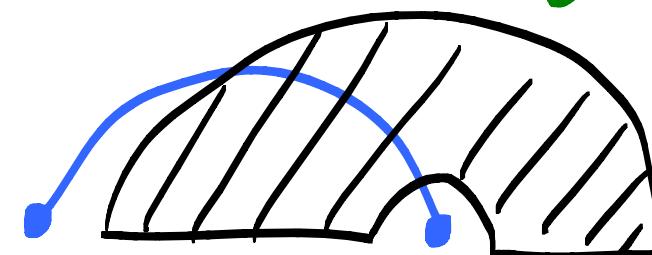
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Proof:



impossible

One piece of size $n-1$

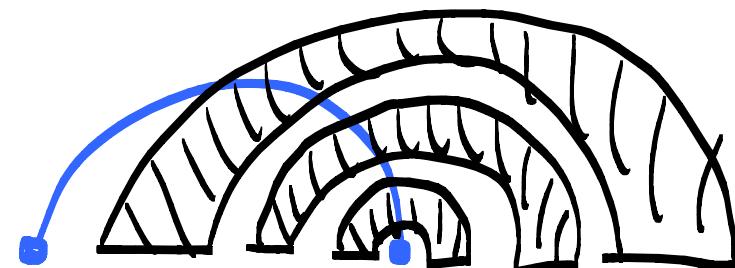


2n-3 possible
locations

ONLY ONE TERMINAL CHORD

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Proof:



impossible



One piece of size $n-1$

2 $n-3$ possible locations

Cor: n^{th} coeff of
the leading-log expansion $= \frac{(2n-3)!!}{n!} f_0^n$

NEXT-TO^l LEADING-LOG TERMS

- "Similar" recursions exist for the diagrams such that $t_1 \geq |C| - l$
- Analytic combinatorics techniques work here.

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Theorem : For $l \geq 0$,
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$$\sim \frac{1}{\sqrt{\pi} 2^{l+1} l!} \times \frac{\ln(n)^l}{n^{\frac{3}{2}}} n!$$

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But how about $\oint_0^{|C|-k} f_{t_1-i} f_{t_2-t_1} f_{t_3-t_2} \dots f_{t_k-t_{k-1}} ?$

THE LAST l CHORDS ARE TERMINAL

- "Similar" recursions exist for the diagrams such that the last l chords are terminal
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Here $f_0^{(c)-k} f_{t_1-t_i} f_{t_2-t_1} f_{t_3-t_2} \dots f_{t_k-t_{k-1}} = f_0^{n-l+1} f_1^{l-1}$

NEXT-TO l LEADING-LOG TERMS

Diagrams such that the last l chords are terminal are dominant among the diagrams such that $t_1 \geq |C| - l -$

Corollary : For $l \geq 0$,

n^{th} coeff of the next-to l leading-log expansion

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Only f_0 and f_1 matter!

NUMBER OF TERMINAL CHORDS

Average number of terminal chords ?

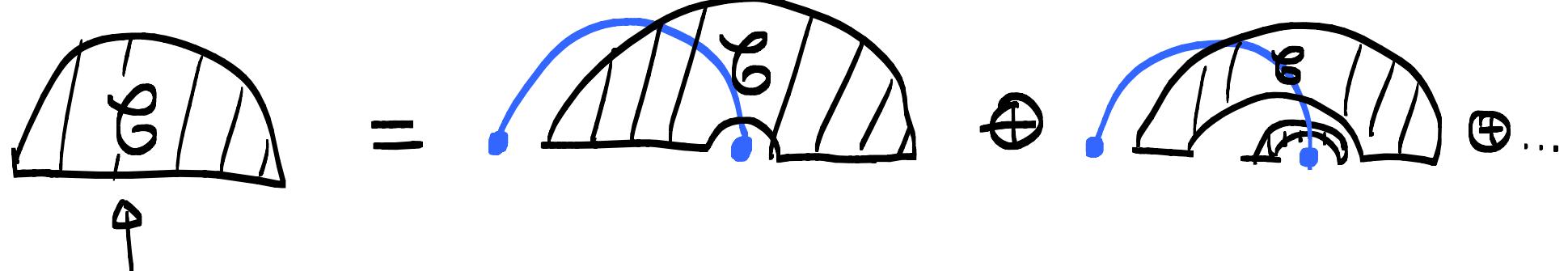
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Idea:



large number
of chords

NUMBER OF TERMINAL CHORDS

Average number of terminal chords ??????

Idea:

$$\text{large number of chords} = \frac{\text{proba}}{\binom{2n}{2}}$$

The diagram illustrates a large circle divided into many sectors by a large number of chords. A blue arc connects two points on the circumference. The word "proba" is written above the arc with an arrow pointing to it.

$\binom{2n}{2}$

NUMBER OF TERMINAL CHORDS

Average number of terminal chords ??????

Idea:

large number of chords

$$= \frac{(2n-3)C_{n-1}}{C_n}$$

proba $\rightarrow 1$

NUMBER OF TERMINAL CHORDS

Average number of terminal chords ??????

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$$\text{large number of chords} = \frac{(2n-3)C_{n-1}}{C_n} \xrightarrow{\text{proba}} 1$$

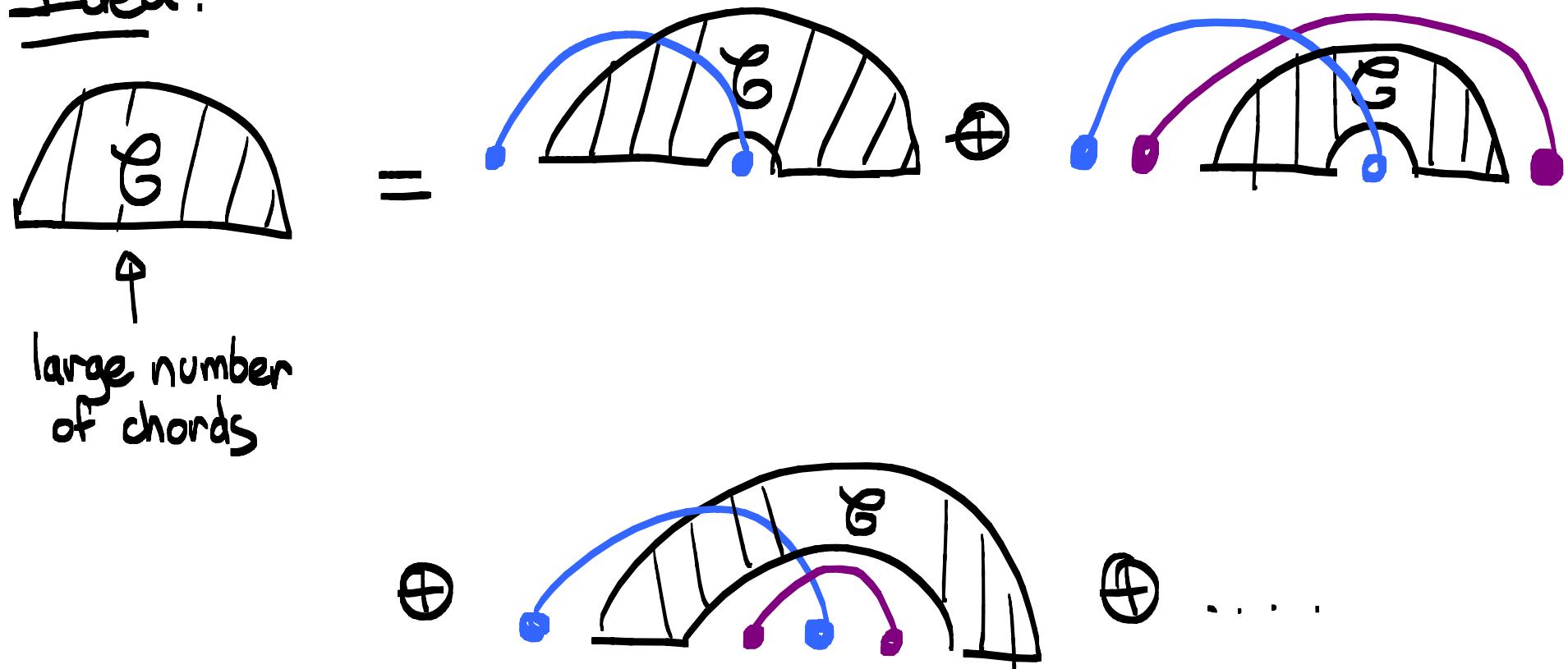
proba $\rightarrow 0$

Interesting but not sufficient...

NUMBER OF TERMINAL CHORDS

Average number of terminal chords ??????

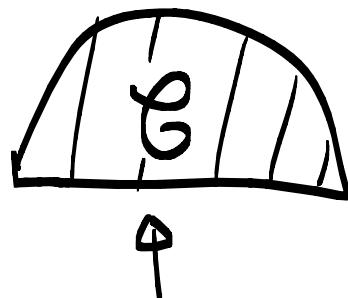
Idea:



NUMBER OF TERMINAL CHORDS

Average number of terminal chords ??????

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large number
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$$= \underbrace{\text{proba} = \frac{(2n-3)c_{n-1}}{c_n}}_{\oplus}$$

A diagram of a circle with multiple chords labeled 'G'. The chords are drawn from various points on the circumference to other points, forming a complex web of lines within the circle.

$$\underbrace{\text{proba} = \frac{(2n-5)c_{n-2}}{c_n}}_{\oplus}$$

A diagram of a circle with several chords labeled 'G'. Some chords are blue and some are purple, showing different configurations of chords within the circle.

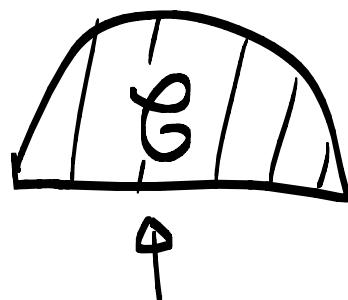
$$\oplus \underbrace{\text{proba} = \frac{(2n-5)c_{n-2}}{c_n}}_{\oplus \dots}$$

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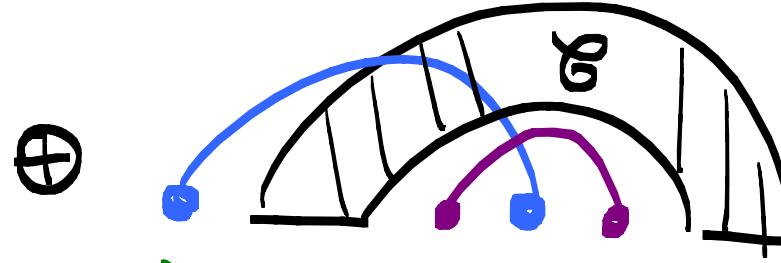
Idea:



large number
of chords

$$= \underbrace{\text{proba} = \frac{(2n-3)c_{n-1}}{c_n}}_{= 1 - \frac{1}{n} + O\left(\frac{1}{n}\right)}$$

$$\underbrace{\text{proba} = \frac{(2n-5)c_{n-2}}{c_n}}_{\sim \frac{1}{2n}}$$



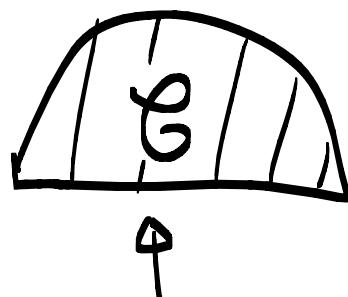
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...
= $O\left(\frac{1}{n}\right)$

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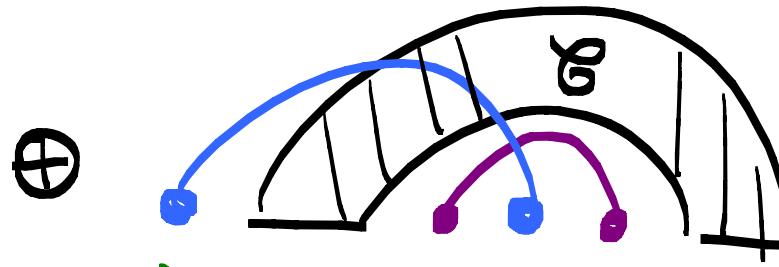
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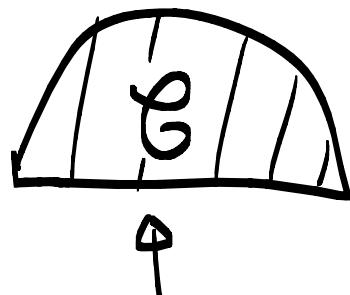
~~Let's forget that~~

Let's forget that

NUMBER OF TERMINAL CHORDS

Set $p_{n,k} = \left(1 - \frac{1}{n}\right) p_{n-1,k} + \frac{1}{n} p_{n-2,k-1}$

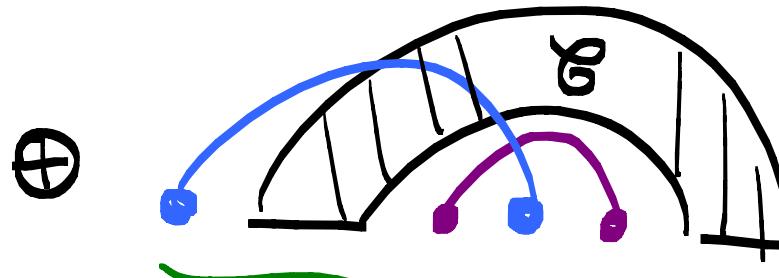
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$$\text{proba} = \frac{(2n-5)c_{n-2}}{c_n} \sim \frac{1}{2n}$$

$$\cancel{\dots} = O\left(\frac{1}{n}\right)$$

Let's forget that

NUMBER OF TERMINAL CHORDS

Set $p_{n,k} = \left(1 - \frac{1}{n}\right) p_{n-1,k} + \frac{1}{n} p_{n-2,k-1}$

Fact 1: Let X_n be the random variable such that $P(X_n=k) = p_{n,k}$

$X_n \rightarrow$ Gaussian law.

Fact 2: The number of terminal chords $\sim X_n$

NUMBER OF TERMINAL CHORDS

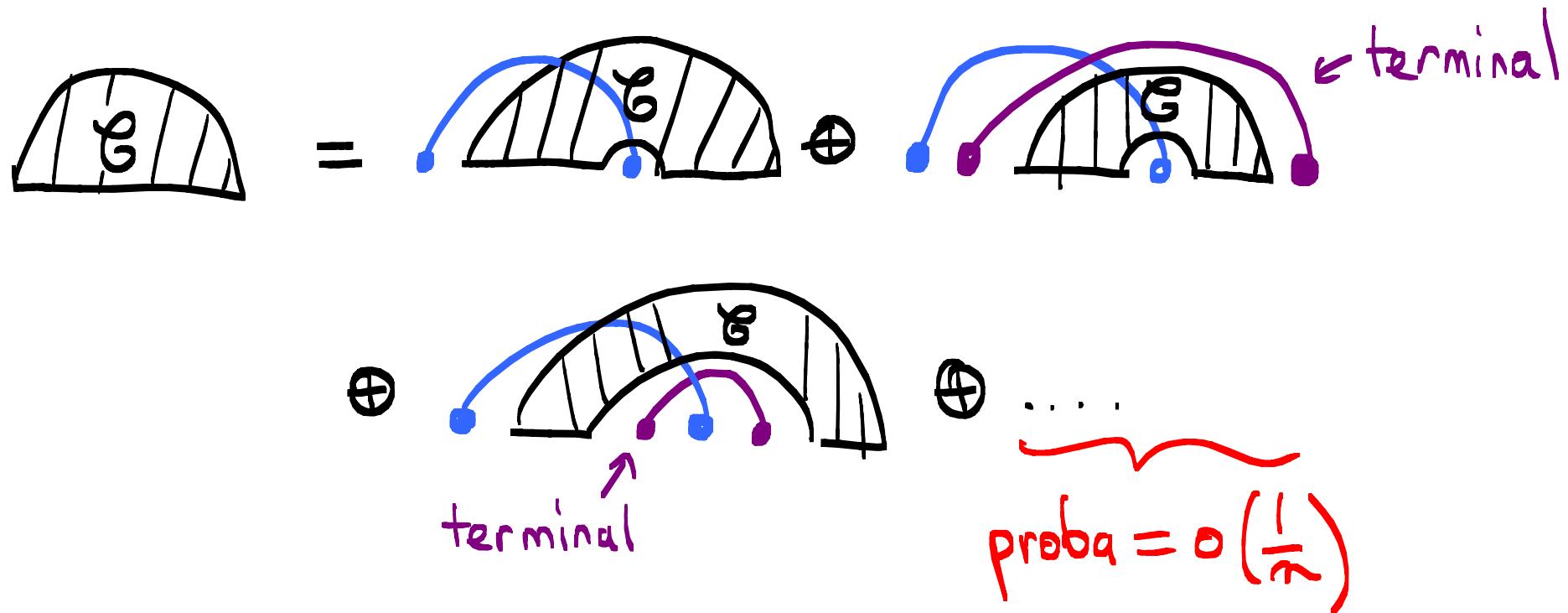
Theorem : The number of terminal chords in a random connected diagram of size n asymptotically obeys to a Gaussian limit law of mean and variance $\sim \ln(n)$ -

NUMBER OF CONSECUTIVE TERMINAL CHORDS

If the terminal chords are in position $t_1 < \dots < t_R$,
how many j 's satisfy $t_j^o - t_{j-1}^o = 1$?

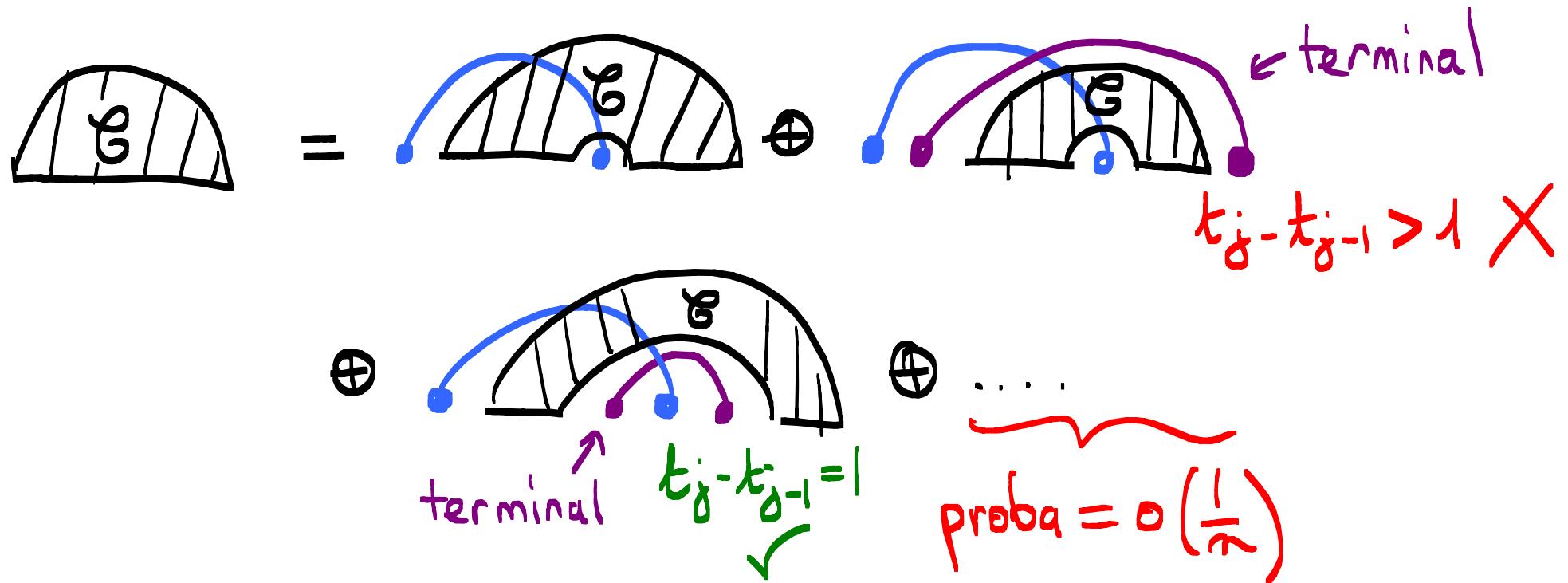
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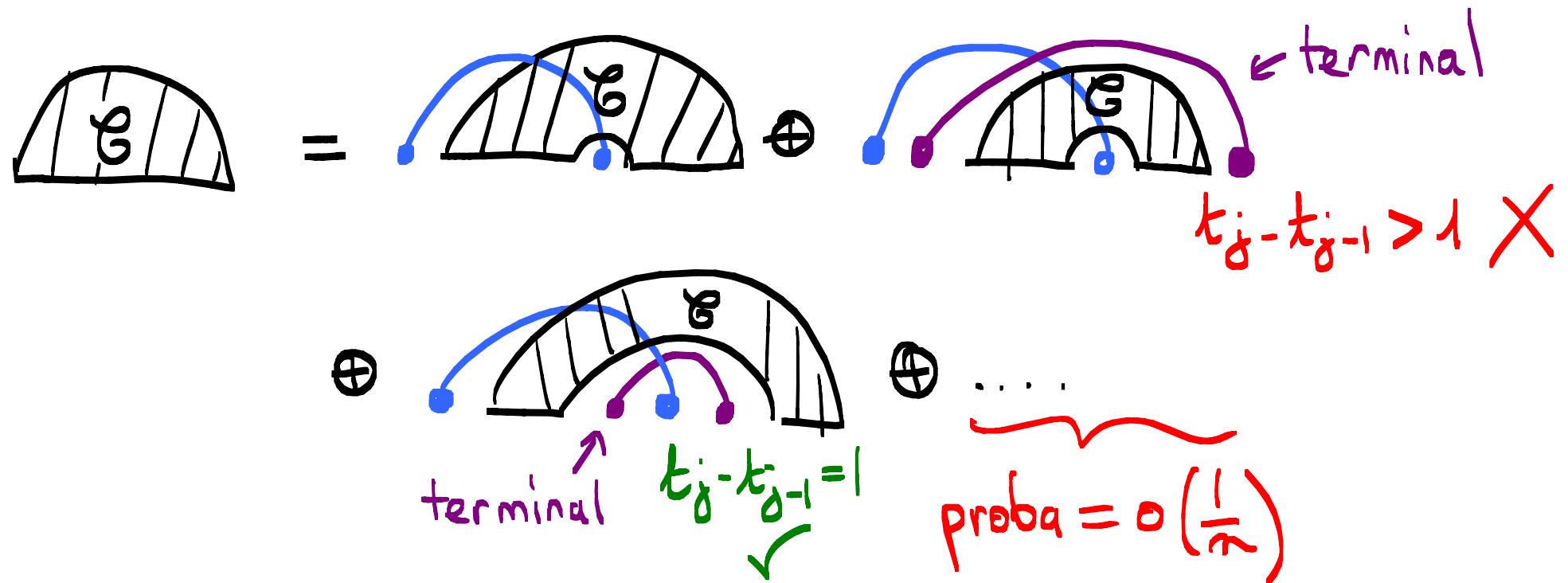
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On average,

$$\int_{t_1}^{t_{R-1}} f_{t_1-i} f_{t_2-t_1} f_{t_3-t_2} \dots f_{t_{R-1}-t_{R-2}} \sim \int_{t_1}^{t_{R-1}} f_{t_1-i} f_1 \frac{\ln n}{2} \dots$$

→ confirms the importance of f_0 and f_1

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POSITION OF THE FIRST TERMINAL CHORD -

t_1 = random variable returning the position
of the 1st terminal chord.

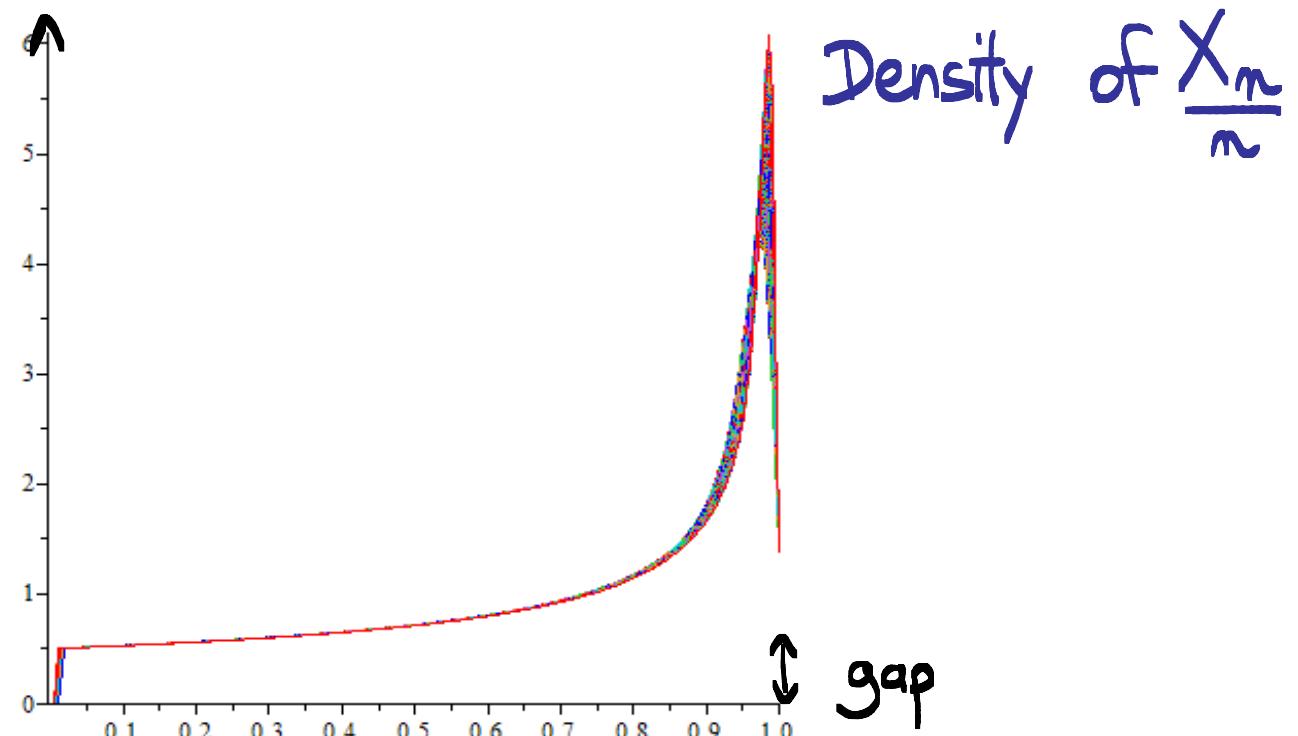
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Limit law?



CONCLUSION

- Recovers the results of Krüger and Kreimer
 - + automaticity of the method
 - + asymptotic behaviour
- New combinatorial approach
- Extension to Hihn-Yeats's results?



11-12 juillet 2016 - VANCOUVER

(après FPSAC)