Julien Courtiel (PIMS, UBC)
Karen Yeats (SFU)

# Terminal Chords in Connected Chord Diagrams

SFU, March 8


Tom Hanks
Catherine Zeta-Jones
The Terminal
Life is waiting.

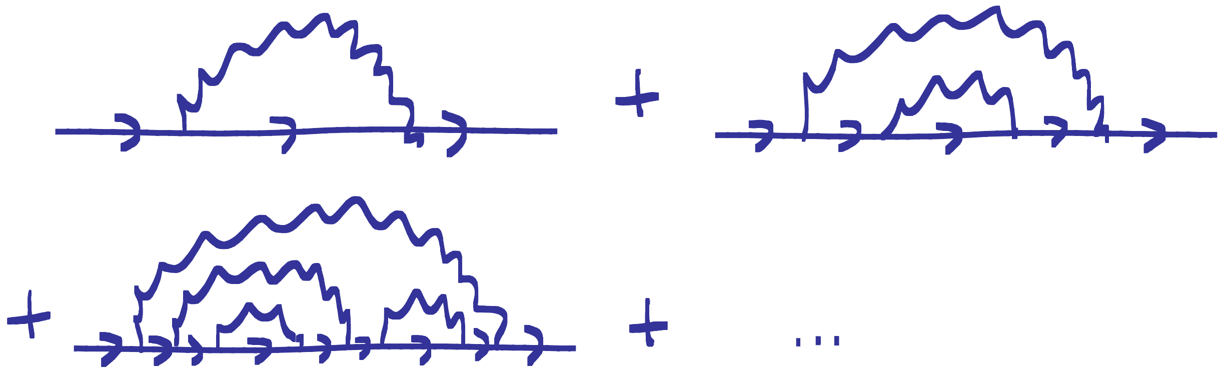
# COMBINATORIAL DYSON-SCHWINGER EQUATIONS

Class of Feynman graphs

$\Big\downarrow$ Feynman rules

solution of Dyson-Schwinger equations

One-loop propagator + recursive iterations



$$G(x,L) = 1 - x\, G\!\left(x, \frac{\partial}{\partial(-\rho)}\right)^{-1} \left(e^{-L\rho} - 1\right) F(\rho)\Big|_{\rho=0}$$

# COMBINATORIAL DYSON-SCHWINGER EQUATIONS

**Theorem** [Marie, Yeats]

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where

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$C$ connected chord diagram with terminal chords in position $t_1 < t_2 < \ldots < t_k$ such that $t_1 \geq i$.

where

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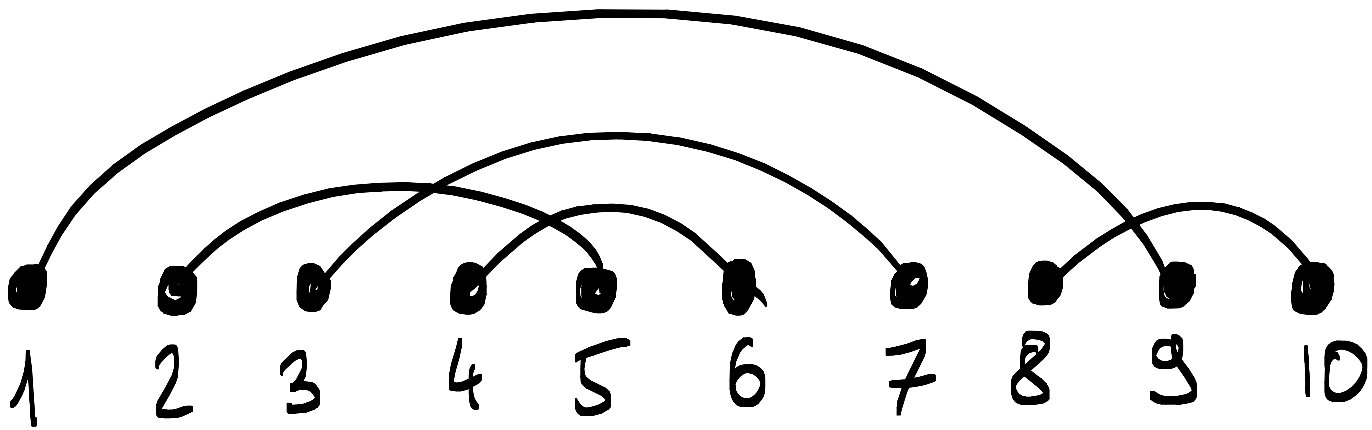
# CONNECTED CHORD DIAGRAMS

Diagram of $n$ chords = matching of $\{1, 2, \dots, 2n\}$



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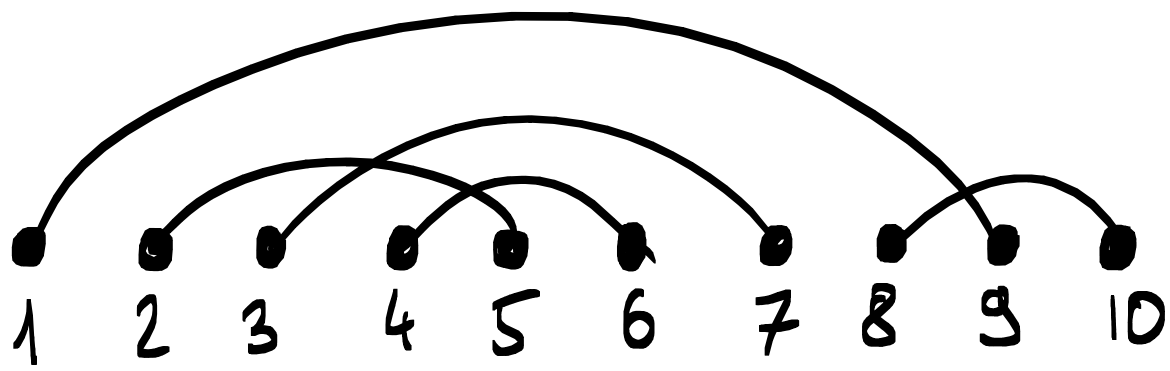


NOT CONNECTED

connected diagram : its representation is in one piece

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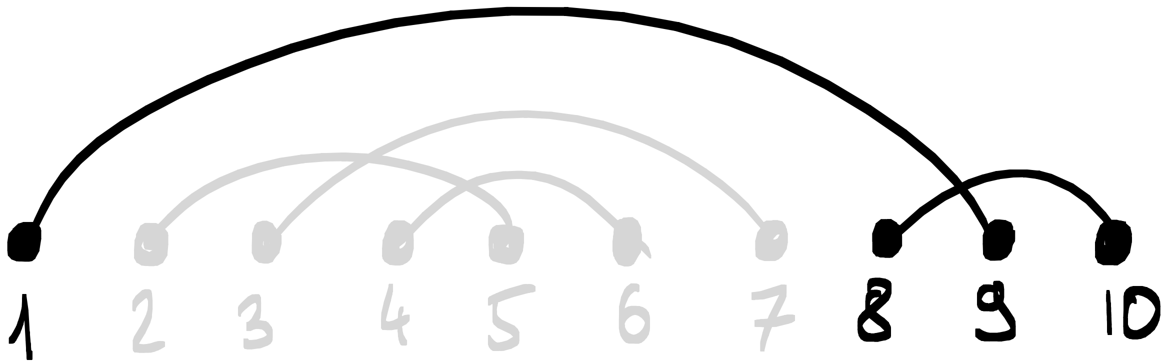


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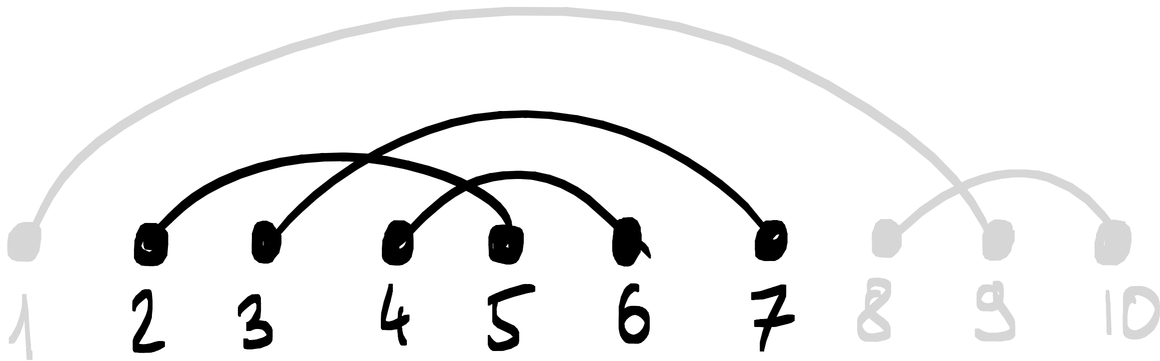


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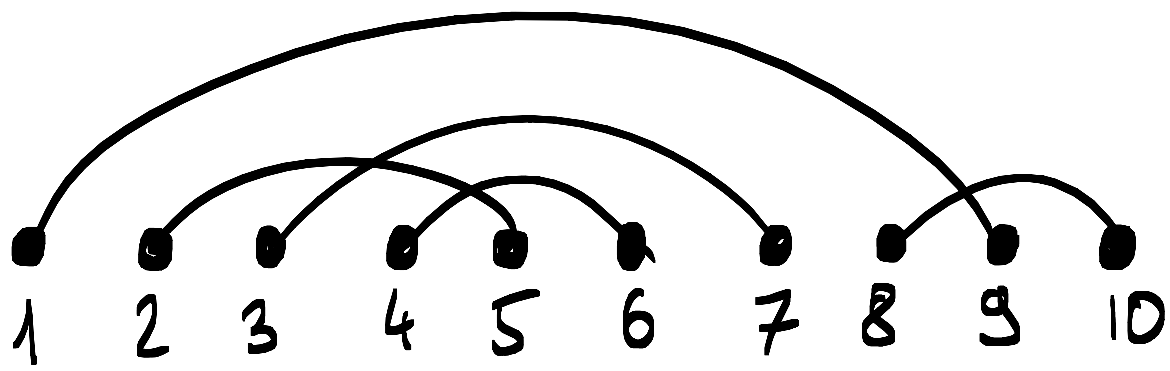


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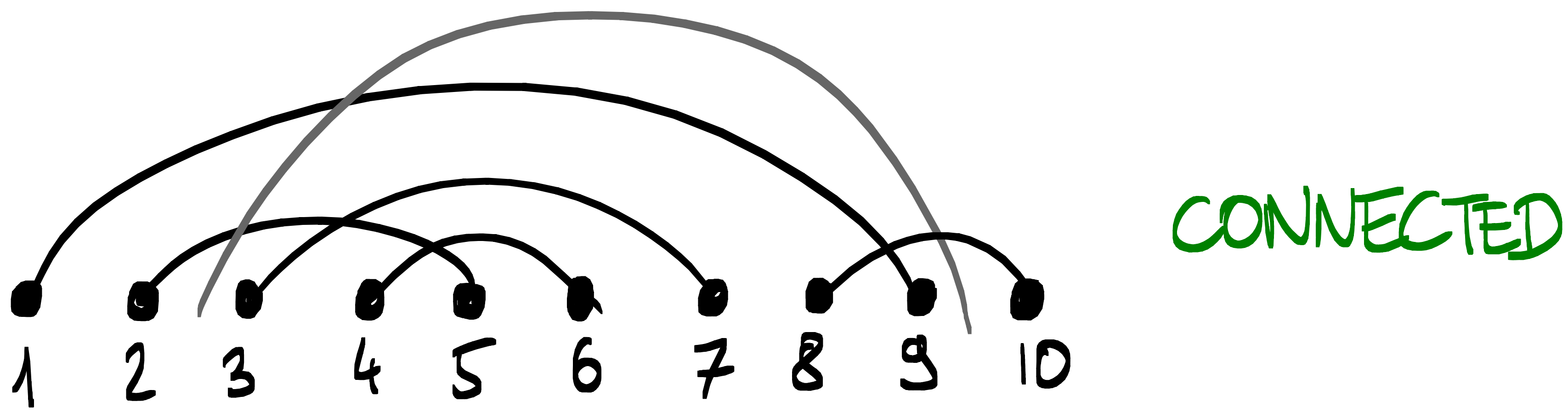


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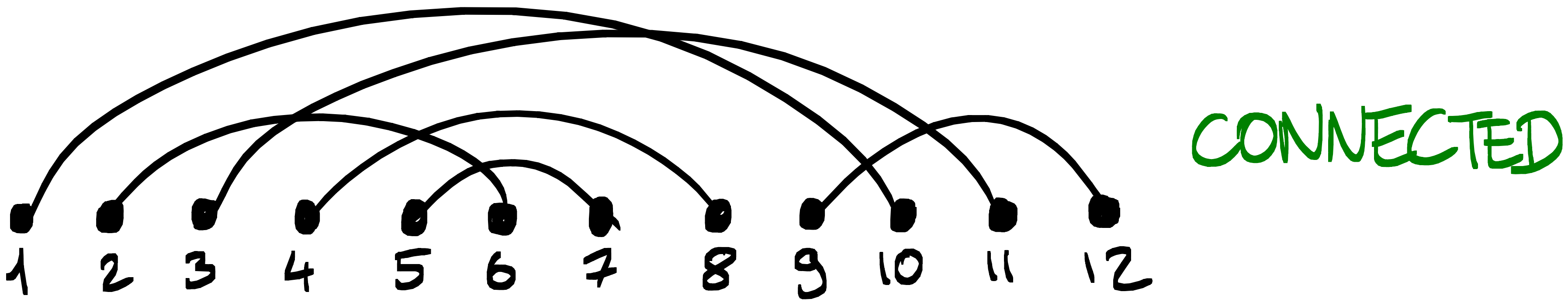


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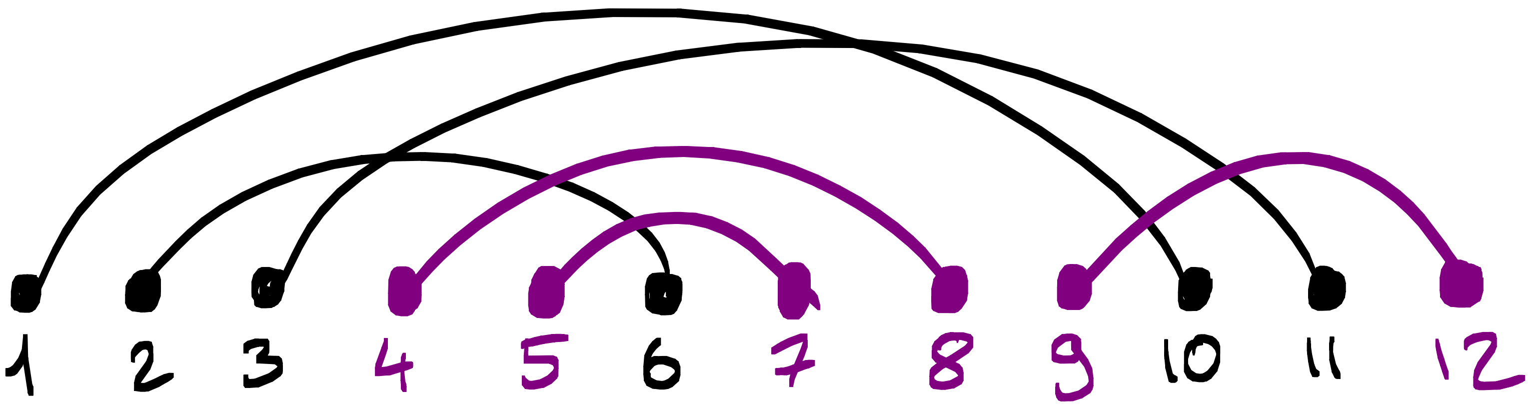
# TERMINAL CHORDS




terminal chord = chord $(a,b)$ such that for every chord $(c,d)$ that intersects it,

$$c < a < d < b.$$

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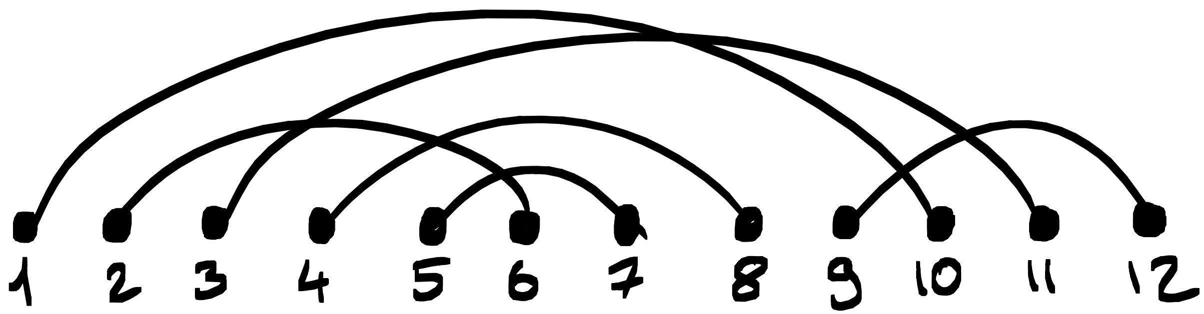
# INTERSECTION ORDER



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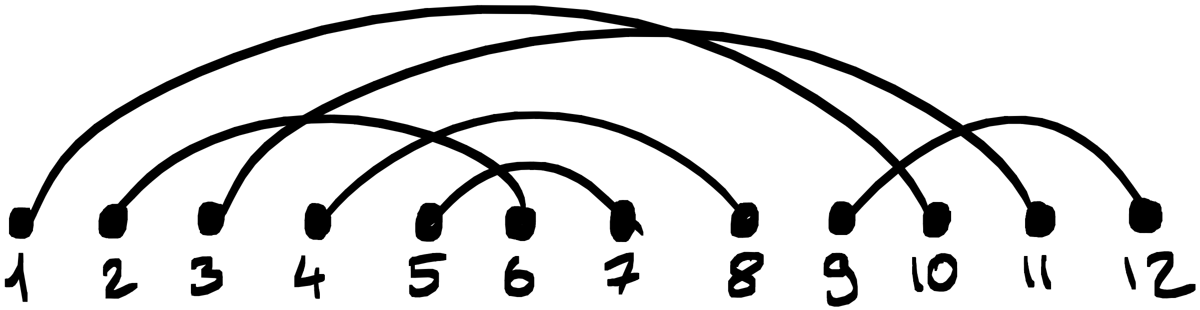


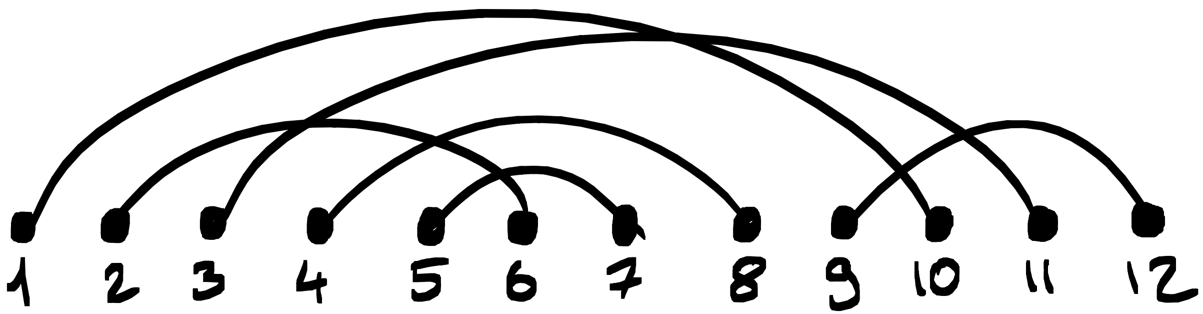

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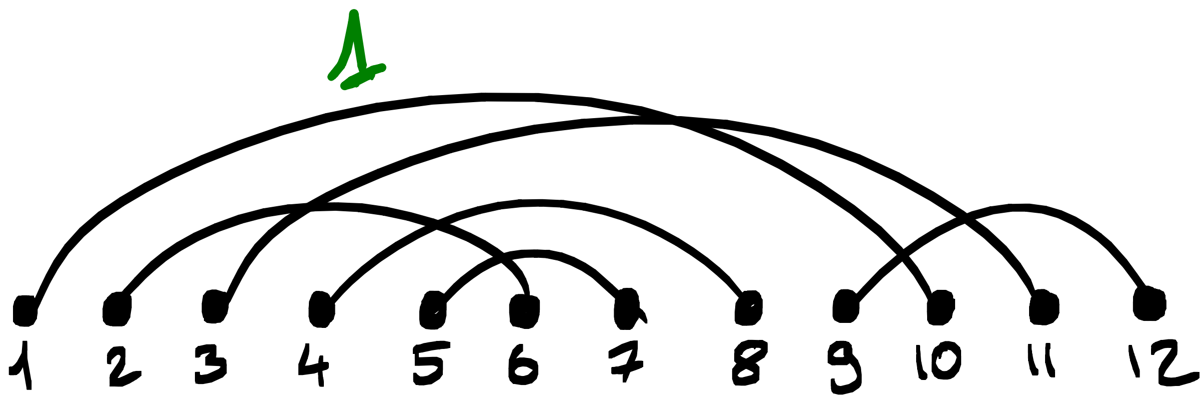


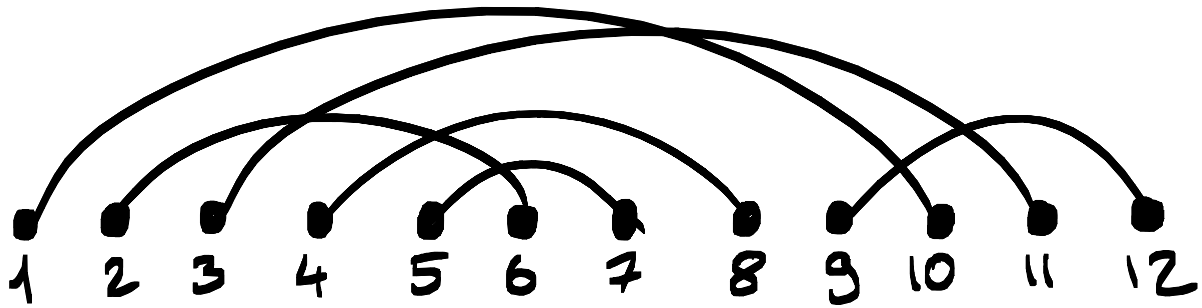

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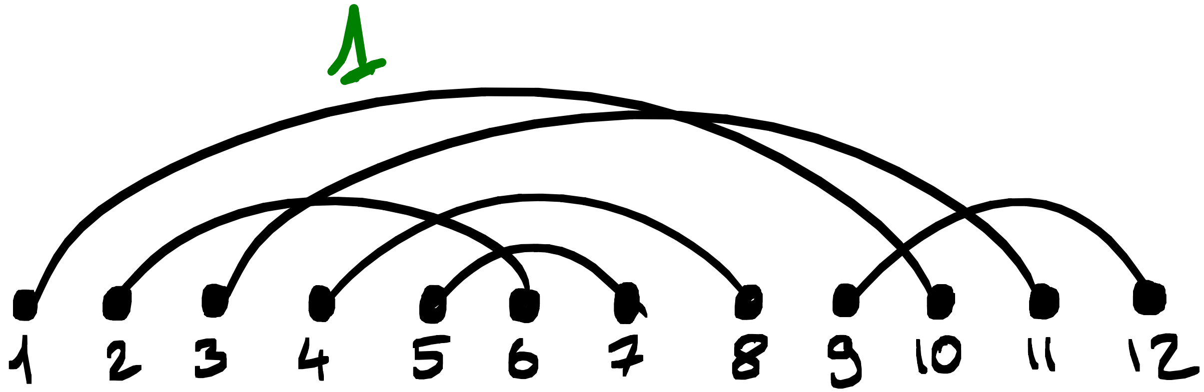


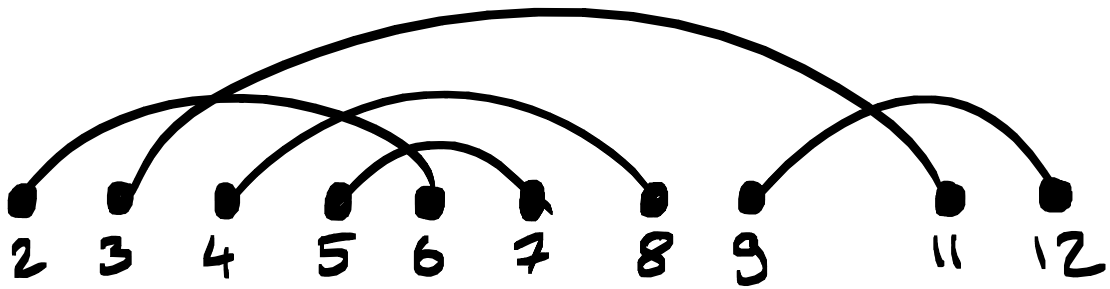

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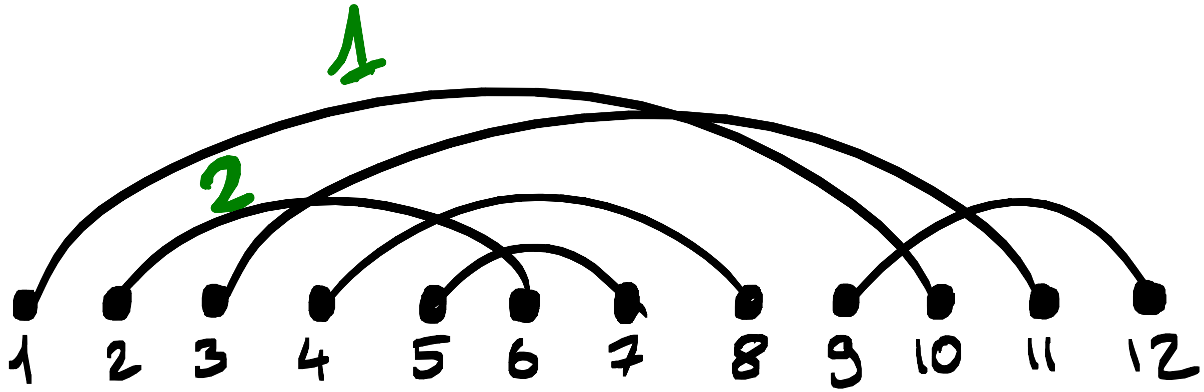


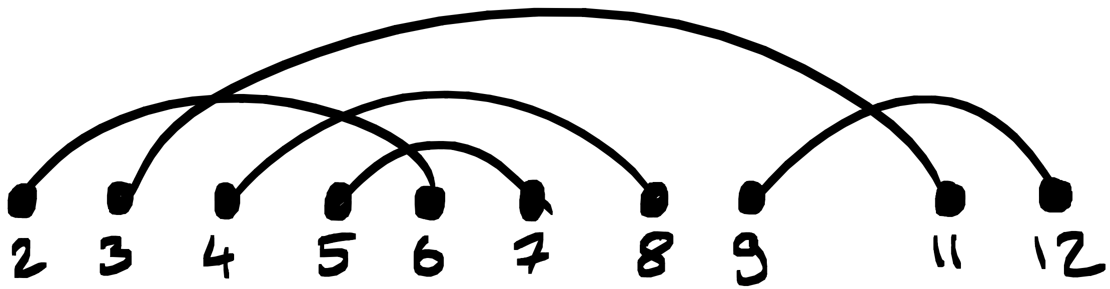

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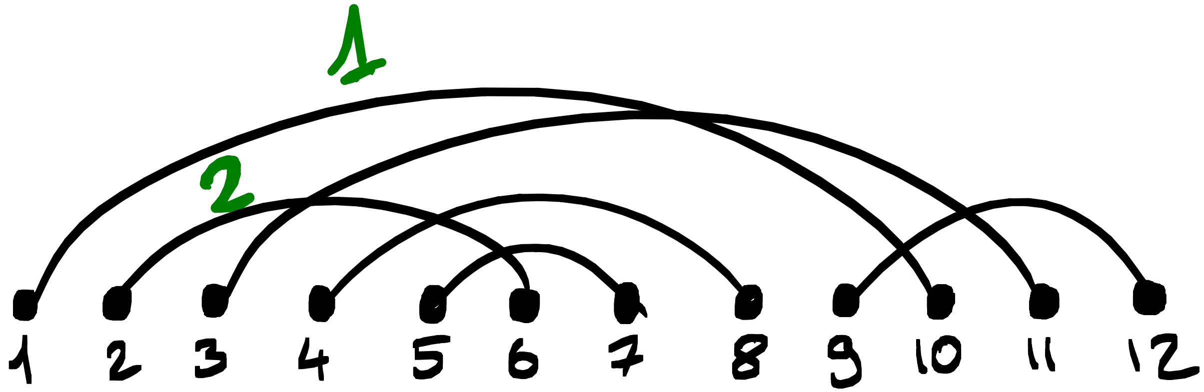


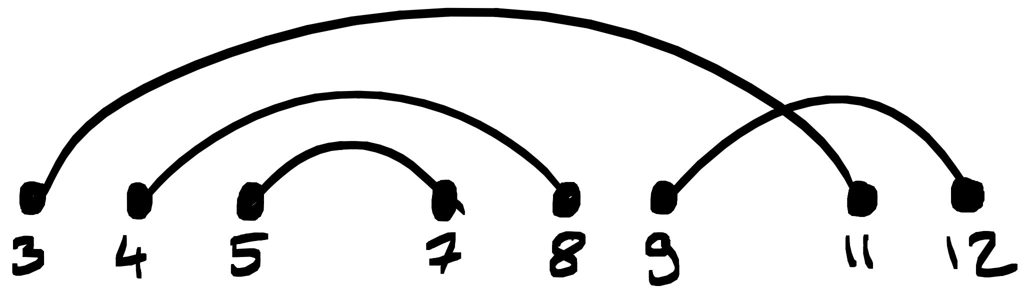

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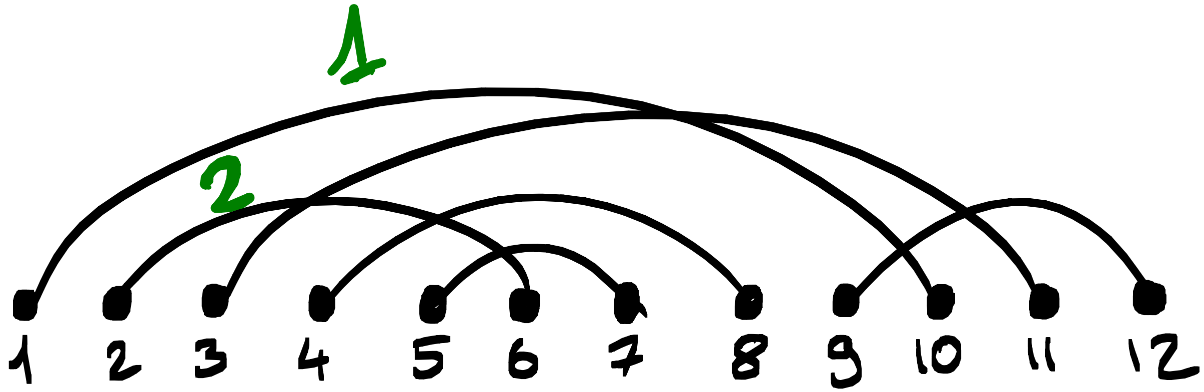


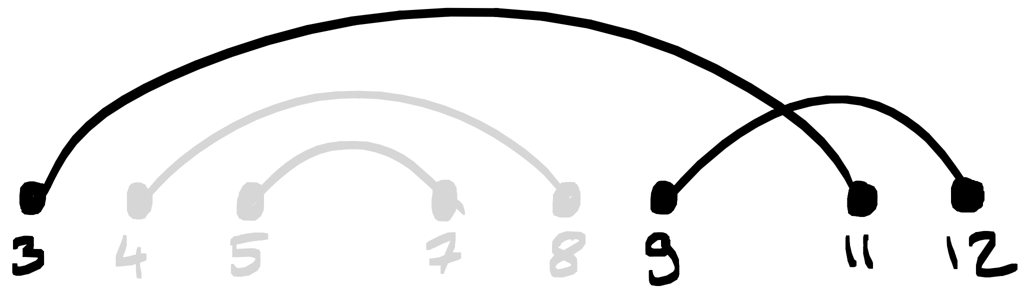

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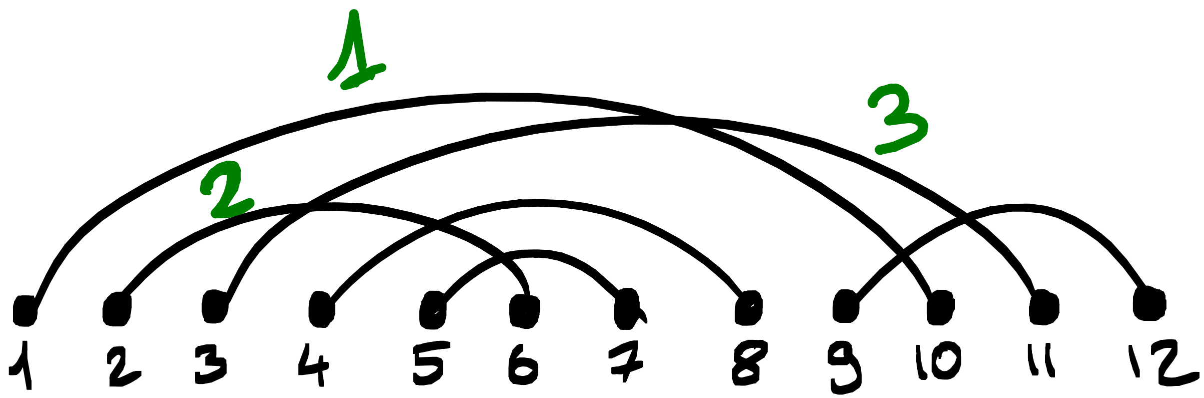


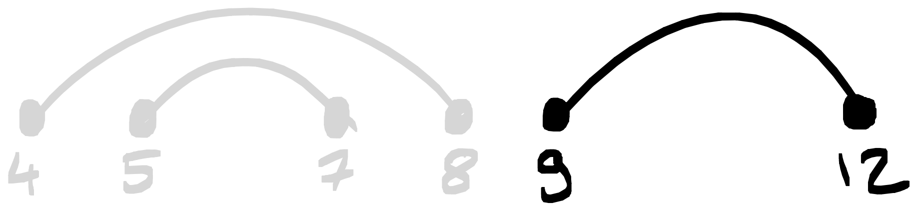

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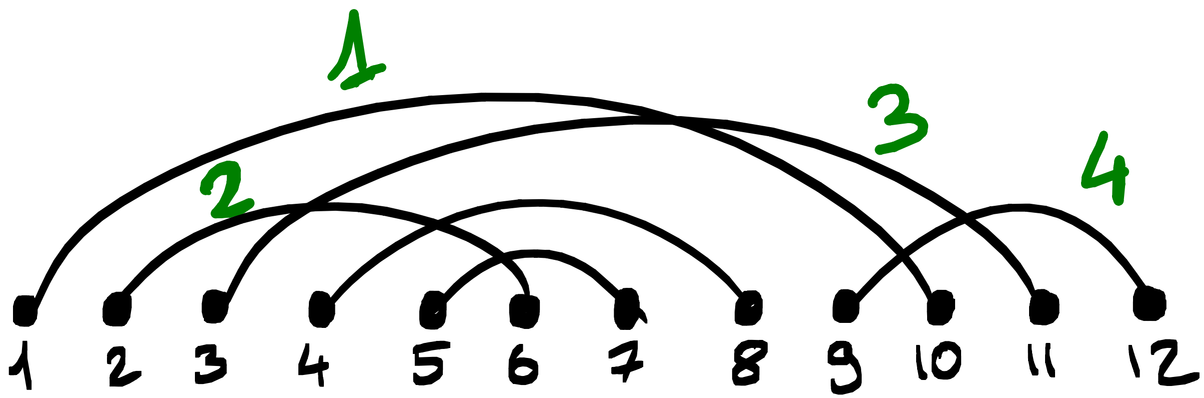

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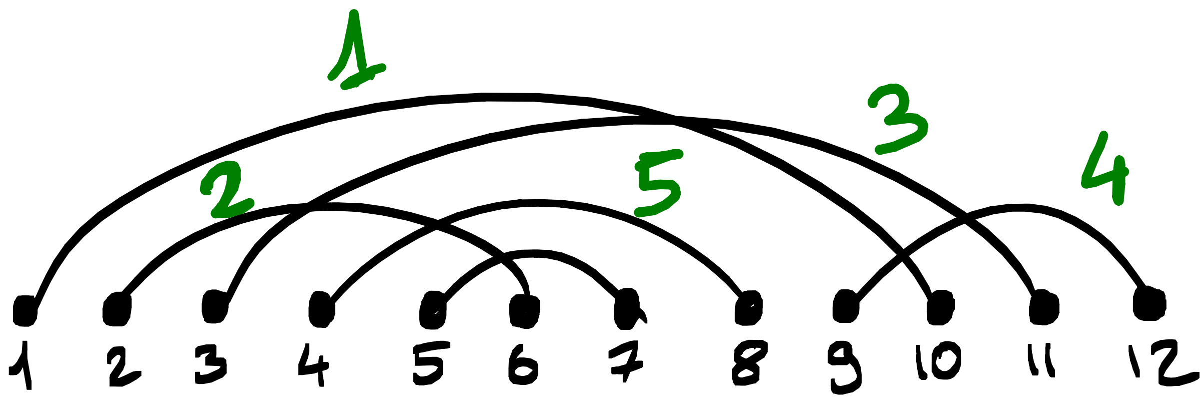


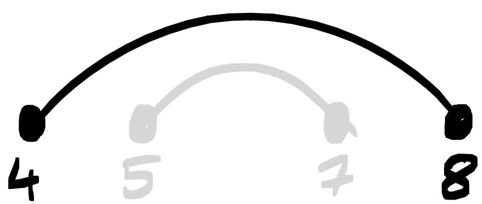

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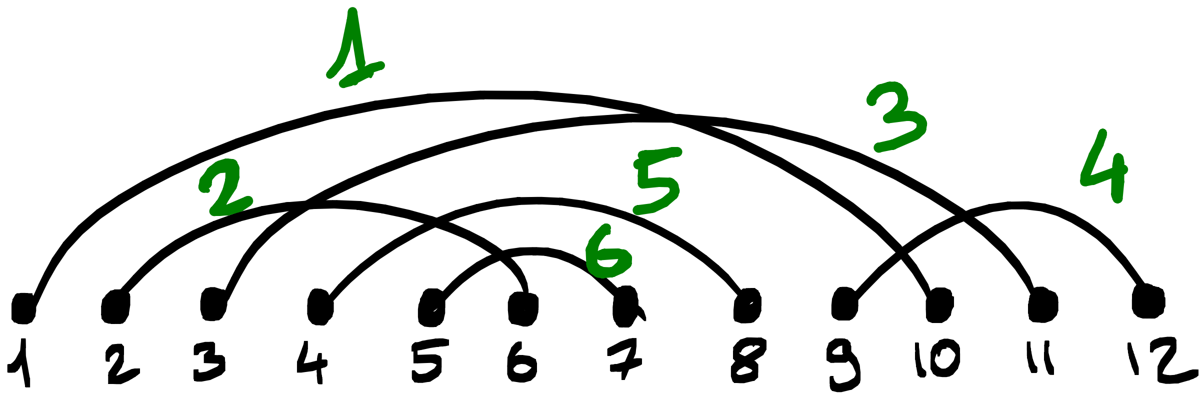


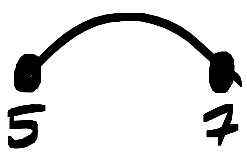

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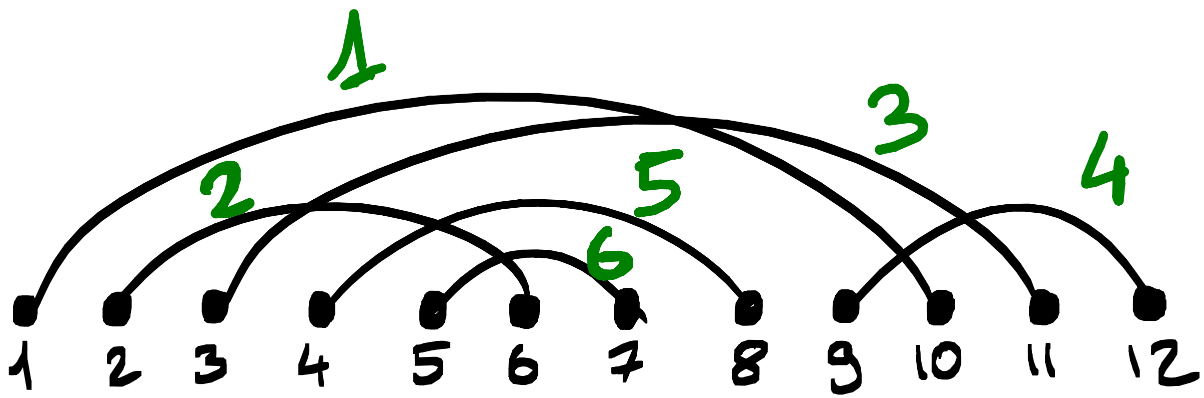


⚠ intersection order ≠ left-right order

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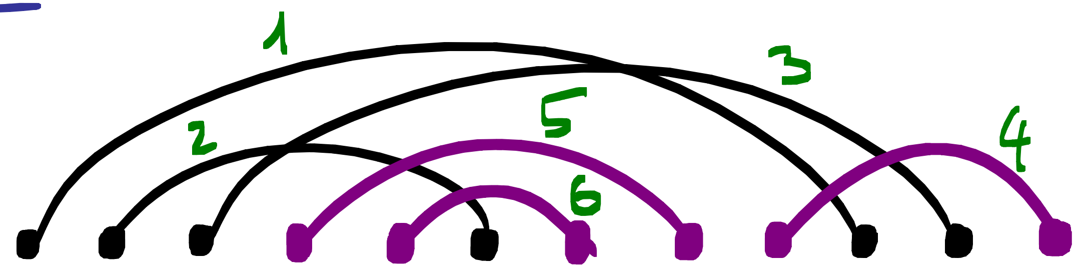


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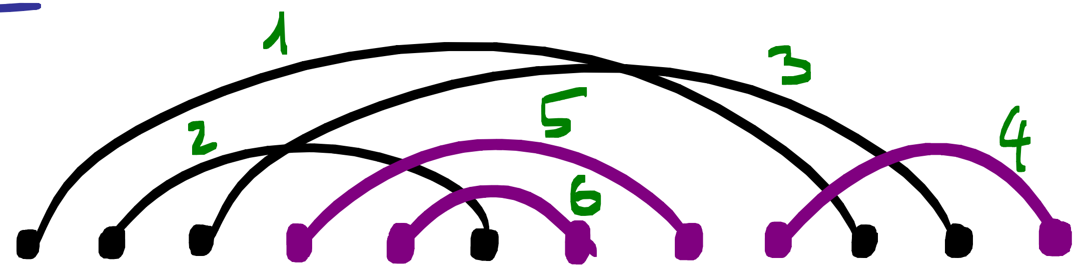


$k=3 \quad t_1=4 \quad t_2=5 \quad t_3=6$

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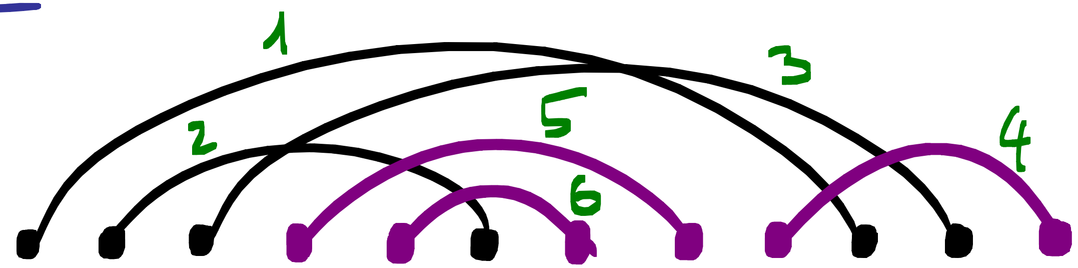


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(for $i \leq 4$)

$$\frac{L^i}{i!} x^6 \times f_0^3 \times f_{4-i} \times f_1 \times f_1$$

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# QUESTIONS

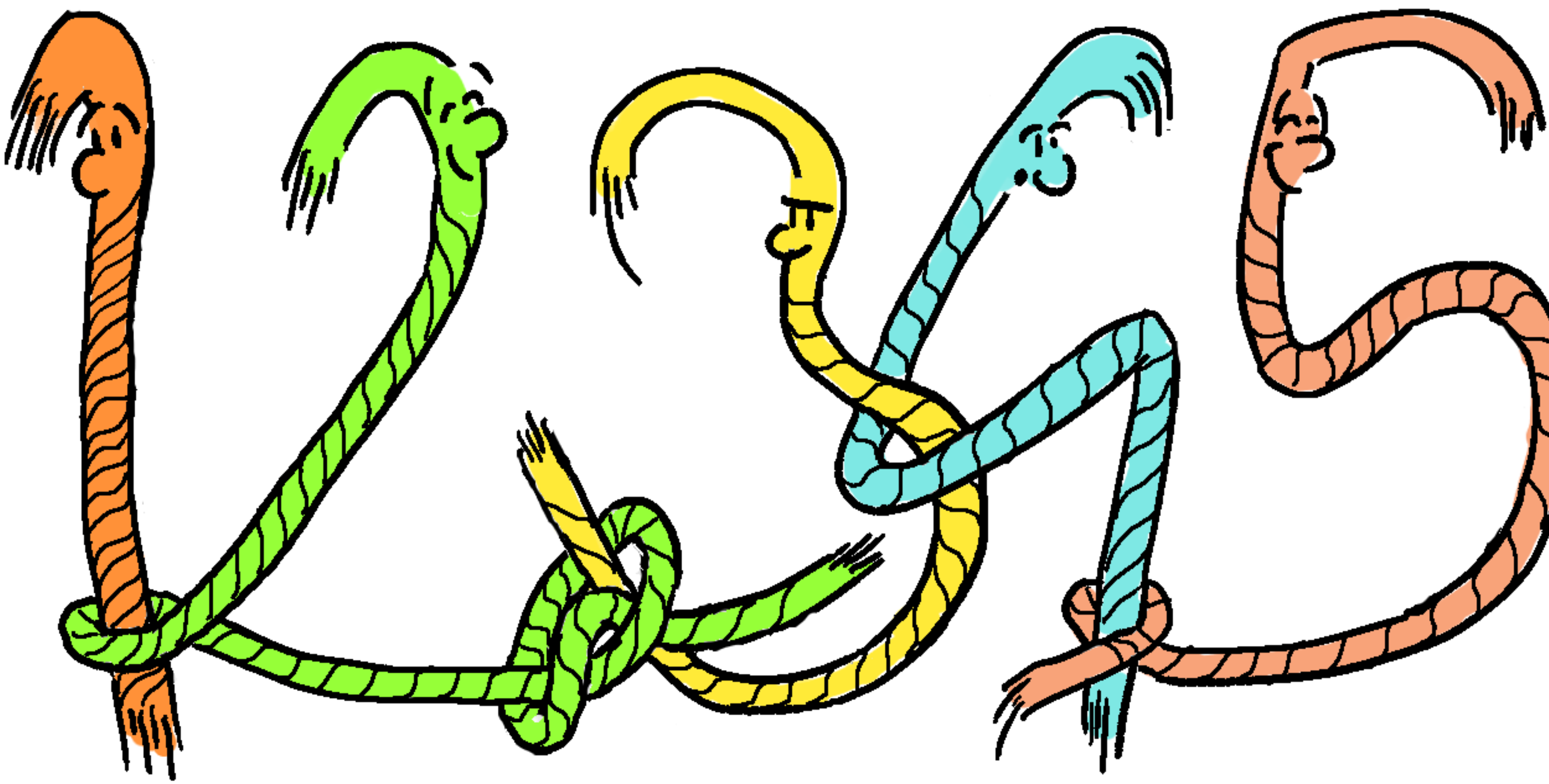
→ leading-log coefficients behaviour?

→ number of terminal chords?

→ position of the first terminal chord?

→ number of consecutive terminal chords?

# ENUMERATION OF CONNECTED CHORD DIAGRAMS



# HISTORICAL BACKGROUND

About the enumeration of chord diagrams:

- → Knot theory (Vassiliev invariants)
- → random graph generation
- → bio-informatics (RNA secondary structures)
- → cumulants
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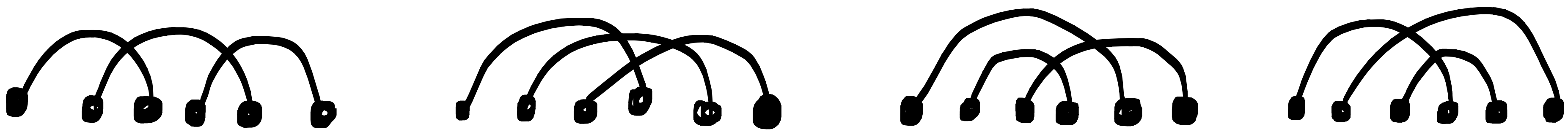
1. [Touchard, 1952] = prehistory
2. [Stein-Everett, 1978] = explicit formulas!
3. [Flajolet-Noy, 2000] = analytic combinatorics!!

# STEIN FORMULA

$c_n$ = number of connected diagrams with $n$ chords

$c_1 = 1 \quad c_2 = 1 \quad c_3 = 4 \quad c_4 = 27$

For $n = 3$,



TRIVIA

number of non-necessarily connected diagrams with $n$ chords $= ?$

TRIVIA

number of non-necessarily connected diagrams with $n$ chords $= (2n-1)!!$

$= (2n-1) \times (2n-3) \times \dots \times 3 \times 1$

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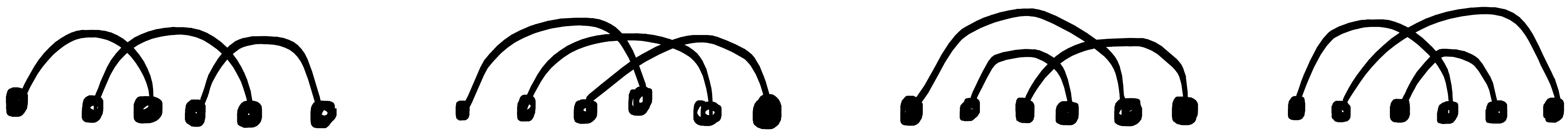

# STEIN FORMULA

$c_n$ = number of connected diagrams with $n$ chords

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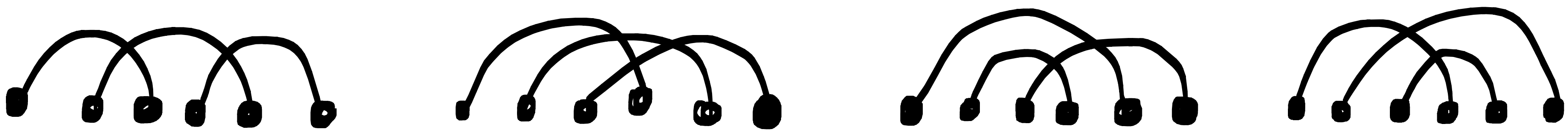
For $n = 3$,



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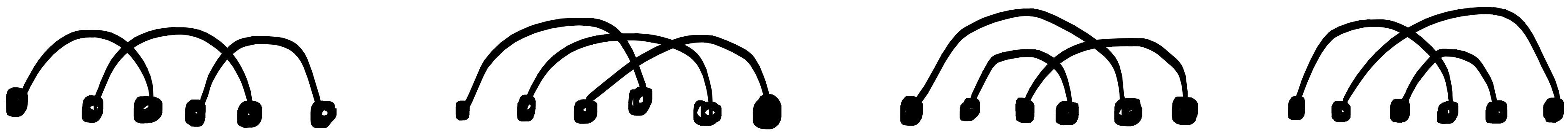

**Theorem** [Stein]

$$c_n = (n-1) \sum_{k=1}^{n-1} c_k \times c_{n-k}$$

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**Theorem:** $$c_n = \sum_{k=1}^{n-1} (2k-1)\, c_k\, c_{n-k}$$

**Corollary:** $$c_n = (n-1) \sum_{k=1}^{n-1} c_k \times c_{n-k}$$

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**Theorem:** $$c_n = \sum_{k=1}^{n-1} (2k-1)\, c_k\, c_{n-k}$$

$\Big\downarrow$ var. change $k \leftarrow n-k$

$$c_n = \sum_{k=1}^{n} (2n-2k-1)\, c_{n-k}\, c_k$$

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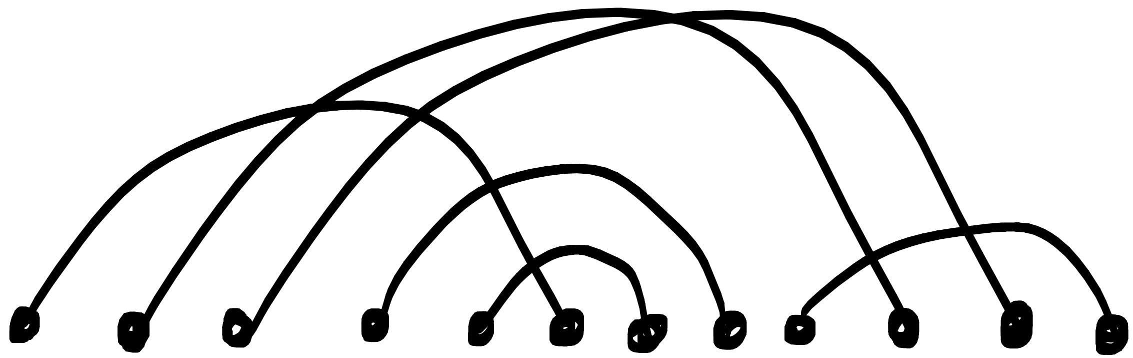

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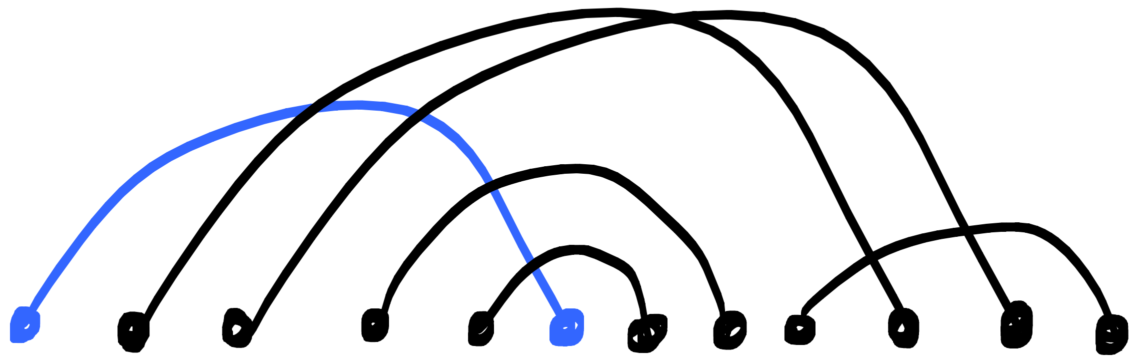

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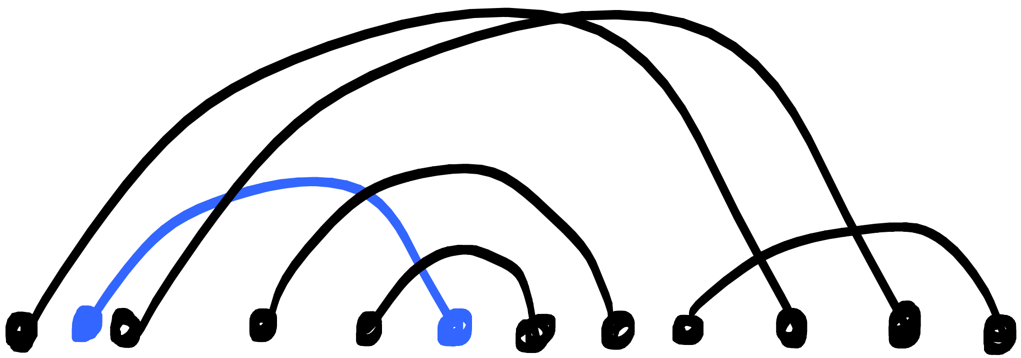

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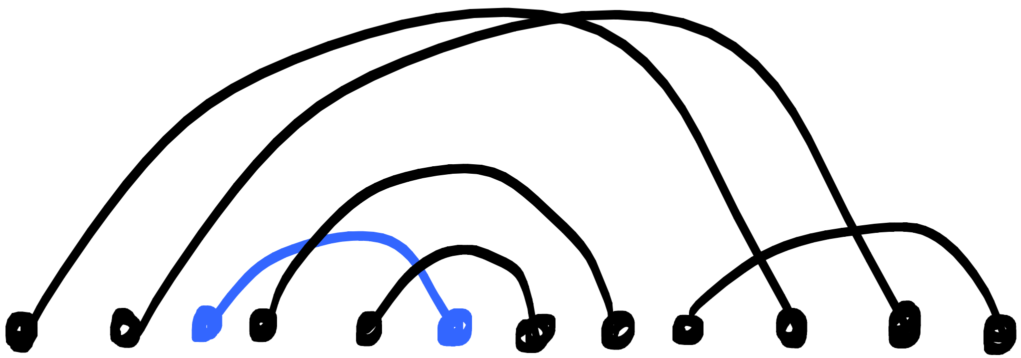

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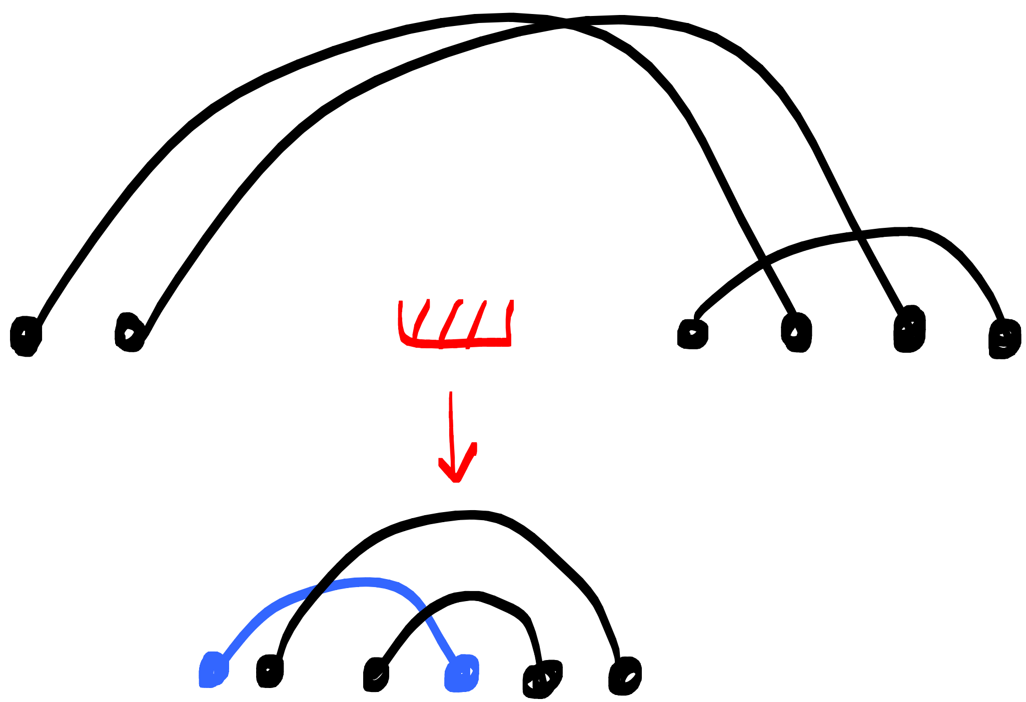

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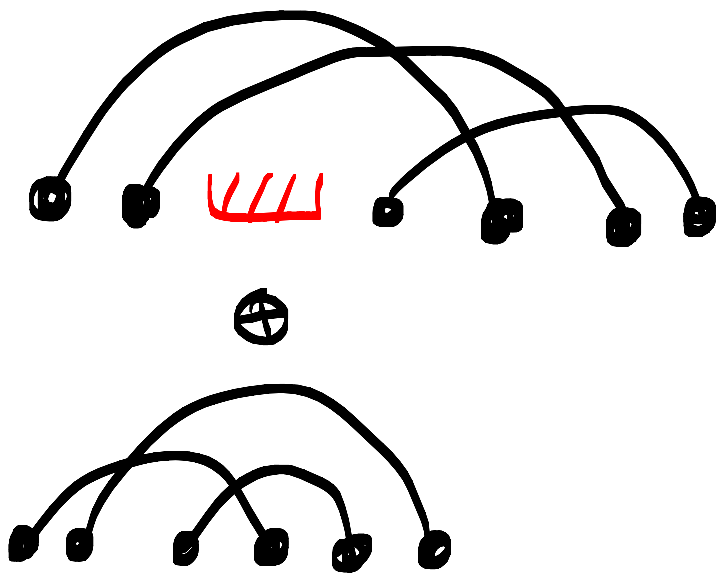

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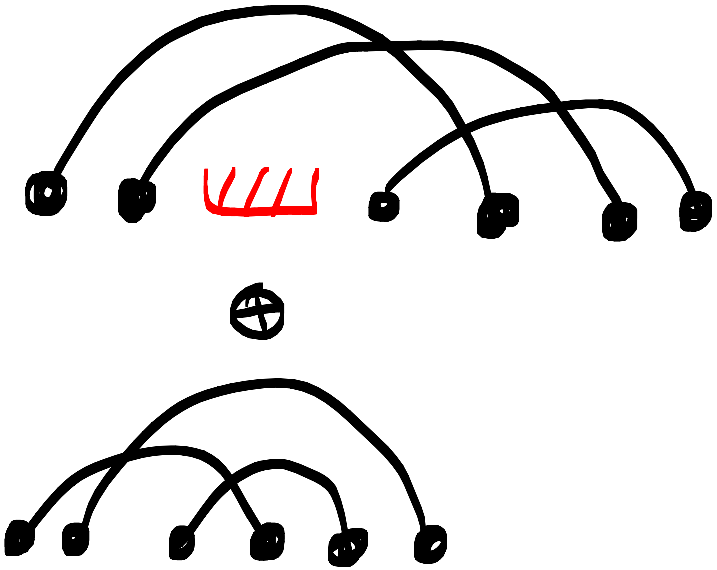
If $k$ chords,

then $(2k-1)$ possible insertions



# $c_n$ VS CATALAN

CONNECTED DIAGRAMS

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Consequence: - Ordinary Generating Functions are not adapted. :(

- No simple equation for the Exponential Generating Functions :(

# ASYMPTOTIC BEHAVIOUR

[Stein-Everett]

$$c_n \sim \frac{1}{e} \times (2n-1)!!$$

Consequence : $\mathbb{P}(\text{diagram is connected}) \to \frac{1}{e}$

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- number of connected components $\sim$ Poisson(1)
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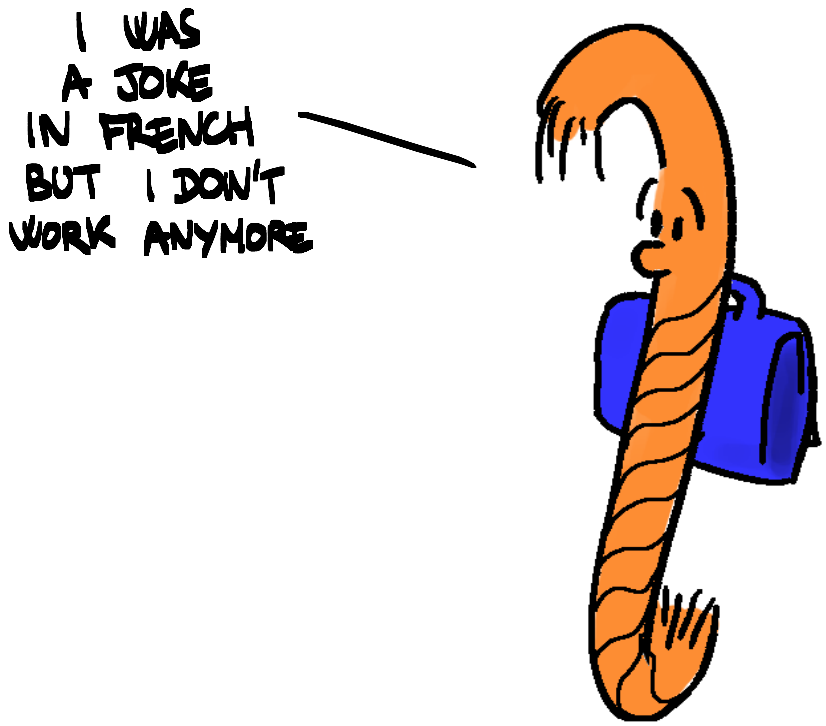
Our humble contribution : $\frac{c_{n-1}}{c_n} = \frac{1}{2n} + \frac{1}{4n^2} - \frac{1}{4n^3} + o\left(\frac{1}{n^3}\right)$

# STATISTICS ON TERMINAL CHORDS



# LEADING-LOG TERMS

$$G(x,L) = 1 - \sum_{i\geq 1} \sum_{\substack{C \text{ connected} \\ \text{chord diagram} \\ \text{with terminal chords} \\ \text{in position } t_1 < t_2 < \ldots < t_k \\ \text{such that } t_1 \geq i}} \frac{L^i}{i!} x^{|C|} f_0^{|C|-k} f_{t_1-i} f_{t_2-t_1} f_{t_3-t_2} \cdots f_{t_k-t_{k-1}}$$

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Coefficient of $(Lx)^i x^{|C|-i}$ for $i$ close to $|C|$?

- $|C| = i$ : leading-log expansion
- $|C| = i+1$ : next-to leading-log expansion
- $|C| = i+l$ : next-to$^l$ leading-log expansion

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Coefficient of $(Lx)^i x^{|C|-i}$ for $i$ close to $|C|$?

- $|C| = i$ : leading-log expansion

$\Leftrightarrow t_1 = |C|$

$\Leftrightarrow$ There is only one terminal chord.

# ONLY ONE TERMINAL CHORD

number of connected diagrams with $n$ chords and only one terminal chord

$= ?$

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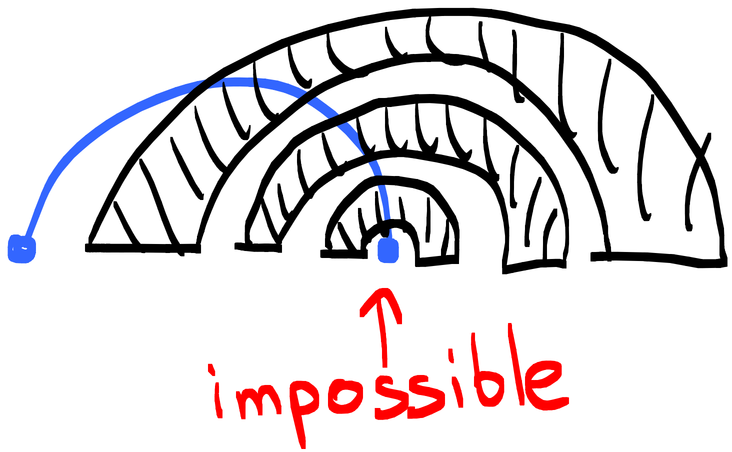


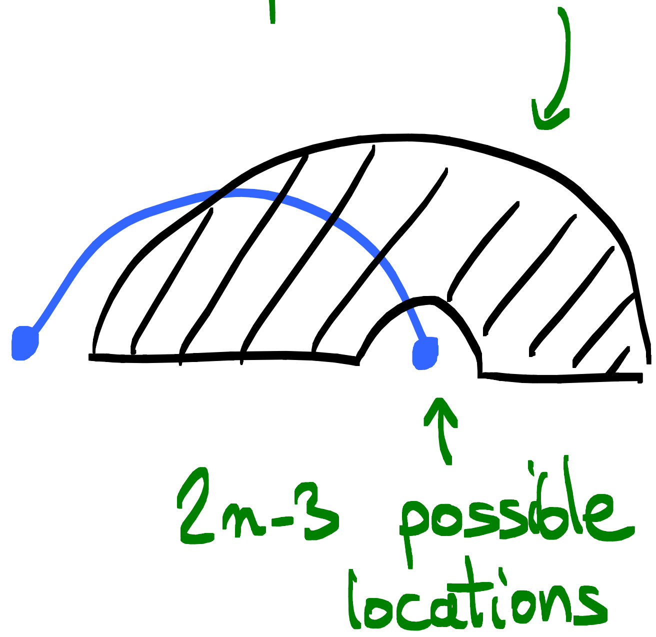

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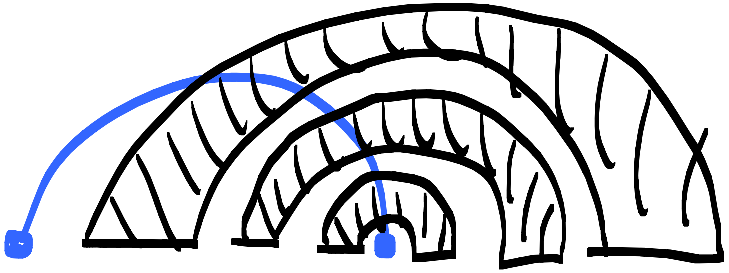


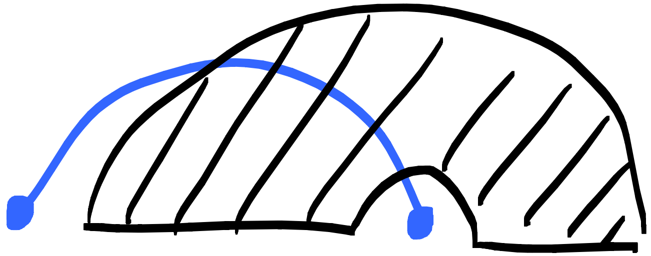


Cor: $n^{th}$ coeff of the leading-log expansion $= \dfrac{(2n-3)!!}{n!}\beta_0^n$

# NEXT-TO$^{\ell}$ LEADING-LOG TERMS

→ "Similar" recursions exist for the diagrams such that $t_1 \geqslant |C| - \ell$

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But how about $\partial_0^{|C|-k} \partial_{t_1 - i} \partial_{t_2 - t_1} \partial_{t_3 - t_2} \cdots \partial_{t_k - t_{k-1}}$ ?

# THE LAST $\ell$ CHORDS ARE TERMINAL

→ "Similar" recursions exist for the diagrams such that the last $\ell$ chords are terminal

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Here $f_0^{|C|-k} f_{t_1-i} f_{t_2-t_1} f_{t_3-t_2} \cdots f_{t_k-t_{k-1}} = f_0^{n-\ell+1} f_1^{\ell-1}$

# NEXT-TO$^{\ell}$ LEADING-LOG TERMS

Diagrams such that the last $\ell$ chords are terminal are dominant among the diagrams such that $t_1 \geq |C| - \ell$.

Corollary: For $\ell \geq 0$,
$n^{th}$ coeff of the next-to$^{\ell}$ leading-log expansion

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Only $f_0$ and $f_1$ matter!

# NUMBER OF TERMINAL CHORDS

Average number of terminal chords ?

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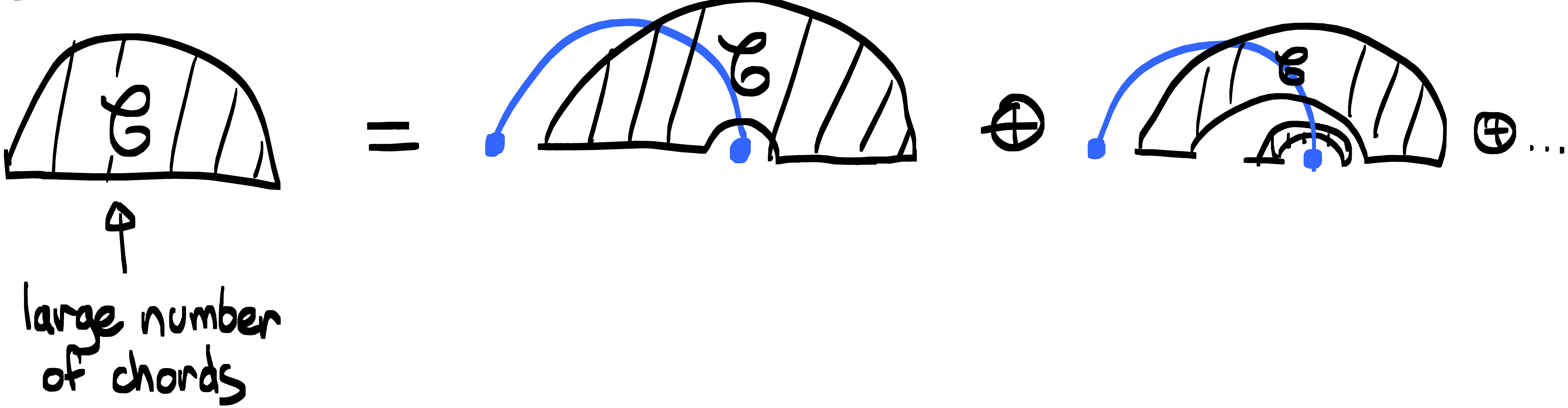

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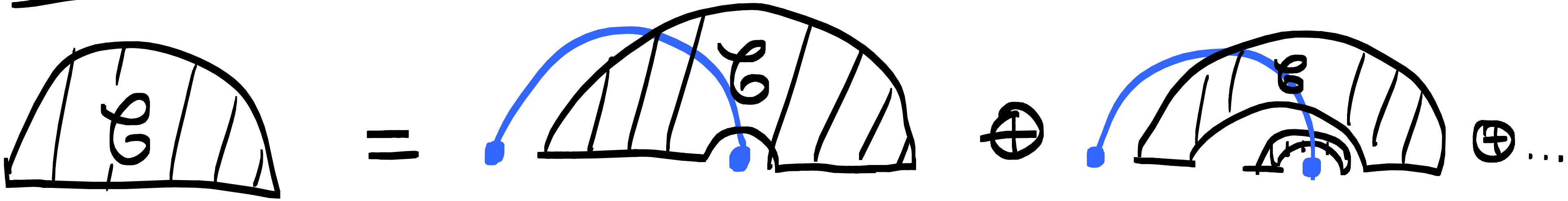


large number of chords

proba

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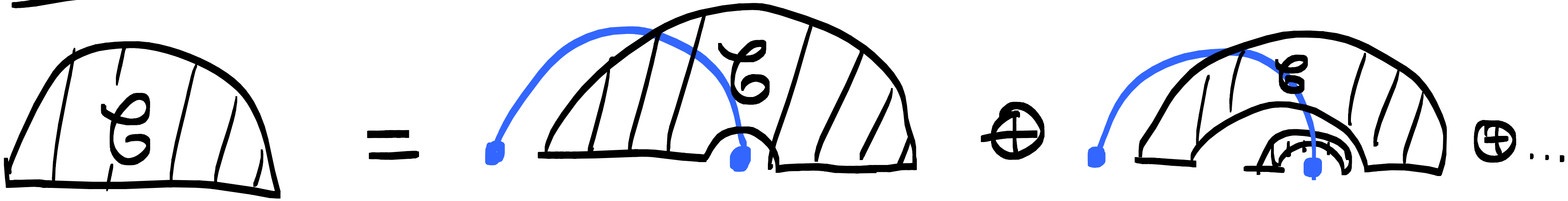


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$$= \underbrace{\frac{(2n-3)c_{n-1}}{c_n}}_{\to 1}$$

proba $\to 0$

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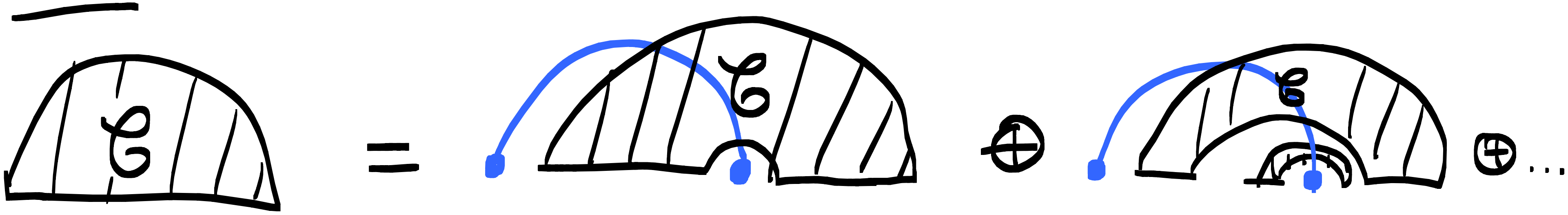


large number of chords

proba $= \underbrace{\frac{(2n-3)c_{n-1}}{c_n}}_{\to 1}$

proba $\to 0$

Interesting but not sufficient...

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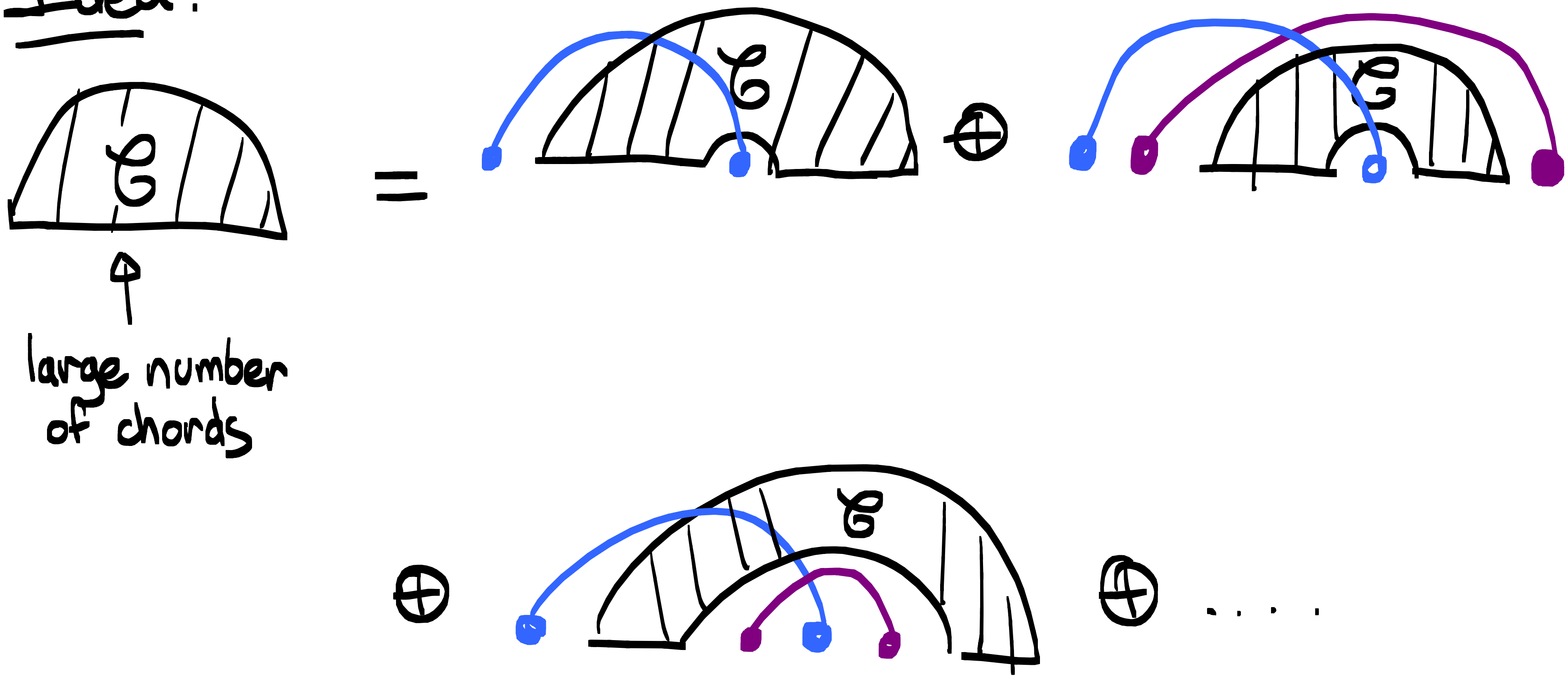

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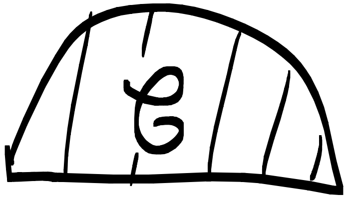


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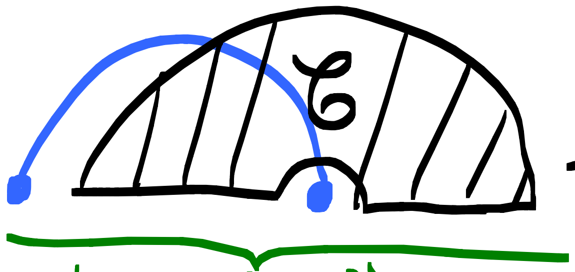
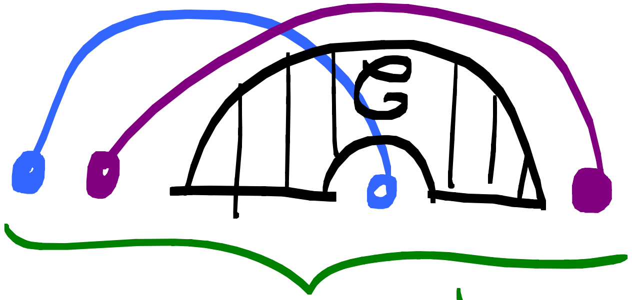



proba $= \dfrac{(2n-3)\,c_{n-1}}{c_n}$

$\oplus$

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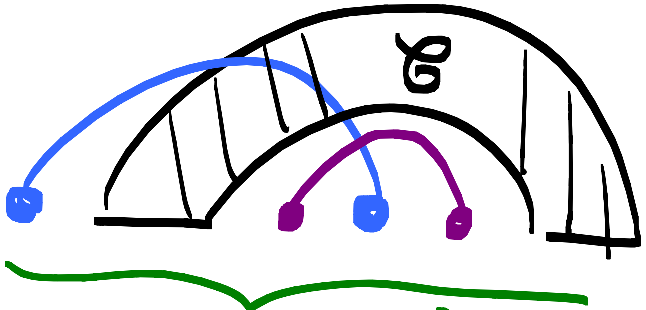


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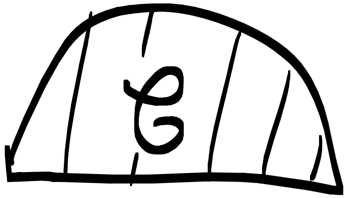


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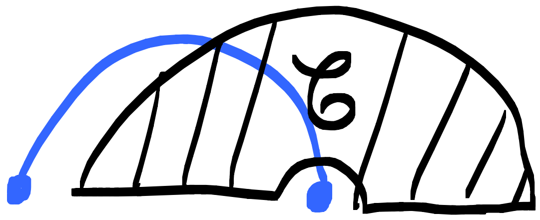
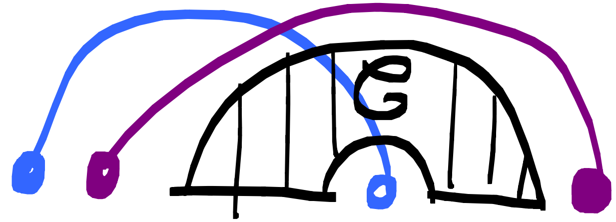


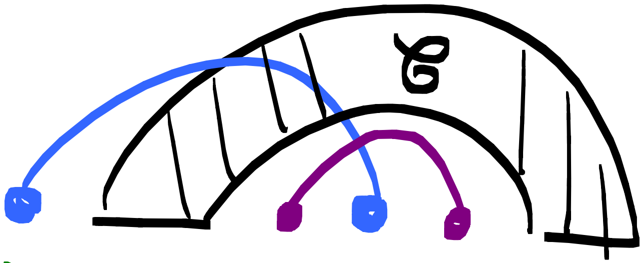

$$\text{proba} = \frac{(2n-3)\,c_{n-1}}{c_n} = 1 - \frac{1}{n} + o\left(\frac{1}{n}\right)$$

$\oplus$




$$\text{proba} = \frac{(2n-5)\,c_{n-2}}{c_n} \sim \frac{1}{2n}$$

$\oplus$

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$\oplus$ .....

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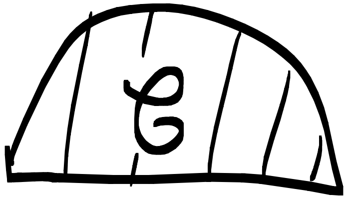


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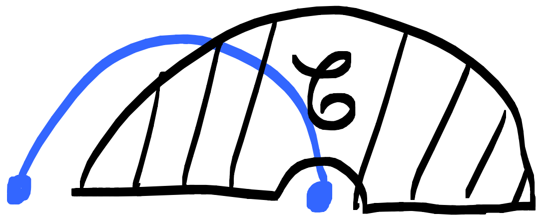


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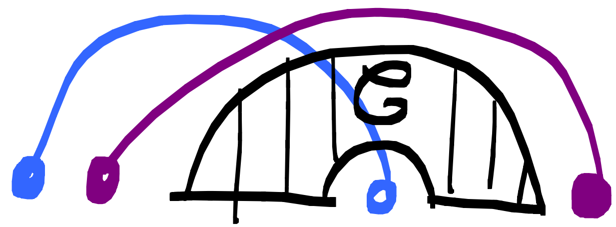


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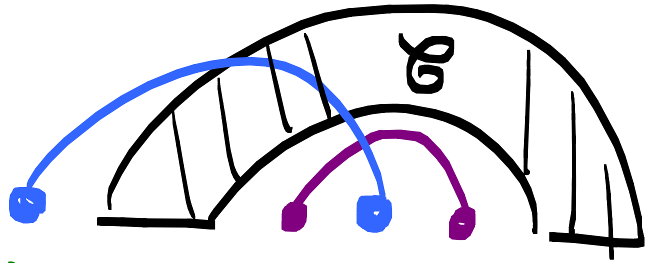



$$\text{proba} = \frac{(2n-5)\,c_{n-2}}{c_n} \sim \frac{1}{2n}$$

$\oplus$

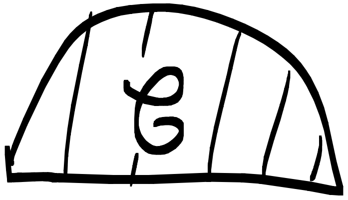
$$\cancel{\cdots = o\left(\frac{1}{n}\right)}$$

Let's forget that

# NUMBER OF TERMINAL CHORDS

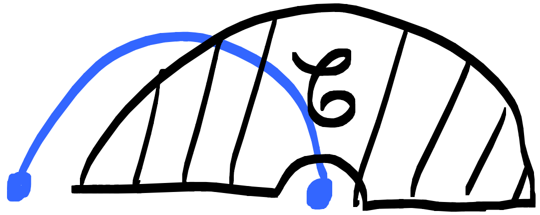
Set $p_{n,k} = \left(1 - \frac{1}{n}\right) p_{n-1,k} + \frac{1}{n} p_{n-2,k-1}$

Idea:



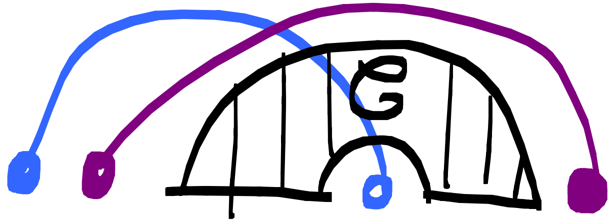

large number of chords

=



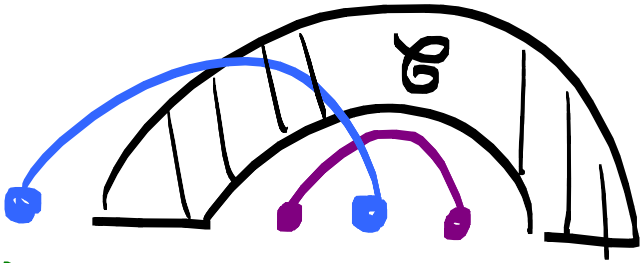

$$\text{proba} = \frac{(2n-3)\,c_{n-1}}{c_n} = 1 - \frac{1}{n} + o\left(\frac{1}{n}\right)$$

⊕




$$\text{proba} = \frac{(2n-5)\,c_{n-2}}{c_n} \sim \frac{1}{2n}$$

⊕

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⊕ ....

$\cancel{= o\left(\frac{1}{n}\right)}$

Let's forget that

# NUMBER OF TERMINAL CHORDS

Set $p_{n,k} = \left(1 - \frac{1}{n}\right) p_{n-1,k} + \frac{1}{n} p_{n-2,k-1}$

Fact 1 : Let $X_n$ be the random variable such that $\mathbb{P}(X_n = k) = p_{n,k}$

$$X_n \longrightarrow \text{Gaussian law.}$$

Fact 2 : The number of terminal chords "$\sim$" $X_n$

# NUMBER OF TERMINAL CHORDS

Theorem : The number of terminal chords in a random connected diagram of size $n$ asymptotically obeys to a Gaussian limit law of mean and variance $\sim \ln(n)$.

# NUMBER OF CONSECUTIVE TERMINAL CHORDS

If the terminal chords are in position $t_1 < \dots < t_k$, how many $j$'s satisfy $t_j - t_{j-1} = 1$ ?

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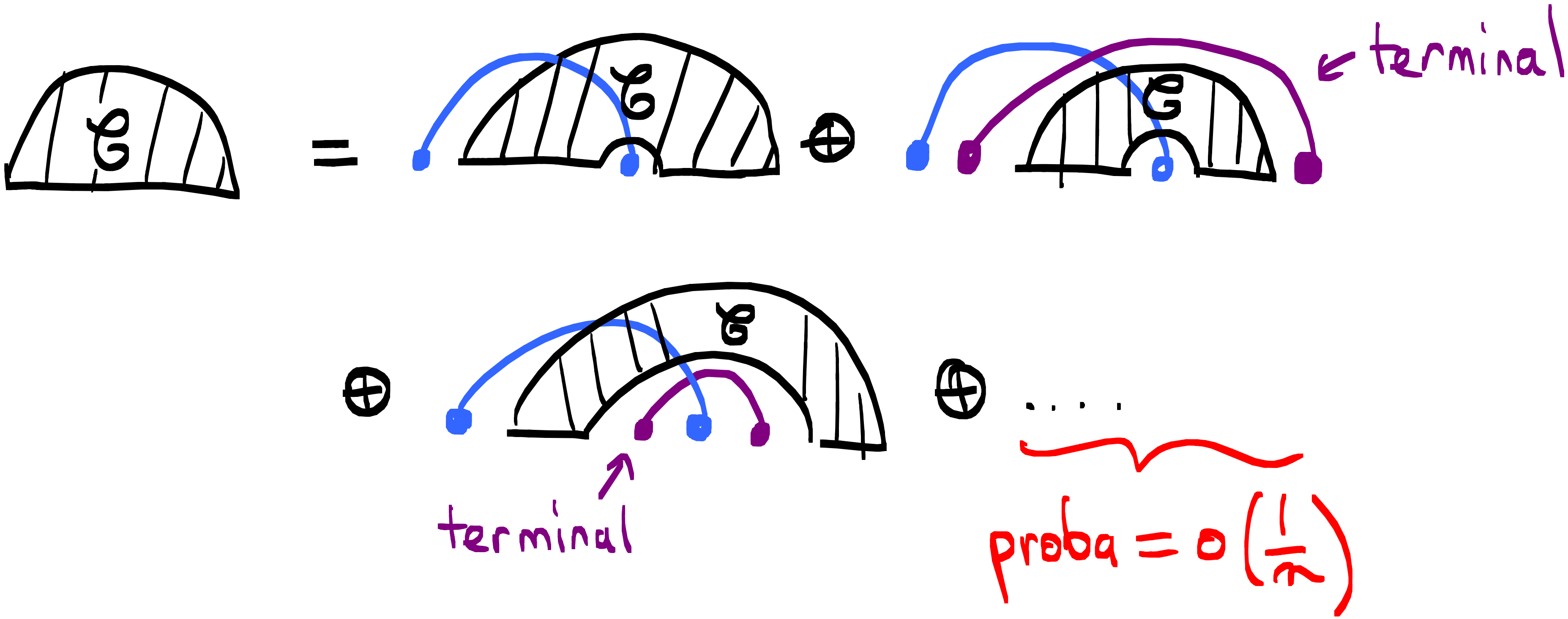

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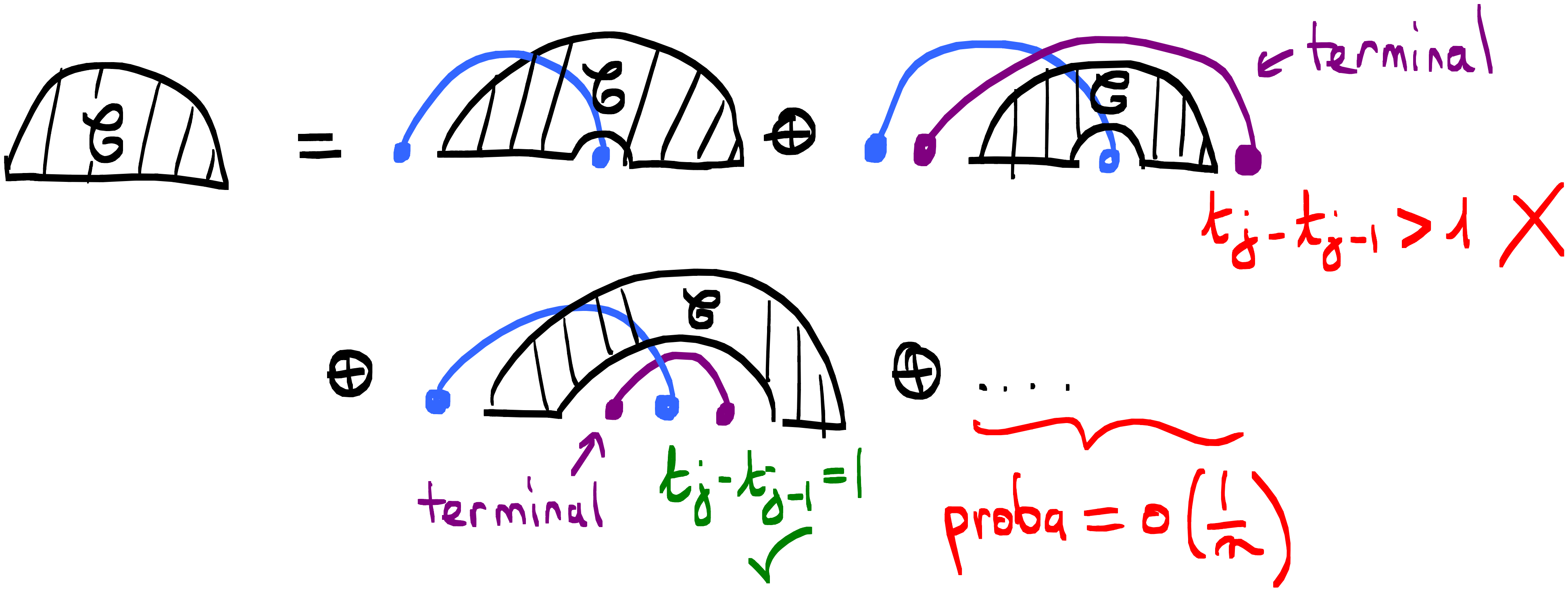

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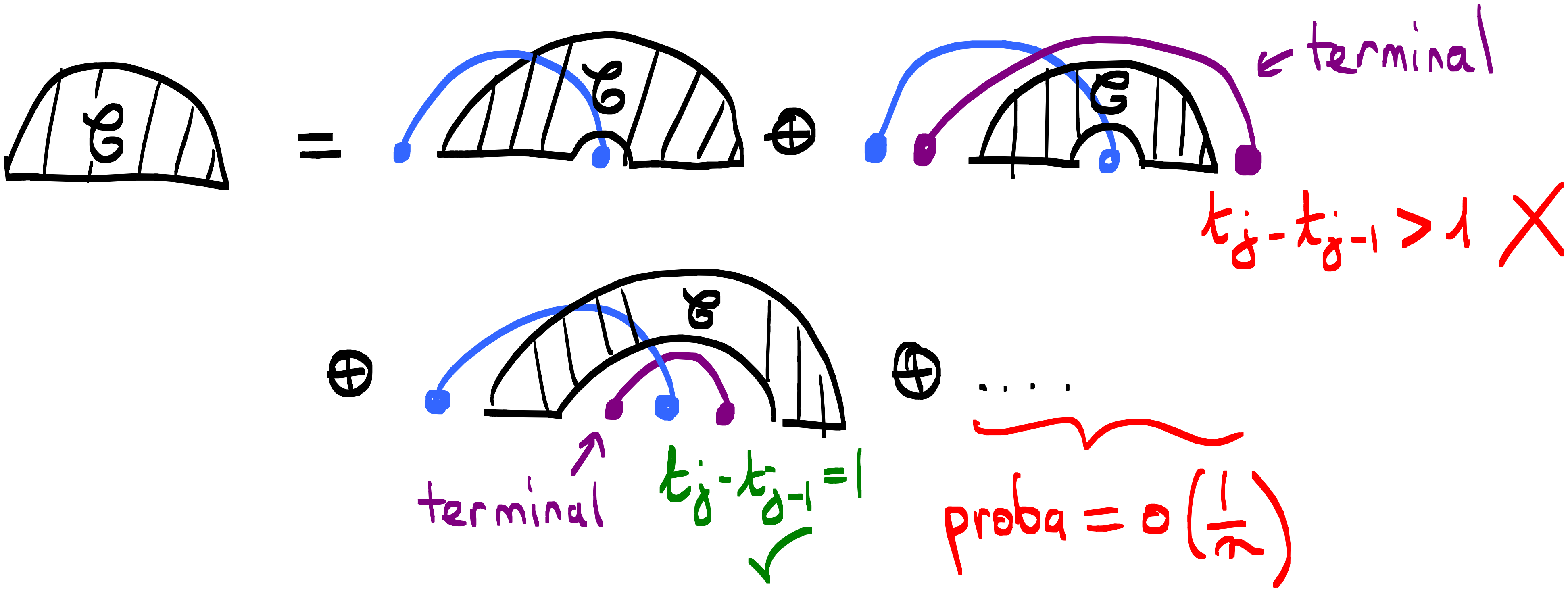


Theorem: Number of consecutive terminal chords

$\longrightarrow$ Gaussian law of mean and variance $\sim \frac{\ln n}{2}$

# NUMBER OF CONSECUTIVE TERMINAL CHORDS

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On average,

$$f_0^{|C|-k} f_{t_1 - i} f_{t_2 - t_1} f_{t_3 - t_2} \cdots f_{t_k - t_{k-1}} \sim f_0^{n - \ln n} f_{t_1 - i} f_1^{\frac{\ln n}{2}} \dots$$

→ confirms the importance of $f_0$ and $f_1$

**Theorem:** Number of consecutive terminal chords → Gaussian law of mean and variance $\sim \frac{\ln n}{2}$

# POSITION OF THE FIRST TERMINAL CHORD.

$t_1$ = random variable returning the position of the $1^{st}$ terminal chord.

**Theorem:** $\mathbb{E}(t_1) \sim \frac{2}{3} n$

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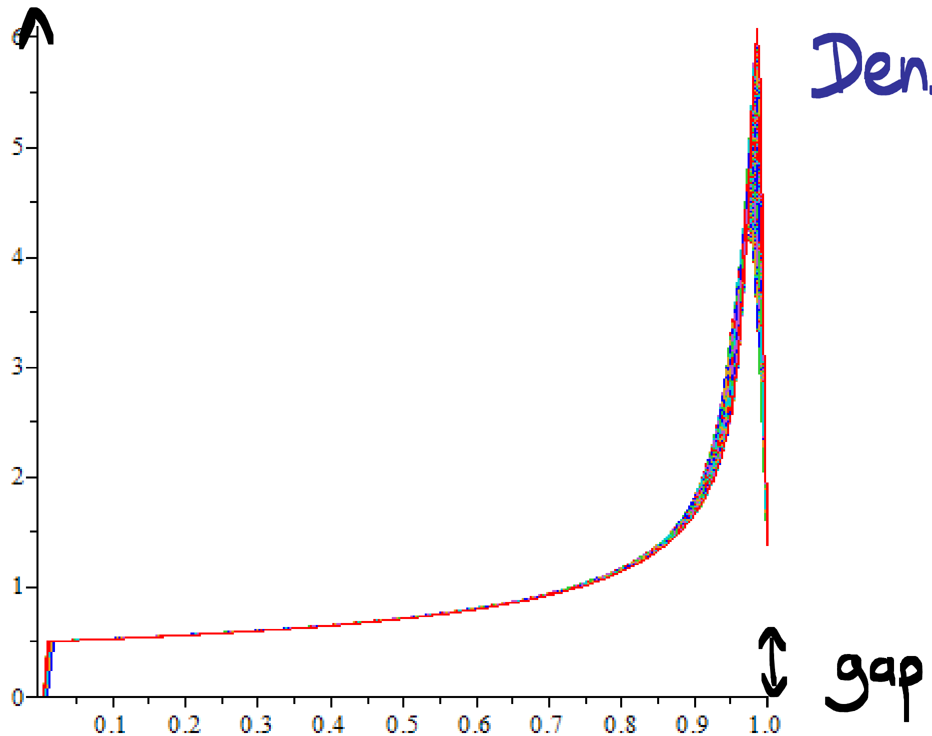
Limit law?




Density of $\frac{X_n}{n}$

# CONCLUSION

→ Recovers the results of Krüger and Kreimer

+ automaticity of the method
+ asymptotic behaviour

→ New combinatorial approach

→ Extension to Hihn-Yeats's results?


11-12 juillet 2016 - VANCOUVER

(après FPSAC)