Terminal Chords in Connected Chord Diagrams

SFU, March 8

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The Terminal

Life is waiting.
COMBINATORIAL DYSON-SCHWINGER EQUATIONS

Class of Feynman graphs

Feynman rules

solution of Dyson-Schwinger equations

One-loop propagator + recursive iterations

\[ G(x, L) = 1 - x G(x, \frac{2}{\Theta(-\rho)}) \left( e^{-L\rho} - 1 \right) F(\rho) \bigg|_{\rho = 0} \]
Theorem [Marie, Yeats]

The solution of the previous equation can be written under the form:

\[ G(x, L) = 1 - \sum_{i \geq 1} \sum_{\text{connected chord diagram with terminal chords in position } k_1 < k_2 < \ldots < k_R} \frac{i^i}{i!} \prod_{j=1}^i c_j^{1-c_j} t_{k_j - i} \]

where \( \frac{b_0}{q^q} + b_1 + b_2 q + b_3 q^2 + \ldots = \text{expansion of a regularized Feynman integral} \)
COMBINATORIAL DYSON-SCHWINGER EQUATIONS

Theorem [Marie, Yeats]
The solution of the previous equation can be written under the form:

\[ G(x, L) = 1 - \sum_{i \geq 1} \sum_{c \text{ connected chord diagram}} \frac{1}{i!} \left[ \frac{L}{\alpha} \right]^{i} \frac{1}{i!} \frac{\Gamma(1+c)}{\Gamma(c+1)} \prod_{k} b_{t_{k}} \right] \]

where

\[ \frac{b_{0}}{L} + b_{1} + b_{2} x + b_{3} x^{2} + \ldots = \text{expansion of a regularized Feynman integral} \]
Diagram of $n$ chords = matching of $\{1, 2, \ldots, 2n\}$
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connected diagram : its representation is in one piece
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CONNECTED CHORD DIAGRAMS

Diagram of \( n \) chords = matching of \( \{1, 2, \ldots, 2n\} \)

connected diagram : its representation is in one piece
Connected Chord Diagrams

Diagram of n chords = matching of \([1,2,\ldots,2n]\)

Connected diagram: its representation is in one piece
Diagram of $n$ chords = matching of $[1, 2, \ldots, 2n]$.

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Theorem [Marie, Yeats]

The solution of the previous equation can be written under the form:

\[ G(x, L) = 1 - \sum_{i \geq 1} \sum_{\text{connected chord diagram}} \frac{i^i}{i!} \frac{1}{x^i} b_{0} \beta_{t_{1}} \beta_{t_{2}} \beta_{t_{3}} \cdots \beta_{t_{k-i}} \]

where

\[
\frac{b_{0}}{x} + b_{1} + b_{2} x + b_{3} x^{2} + \cdots = \text{expansion of a regularized Feynman integral}
\]
Theorem [Marie, Yeats]

The solution of the previous equation can be written under the form:

\[ G(x, L) = 1 - \sum_{i \geq 1} \sum_{\text{connected chord diagram with terminal chords in position } t_1 < t_2 < \ldots < t_k} \frac{i}{i!} \prod_{j=1}^{k} b_{t_j} \left( b_{t_{j-1}} - b_{t_j} \right) \]

where

\[ \frac{b_0}{g} + b_1 + b_2 p + b_3 p^2 + \ldots = \text{expansion of a regularized Feynman integral} \]
**Terminal Chords**

Terminal chord = chord \((a, b)\) such that for every chord \((c, d)\) that intersects it,
\[
c < a < d < b.
\]
COMBINATORIAL DYSON-SCHWINGER EQUATIONS

Theorem [Marie, Yeats]

The solution of the previous equation can be written under the form:

\[ G(x, L) = 1 - \sum \sum \frac{i^i}{i!} \left( \sum_{\text{connected chord diagram}} \frac{1}{x^i} \right) b_0 \beta_{t_1} \beta_{t_2-1} \beta_{t_3-2} \beta_{t_4-3} \ldots \beta_{t_k-t_{k-1}} \]

where

\[ \frac{b_0}{x^i} + b_1 + b_2 x + b_3 x^2 + \ldots = \text{expansion of a regularized Feynman integral} \]
**Theorem [Marie, Yeats]**

The solution of the previous equation can be written under the form:

\[ G(x, L) = 1 - \sum_{i \geq 1} \sum_{C \text{ connected chord diagram}} \frac{1}{i!} \left( \sum_{1 \leq j_1 < j_2 < \ldots < j_k} \frac{1}{c_1^{j_1} c_2^{j_2} \cdots c_k^{j_k}} \right) \frac{b_0}{i!} + \frac{b_1}{x^2} + \frac{b_2}{x^4} + \frac{b_3}{x^6} + \cdots = \text{expansion of a regularized Feynman integral} \]
INTERSECTION ORDER
INTERSECTION ORDER
INTERSECTION ORDER

Diagram of two sets with numbers 1 to 12.
INTERSECTION ORDER
INTERSECTION ORDER

1
2

2 3 4 5 6 7 8 9 10 11 12

2 3 4 5 6 7 8 9 11 12
INTERSECTION ORDER

1

2

3 4 5 6 7 8 9 10 11 12

3 4 5 7 8 9 11 12
INTERSECTION ORDER
INTERSECTION ORDER
INTERSECTION ORDER
INTERSECTION ORDER

1

2 3 4 5 6 7 8 9 10 11 12

5 7
INTERSECTION ORDER

⚠️ intersection order ≠ left-right order
Theorem [Marie, Yeats]

The solution of the previous equation can be written under the form:

\[
G(x, \lambda) = 1 - \sum_{i \geq 1} \sum_{\text{connected chord diagram with terminal chords in position } k_1 < k_2 < \ldots < k_R \text{ such that } k_1 \geq i} \frac{i^{l-1} c_1 \ldots c_{l-1}}{i!} b_0^{t_1} b_{t_2 - k_1} b_{t_3 - t_2} \ldots b_{t_R - t_{R-1}}
\]

where

\[
\frac{b_0}{\lambda} + b_1 + b_2 \lambda + b_3 \lambda^2 + \ldots = \text{expansion of a regularized Feynman integral}
\]
\[ G(x, L) = 1 - \sum_{i \geq 1} \sum_{c \text{ connected chord diagram}} \frac{1}{i!} \frac{i!}{\ell^i} \prod_{t} c_{t_i}^{t_i^1 - \delta_{t_i}} \delta_{t_i - i} \delta_{t_i + k} \delta_{t_i + 2k} \delta_{t_i + 3k} \cdots \delta_{t_i + tk} \delta_{t_i + tk - 1} \]

where
\[ \frac{b_0}{\ell} + b_1 + b_2 \ell + b_3 \ell^2 + \cdots = \text{expansion of a regularized Feynman integral} \]
$k=3 \quad k_1 = 4 \quad k_2 = 5 \quad k_3 = 6$

$$G(x; L) = 1 - \sum_{i \geq 1} \sum_{C \text{ connected chord diagram}} \frac{L^i}{i!} \times \frac{i! c_1^{l_1} c_1^{l_1-k}}{b_0^{l_1-i} b_1^{l_2-t_1} b_2^{l_3-t_2} \cdots b_{k-1}^{l_{k-1}}}$$

where

$$\frac{b_0}{b} + b_1 + b_2 p + b_3 p^2 + \cdots = \text{expansion of a regularized Feynman integral}$$
**Ex:**

\[ G(x,L) = 1 - \sum_{i \geq 1} \sum_{\text{connected chord diagram}} \frac{\binom{i}{c} \binom{i}{c-1}}{i!} \cdot b_0 \cdot b_{t_1-1} \cdot b_{t_2-t_1} \cdot b_{t_3-t_2} \cdot \ldots \cdot b_{t_k-t_{k-1}} \]

(for \( i \leq 4 \))

\[ \frac{i^i}{i!} \cdot x^6 \cdot b_0 \cdot b_{4-2} \cdot b_1 \cdot b_1 \]

\( k = 3 \quad t_1 = 4 \quad t_2 = 5 \quad t_3 = 6 \)

---

where

\[ \frac{b_0}{\phi} + b_1 + b_2 \phi + b_3 \phi^2 + \ldots = \text{expansion of a regularized Feynman integral} \]
QUESTIONS

-> leading-log coefficients behaviour?

-> number of terminal chords?

-> position of the first terminal chord?

-> number of consecutive terminal chords?
ENUMERATION OF CONNECTED CHORD DIAGRAMS

1235
HISTORICAL BACKGROUND

About the enumeration of chord diagrams:

- knot theory (Vassiliev invariants)
- random graph generation
- bio-informatics (RNA secondary structures)
- cumulants
- ...
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About the enumeration of connected chord diagrams:

3 papers:
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3 papers:
1. [Touchard, 1952] = prehistory
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About the enumeration of connected chord diagrams:

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1. [Touchard, 1952] = prehistory
2. [Stein-Everett, 1978] = explicit formulas!
HISTORICAL BACKGROUND

About the enumeration of chord diagrams:
- Knot theory (Vassiliev invariants)
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About the enumeration of connected chord diagrams:
3 papers:
1. [Touchard, 1952] = prehistory
2. [Stein-Everett, 1978] = explicit formulas!
3. [Flajolet-Noy, 2000] = analytic combinatorics!!
Stein Formula

\[ c_n = \text{number of connected diagrams with } n \text{ chords} \]

\[ c_1 = 1 \quad c_2 = 1 \quad c_3 = 4 \quad c_4 = 27 \]

For \( n = 3 \),

![Diagrams](image-url)
number of non-necessarily connected diagrams with $n$ chords = ?
number of non-necessarily connected diagrams with \( n \) chords

\[ = \left( 2n - 1 \right)!! \]

\[ = (2n-1)(2n-3)(2n-5) \ldots 3 \times 1 \]
STEIN FORMULA

$c_n = \text{number of connected diagrams with } n \text{ chords}$

$c_1 = 1 \quad c_2 = 1 \quad c_3 = 4 \quad c_4 = 27$

For $n = 3$,}

\begin{align*}
\text{Diagram 1} & \quad \text{Diagram 2} & \quad \text{Diagram 3} & \quad \text{Diagram 4}
\end{align*}
$c_n = \text{number of connected diagrams with } n \text{ chords}$

$c_1 = 1, \quad c_2 = 1, \quad c_3 = 4, \quad c_4 = 27, \quad c_5 = 248$

For $n = 3$, the connected diagrams are shown in the image.
STEIN FORMULA

\[ c_n = \text{number of connected diagrams with } n \text{ chords} \]

\[ c_1 = 1 \quad c_2 = 1 \quad c_3 = 4 \quad c_4 = 27 \quad c_5 = 248 \]

For \( n = 3 \),

\[ \text{Theorem [Stein]} \quad c_n = (n-1) \sum_{k=1}^{n-1} c_k \times c_{n-k} \]
Theorem: $c_m = (m-1) \sum_{k=1}^{n-1} c_k \times c_{m-k}$
Theorem: \( c_n = \sum_{k=1}^{n-1} (2k-1) \cdot c_k \cdot c_{n-k} \)

Corollary: \( c_n = (n-1) \sum_{k=1}^{n-1} c_k \times c_{n-k} \)
Theorem: \[ c_n = \sum_{k=1}^{n-1} (2k-1) c_k c_{n-k} \]

( var. change \( k \to n-k \) )

\[ c_n = \sum_{k=1}^{n-1} (2n-2k-1) c_{n-k} c_k \]

Corollary: \[ c_n = (n-1) \sum_{k=1}^{n-1} c_k \times c_{n-k} \]
**Theorem:** \( c_n = \sum_{k=1}^{n-1} (2k-1) \binom{k}{c_k} c_{n-k} \)

\[ c_n = \sum_{k=1}^{n-1} (2n-2k-1) \binom{k}{c_k} c_{n-k} \]

\[ 2c_n = \sum_{k=1}^{n-1} (2n-2) \binom{k}{c_k} c_{n-k} \]

**Corollary:** \( c_n = (n-1) \sum_{k=1}^{n-1} c_k \times c_{n-k} \)
STEIN FORMULA

**Theorem:** \( c_m = \sum_{k=1}^{n-1} (2k-1) c_k \cdot c_{n-k} \)

\[ c_m = \sum_{k=1}^{n-1} (2n-2k-1) c_{n-k} \cdot c_k \]

\[ 2c_m = \sum_{k=1}^{n-1} (2n-2) c_k \cdot c_{n-k} \]

\[ \frac{1}{2} \left( \sum_{k=1}^{n-1} c_k \cdot c_{n-k} \right) \]

**Corollary:** \( c_m = (n-1) \sum_{k=1}^{n-1} c_k \cdot c_{n-k} \)
Theorem: \[ c_n = \sum_{k=1}^{n-1} (2k-1) c_k c_{n-k} \]

Proof:
Theorem: \( c_n = \sum_{k=1}^{n-1} (2k-1) \cdot c_k \cdot c_{n-k} \)

Proof:
Theorem: \[ c_n = \sum_{k=1}^{n-1} (2k-1) \cdot c_k \cdot c_{n-k} \]

Proof:
Theorem: \( c_n = \sum_{k=1}^{\frac{n-1}{2}} (2k-1) c_k c_{n-k} \)

Proof:
Theorem: \[ c_n = \sum_{k=1}^{n-1} (2k-1) \cdot c_k \cdot c_{n-k} \]

Proof: [Diagram showing derivation or explanation]
Theorem: \[ c_n = \sum_{k=1}^{n-1} (2k-1) c_k c_{n-k} \]
Theorem: \( c_n = \sum_{k=1}^{n-1} (2k-1) c_k c_{n-k} \)

Proof:

If \( k \) chords,

then \( (2k-1) \) possible insertions
\[ C_m = (n-1) \sum_{k=1}^{n-1} C_k C_{n-k} \]
\[ C_m \ 	ext{US CATALAN} \]

**CONNECTED DIAGRAMS**

\[ C_m = (n-1) \sum_{k=1}^{n-1} C_k C_{n-k} \]

\[ C_m \geq (n-1) \times C_1 \times C_{n-1} \]

**CATALAN**

\[ C_m = \sum_{k=1}^{n-1} C_k C_{n-k} \]
\[ C_n = (n-1) \sum_{k=1}^{n-1} C_k C_{n-k} \]

\[ C_n \geq (n-1) \times C_1 \times C_{n-1} \]

\[ C_n \geq (n-1)! \]
\[ C_n = (n-1) \sum_{k=1}^{n-1} C_k C_{n-k} \]

\[ C_n \geq (n-1) \times C_1 \times C_{n-1} \]

\[ C_n \geq (n-1)! \]

\[ \rightarrow \text{not analytic} \]

\[ C_n = \sum_{k=1}^{n-1} C_k C_{n-k} \rightarrow \text{analytic} \]
\( C_n \) US CATALAN

CONNECTED DIAGRAMS

\[ C_n = (n-1) \sum_{k=1}^{n-1} c_k c_{n-k} \]

\[ C_n \geq (n-1) c_1 c_{n-1} \]

\[ C_n \geq (n-1)! \]

\[ \rightarrow \text{not analytic} \]

Consequence: - Ordinary Generating Functions \( C \) are not adapted.

CATALAN

\[ C_n = \sum_{k=1}^{n-1} c_k c_{n-k} \]

\[ \rightarrow \text{analytic} \]
\( C_m \) US CATALAN

**CONNECTED DIAGRAMS**

\[
C_m = (n-1) \sum_{k=1}^{n-1} C_k C_{n-k}
\]

\[
C_m \geq (n-1) \times C_1 \times C_{n-1}
\]

\[
C_m \geq (n-1)! \\
\rightarrow \text{not analytic}
\]

**Consequence:** - Ordinary Generating Functions \( \square \) are not adapted.

- No simple equation for the Exponential Generating Functions.
ASYMPTOTIC BEHAVIOUR

[Stein-Everett]

\[ c_m \sim \frac{1}{e} \times (2m-1)!! \]

Consequence: \( P(\text{diagram is connected}) \rightarrow \frac{1}{e} \)
ASYMPTOTIC BEHAVIOUR

[Stein-Everett]

\[ c_m \sim \frac{1}{e} \times (2m-1)!! \]

Consequence: \[ P(\text{diagram is connected}) \to \frac{1}{e} \]

[Flajolet-Noy]

- number of connected components \( \sim \text{Poisson}(1) \)
- \( n \) - size of the largest component \( \sim \text{Poisson}(1) \)
ASYMPOTIC BEHAVIOUR

[Stein-Everett]

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[Flajolet-Noy]

- number of connected components \( \sim \text{Poisson}(1) \)
- \( n \)-size of the largest component \( \sim \text{Poisson}(1) \)

Our humble contribution: \( \frac{c_{n-1}}{c_n} = \frac{1}{2n} + \frac{1}{4n^2} - \frac{1}{4n^3} + o\left(\frac{1}{n^3}\right) \)
STATISTICS ON TERMINAL CHORDS

I was a joke in French but I don't work anymore.
\[ G(x, L) = 1 - \sum_{i \geq 1} \sum_{\text{connected chord diagram}} \frac{L^i}{i!} C_{\text{connected chord diagram}} \times \frac{1}{\alpha} \prod_{k=0}^{i} \beta_{t_1 - i} \beta_{t_2 - t_1} \beta_{t_3 - t_2} \cdots \beta_{t_{k-1} - t_{k-2}} \beta_{t_k - t_{k-1}} \]

with terminal chords in position \( t_1 < t_2 < \ldots < t_k \) such that \( t_1 \geq i \)
$G(x,L) = 1 - \sum_{i \geq 1} \sum_{\mathcal{C} \text{ connected chord diagram}} \frac{(Lx)^i}{i!} \frac{1}{c_1^{i-1} c_2^{i-2}} \cdot f_{t_1-i} f_{t_2-t_1} f_{t_3-t_2} \cdots f_{t_k-t_{k-1}}$

where $\mathcal{C}$ is a chord diagram with terminal chords in position $t_1 < t_2 < \cdots < t_k$ such that $t_1 \geq i$.
$G(x,L) = 1 - \sum_{i \geq 1} \sum_{\text{connected chord diagram}} (Lx)^i \frac{1}{i!} \frac{1}{cl-i} \prod_{b_1, b_2, \ldots, b_{k_R}} b_{t_1-i} b_{t_2-t_1} b_{t_3-t_2} \cdots b_{t_{k_R}-t_{k_R-1}}$

Coefficient of $(Lx)^i \frac{1}{cl-i}$ for $i$ close to $|cl|$?
\[ G(i,L) = 1 - \sum_{i \geq 1} \sum_{\text{connected chord diagram}} \frac{(Lx)^i}{i!} \prod_{j=1}^{i} B(x^{-1}) \cdot B(t_1, t_1, t_2, t_2, \ldots, t_{i-1}, t_{i-1}) \]

Coefficient of \((Lx)^i\) vs \(|C|-i\) for \(i\) close to \(|C|\)?

- \(|C| = i\) : leading-log expansion
- \(|C| = i+1\) : next-to leading-log expansion
- \(|C| = i+2\) : next-to\(^2\) leading-log expansion
\[ G(x, L) = 1 - \sum_{i \geq 1} \sum_{\text{connected chord diagram}} \left( Lx \right)^i \frac{1}{i!} x \beta_{t_1-i} \beta_{t_2-t_1} \beta_{t_3-t_2} \cdots \beta_{t_k-t_{k-1}} \]

Coefficient of \( (Lx)^i \) \( \beta_{t_1-i} \) for \( i \) close to \( |C| \)?

- \( |C| = i \) : leading-log expansion

\[ \Leftrightarrow t_1 = |C| \]

\[ \Leftrightarrow \text{There is only one terminal chord.} \]
ONLY ONE TERMINAL CHORD

number of connected diagrams with $n$ chords and only one terminal chord

= ?
ONLY ONE TERMINAL CHORD

number of connected diagrams with n chords and only one terminal chord

= \((2n - 3)!!\)
ONLY ONE TERMINAL CHORD

number of connected diagrams with n chords and only one terminal chord

= \( (2n - 3)!! \)

Proof:

One piece of size \( n-1 \)

impossible

2n-3 possible locations
ONLY ONE TERMINAL CHORD

number of connected diagrams with $n$ chords and only one terminal chord

$$= (2n-3)!!$$

Proof:

One piece of size $n-1$

impossible

2n-3 possible locations

Cor:

$n$th coeff of the leading-log expansion

$$= \frac{(2n-3)!!}{n!} f_0^n$$
**Next-to-leading-log terms**

→ "Similar" recursions exist for the diagrams such that \( t_1 \geq |C|-\ell \)

→ Analytic combinatorics techniques work here.
**Next-to**$^l$ **Leading-log Terms**

→ “Similar” recursions exist for the diagrams such that $t_1 \geq |C|-l$

→ Analytic combinatorics techniques work here.

**Theorem**: For $l \geq 0$,

number of connected diagrams with $n$ chords such that $t_1 \geq |C|-l$

\[
\sim \frac{1}{\sqrt{\pi}} \prod_{l=1}^{l} \frac{2^{l+1}}{l!} \frac{\ln(n)}{n^{\frac{3}{2}}} \cdot n!.
\]
NEXT-TO$^l$ LEADING-LOG TERMS

→ "Similar" recursions exist for the diagrams such that $t_1 \geq |C|-l$

→ Analytic combinatorics techniques work here.

**Theorem**: For $l \geq 0$,

the number of connected diagrams with $n$ chords such that $t_1 \geq |C|-l$

\[ \sim \frac{1}{\sqrt{2\pi}^{l+1} l!} \times \frac{\ln(n)^l}{n^{3/2}} \times n! \]

But how about $b_t^{l_1} b_{t_1} i b_{t_2} i b_{t_2} i b_{t_3} i b_{t_3} i b_{t_k} i b_{t_k} i$?
THE LAST $l$ CHORDS ARE TERMINAL

"Similar" recursions exist for the diagrams such that the last $l$ chords are terminal.

Analytic combinatorics techniques work here.

Theorem: For $l \geq 0$,

number of connected diagrams with $n$ chords such that the last $l$ chords are terminal

$$\sim \frac{1}{\sqrt{\pi}} \frac{1}{2^{l+1} l!} \times \frac{\ln(n)^l}{n^{3/2}} n!.$$
THE LAST $l$ CHORDS ARE TERMINAL

→ "Similar" recursions exist for the diagrams such that the last $l$ chords are terminal
→ Analytic combinatorics techniques work here.

**Theorem:** For $l \geq 0$,
the number of connected diagrams with $n$ chords such that the last $l$ chords are terminal

\[
\sim \frac{1}{\sqrt{\pi}} 2^{\frac{l+1}{2}} \frac{\ln(n)^l}{n^{\frac{3}{2}}} n!
\]

Here $b_0^{l_0} b_{t_1-l_1} b_{t_2-t_1} b_{t_3-t_2} \cdots b_{t_{l-1}-t_{l-2}} = b_0^{n-l+1} b_l^{l-1}$
**NEXT-TO-$l$ LEADING-LOG TERMS**

Diagrams such that the last $l$ chords are terminal are dominant among the diagrams such that $t_1 > |c_l - l|$

**Corollary:** For $l \geq 0$,

$n^{th}$ coeff of the next-to-$l$ leading-log expansion

\[ \sim \frac{1}{\sqrt{\pi}} \frac{1}{2^{l+1} l!} \times \frac{\ln(n)^l}{n^{3/2}} \times n^{n-l+1} \left( \frac{l-1}{n} \right)^{l-1} \]

$B_0 \quad B_1$
**NEXT-TO-LO** LEADING-LOG TERMS

Diagrams such that the last \( l \) chords are terminal are dominant among the diagrams such that \( n_1 \geq |C|-l \).

**Corollary**: For \( l \geq 0 \),

\( n^{\text{th}} \) coeff of the next-to-\( l \) leading-log expansion

\[
\sim \frac{1}{\sqrt{\pi} 2^{l+1} l!} \times \frac{\ln(n)^l}{n^{\frac{3l}{2}}} n! \cdot \beta_0 \beta_1^{l-1}
\]

Only \( \beta_0 \) and \( \beta_1 \) matter!
NUMBER OF TERMINAL CHORDS

Average number of terminal chords?
NUMBER OF TERMINAL CHORDS

Average number of terminal chords ?????
NUMBER OF TERMINAL CHORDS

Average number of terminal chords ????

Idea:

large number of chords
NUMBER OF TERMINAL CHORDS

Average number of terminal chords

Idea:

large number of chords

\[ (2n-3)c_{n-1} \]

\[ c_n \]
NUMBER OF TERMINAL CHORDS

Average number of terminal chords

Idea:

large number of chords

\[ \text{proba} \]

\[ \frac{(2n-3)C_{n-1}}{C_n} \]

\[ \rightarrow 0 \]

\[ \rightarrow 1 \]
NUMBER OF TERMINAL CHORDS

Average number of terminal chords

Idea:

\[
\frac{\text{proba}}{(2n-3)c_{n-1}} \Rightarrow 0
\]

\[
\Rightarrow 1
\]

Interesting but not sufficient...
NUMBER OF TERMINAL CHORDS

Average number of terminal chords ?????

Idea:

large number of chords
NUMBER OF TERMINAL CHORDS

Average number of terminal chords

Idea:

large number of chords

\[
\text{proba} = \frac{(2n-3) \cdot C_{n-1}}{C_n}
\]

\[
\text{proba} = \frac{(2n-5) \cdot C_{n-2}}{C_n}
\]

\[
\text{proba} = \frac{(2n-5) \cdot C_{n-2}}{C_n}
\]
NUMBER OF TERMINAL CHORDS

Average number of terminal chords

Idea:

\[
\begin{align*}
\text{proba} &= \frac{(2n-3)cn-1}{cn} \\
&= 1 - \frac{1}{n} + o\left(\frac{1}{n}\right)
\end{align*}
\]

\[
\begin{align*}
\text{proba} &= \frac{(2n-5)cn-2}{cn} \\
&\sim \frac{1}{2n}
\end{align*}
\]

\[
\begin{align*}
\text{proba} &= \frac{(2n-5)cn-2}{cn} \sim \frac{1}{2n} \\
&= o\left(\frac{1}{n}\right)
\end{align*}
\]
NUMBER OF TERMINAL CHORDS

Average number of terminal chords

Idea:

large number of chords

\[ \text{proba} = \frac{(2n-3) \cdot \binom{n-1}{n}}{\binom{n}{n}} \]
\[ = 1 - \frac{1}{n} + o\left(\frac{1}{n}\right) \]

\[ \text{proba} = \frac{(2n-5) \cdot \binom{n-2}{n}}{\binom{n}{n}} \]
\[ \sim \frac{1}{2n} \]

\[ \text{proba} = \frac{(2n-5) \cdot \binom{n-2}{n}}{\binom{n}{n}} \sim \frac{1}{2n} \]

\[ = o\left(\frac{1}{n}\right) \]

Let's forget that
NUMBER OF TERMINAL CHORDS

Set \( p_{n,k} = (1 - \frac{1}{n}) p_{n-1,k} + \frac{1}{n} p_{n-2,k-1} \)

Idea:

\[
\begin{align*}
\text{large number of chords} & \quad \text{\large number of choices} \\
\text{large number of chords} & \quad \text{large number of choices} \\
\end{align*}
\]

\[
\text{large number of chords} = 1 - \frac{4}{n} + o\left(\frac{1}{n}\right)
\]

\[
\text{large number of chords} \sim \frac{1}{2n}
\]

\[
\text{large number of chords} = o\left(\frac{1}{n}\right)
\]

Let's forget that.
NUMBER OF TERMINAL CHORDS

Set \( p_{n,k} = \left(1 - \frac{1}{n}\right) p_{n-1,k} + \frac{1}{n} p_{n-2,k-1} \)

**Fact 1:** Let \( X_n \) be the random variable such that \( \mathbb{P}(X_n = k) = p_{n,k} \)

\( X_n \rightarrow \text{Gaussian law} \)

**Fact 2:** The number of terminal chords \( \sim X_n \)
Theorem: The number of terminal chords in a random connected diagram of size $n$ asymptotically obeys to a Gaussian limit law of mean and variance $\sim \ln(n)$. 
NUMBER OF CONSECUTIVE TERMINAL CHORDS

If the terminal chords are in position \( x_1 < \ldots < x_r \), how many \( j \)'s satisfy \( t_j^g - t_{j-1}^g = 1 \) ?
If the terminal chords are in position $x_1 < \ldots < x_k$, how many $j$'s satisfy $t_j^g - t_{j-1}^g = 1$?

\[ \text{terminal} \]

\[ \text{terminal} \]

\[ \text{proba} = o \left( \frac{1}{n} \right) \]
NUMBER OF CONSECUTIVE TERMINAL CHORDS

If the terminal chords are in position \( t_1 < \ldots < t_k \), how many \( j \)'s satisfy \( t_0^j - t_{0,j-1} = 1 \) ?

\[ \begin{align*}
&\text{terminal} \\
&t_{0,j} - t_{0,j-1} > 1 \times \\
&\text{terminal} \\
&\text{prob} = o \left( \frac{1}{n} \right)
\end{align*} \]
NUMBER OF CONSECUTIVE TERMINAL CHORDS

If the terminal chords are in position \( k_1 < \ldots < k_r \), how many \( j \)'s satisfy \( t^o_j - t^o_{j-1} = 1 \)?

\[
\begin{align*}
\text{terminal} & \quad t^o_j - t^o_{j-1} > 1 \times \\
\text{terminal} & \quad t^o_j - t^o_{j-1} = 1 \checkmark
\end{align*}
\]

Theorem: Number of consecutive terminal chords

\( \longrightarrow \) Gaussian law of mean and variance \( \sim \frac{\ln n}{2} \)
NUMBER OF CONSECUTIVE TERMINAL CHORDS

If the terminal chords are in position $k_1 < \ldots < k_r$, how many $j$'s satisfy $t_{j^*}^k - t_{j^*}^{k-1} = 1$?

On average,

$$b_0 b_{t_1} b_{t_2} b_{t_3} \ldots b_{t_k} \sim b_0 b_{t_1}^2$$

→ confirms the importance of $b_0$ and $b_1$

Theorem: Number of consecutive terminal chords

→ Gaussian law of mean and variance $\sim \frac{\ln n}{2}$
POSITION OF THE FIRST TERMINAL CHORD -

\[ t_1 = \text{random variable returning the position of the 1}^{st} \text{ terminal chord.} \]

**Theorem:** \[ E(t_1) \sim \frac{2}{3} n \]
**POSITION OF THE FIRST TERMINAL CHORD** -

\[ t_1 = \text{random variable returning the position of the 1st terminal chord} \]

**Theorem:** \[ E(t_1) \sim \frac{2}{3} n \]

**Limit law?**

Density of \( \frac{X_n}{n} \)
CONCLUSION

- Recovers the results of Krüger and Kreimer
  + automaticity of the method
  + asymptotic behaviour

- New combinatorial approach

- Extension to Hihn-Yeats’s results?
Journées Combinatoires Françoises
11-12 juillet 2016 - VANCOUVER
(après FPSAC)