Julien Courtiel (PIMS, UBC)
Karen Yeats (SFU)

Terminal Chords in Connected Chord Diagrams

UBC, April 12


Tom Hanks
Catherine Zeta-Jones
The Terminal
Life is waiting.

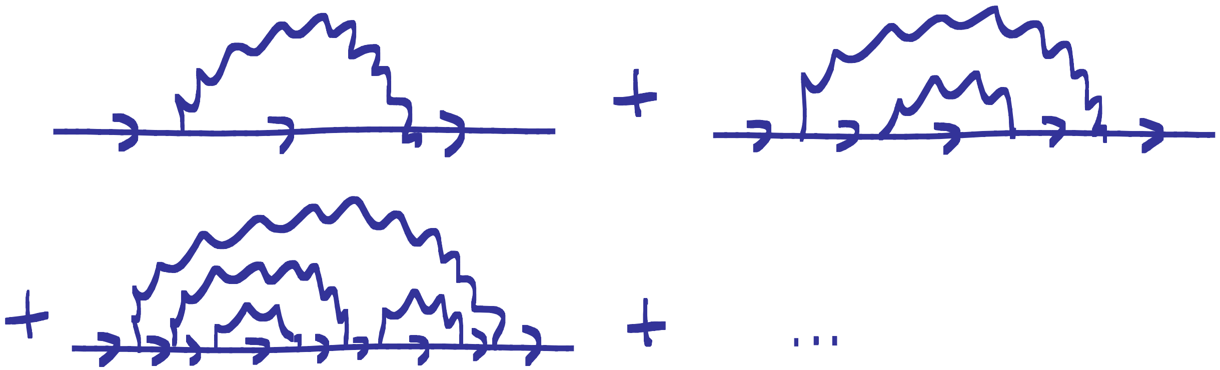
COMBINATORIAL DYSON-SCHWINGER EQUATIONS

Class of Feynman graphs

Feynman rules

solution of Dyson-Schwinger equations

One-loop propagator + recursive iterations



$$G(x,L) = 1 - x\, G\!\left(x, \frac{\partial}{\partial(-\rho)}\right)^{-1} \left(e^{-L\rho} - 1\right) F(\rho)\Big|_{\rho=0}$$

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Theorem [Marie, Yeats]

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where

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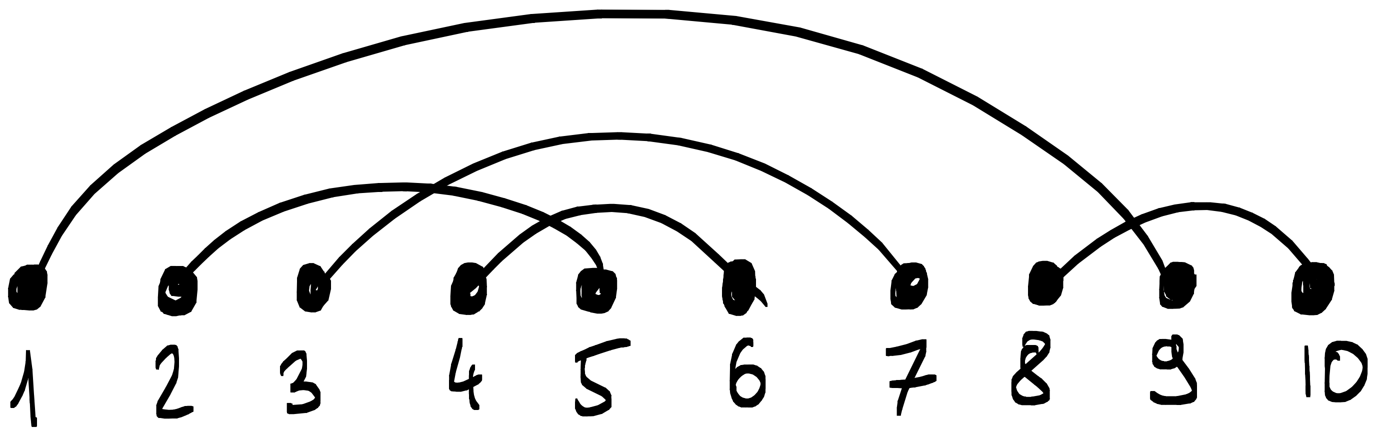
CONNECTED CHORD DIAGRAMS

Diagram of n chords = matching of $\{1, 2, \dots, 2n\}$



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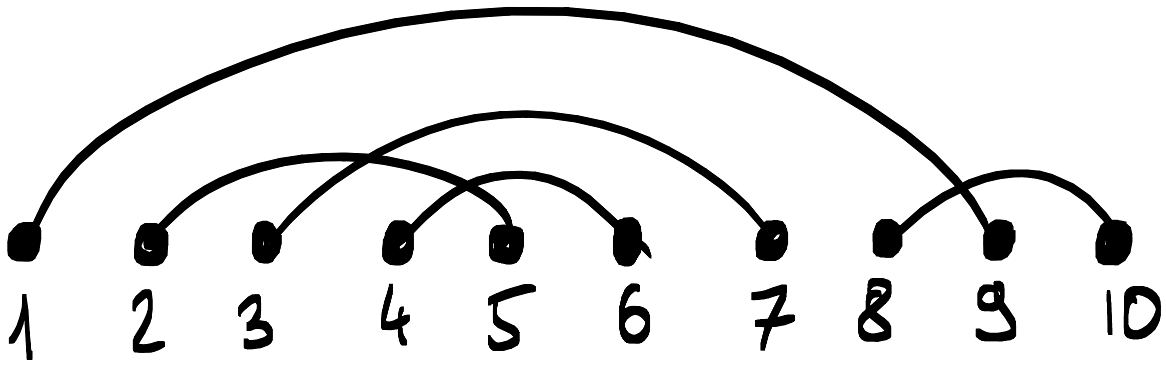

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connected diagram : its representation is in one piece

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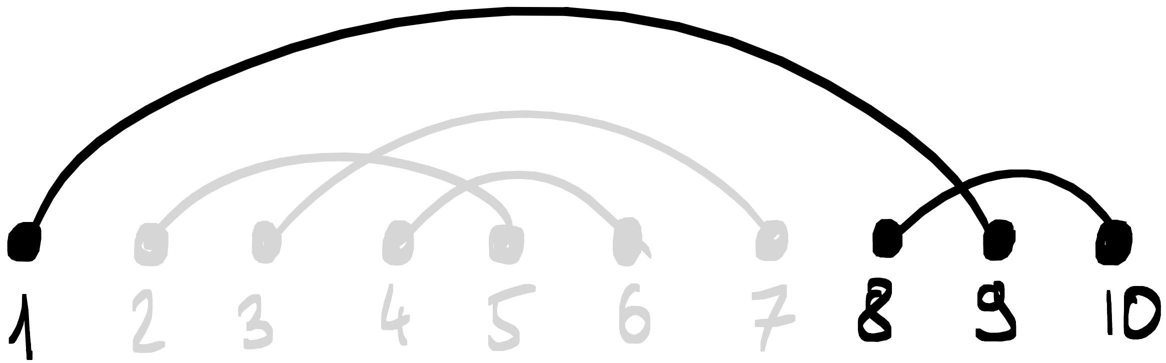

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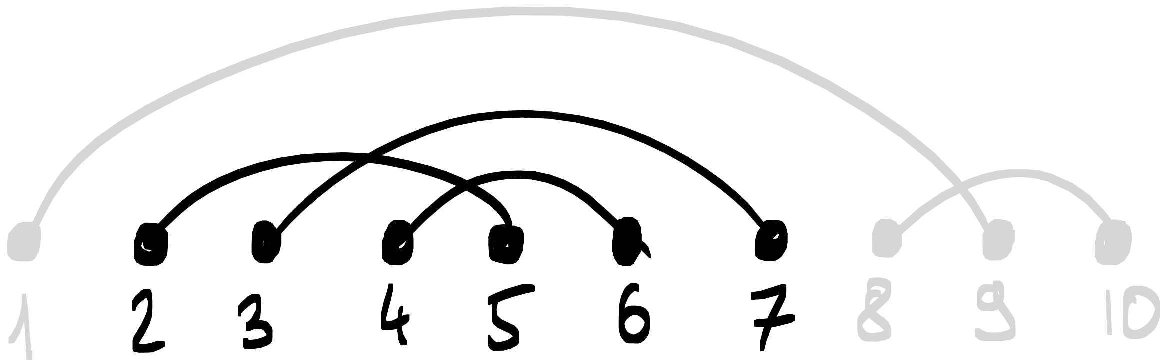

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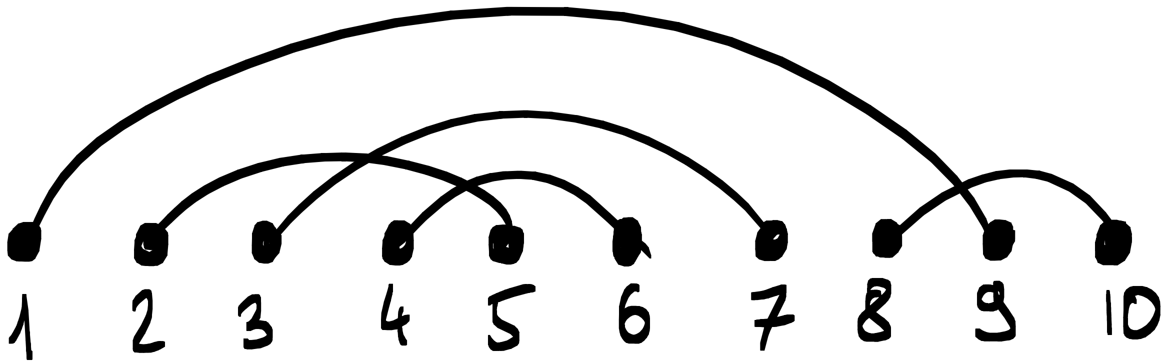

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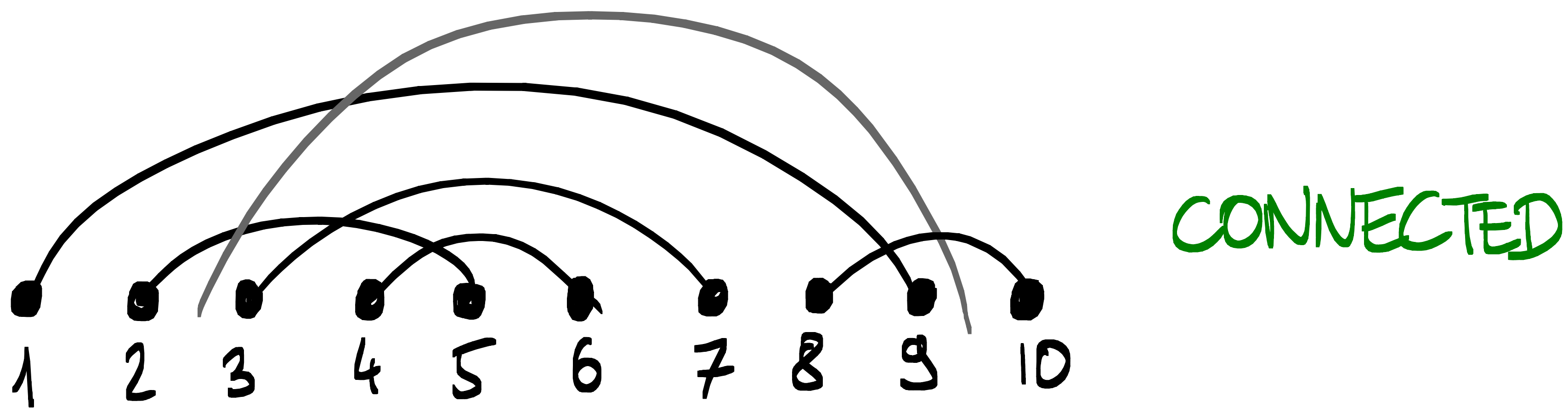

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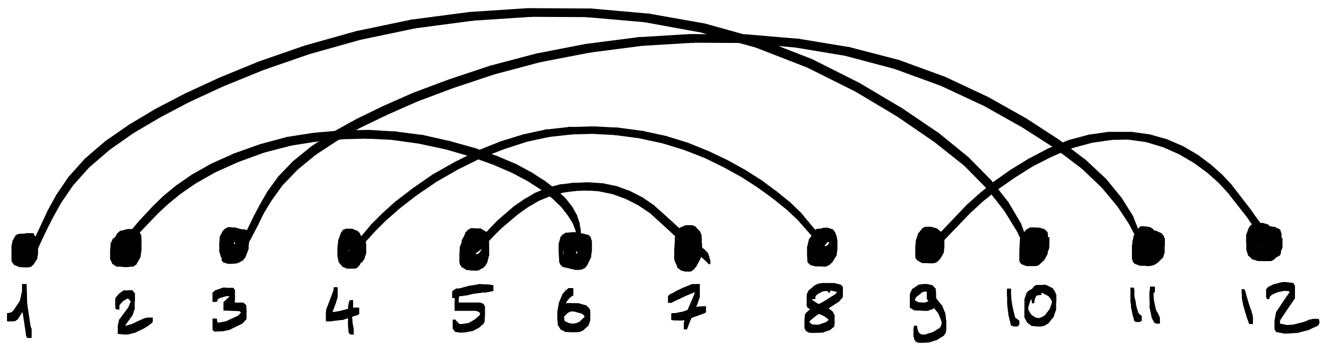

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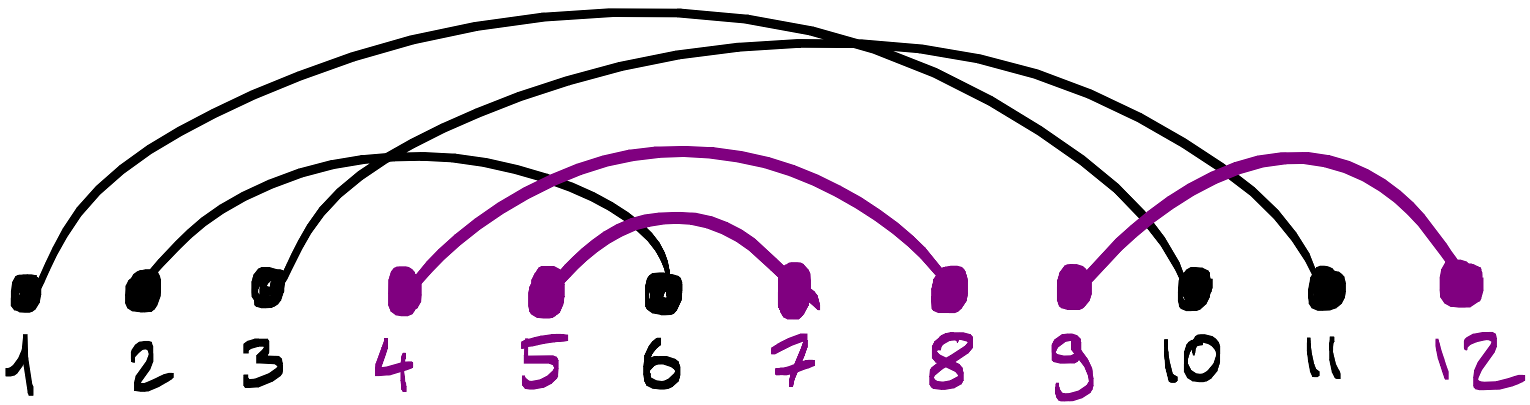
TERMINAL CHORDS



terminal chord = chord (a,b) such that for every chord (c,d) that intersects it,

$$c < a < d < b.$$

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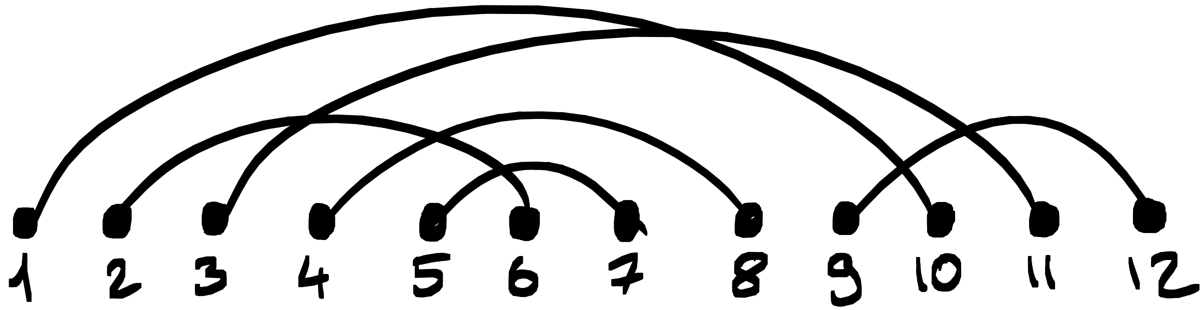
INTERSECTION ORDER



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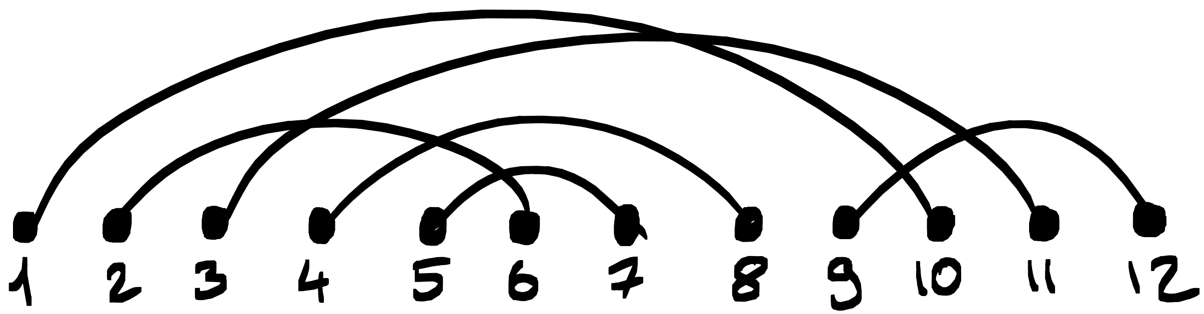

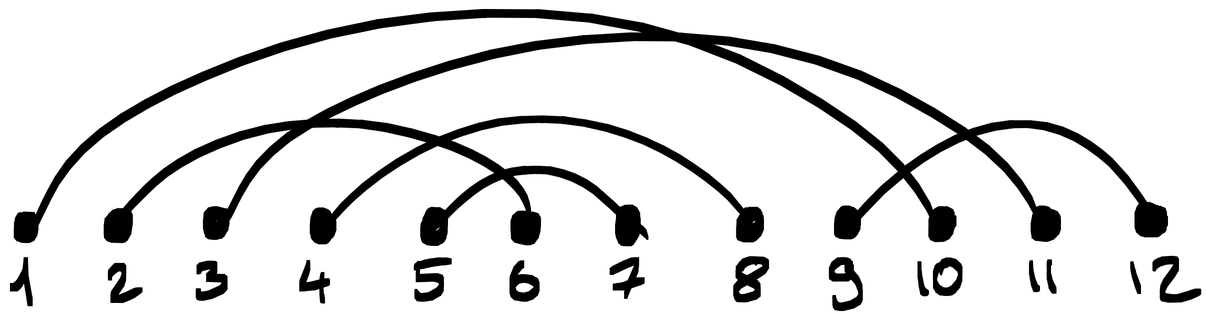

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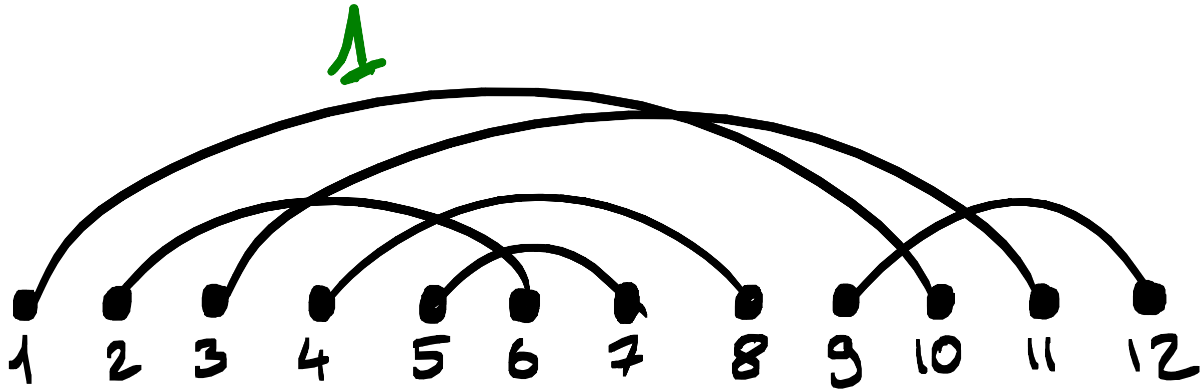

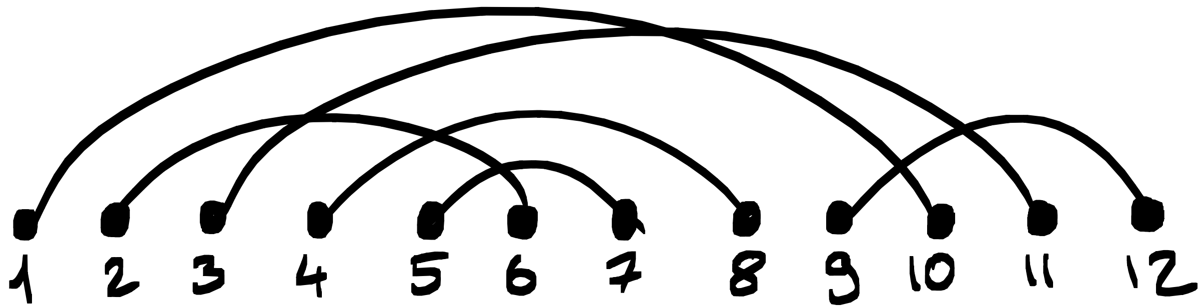

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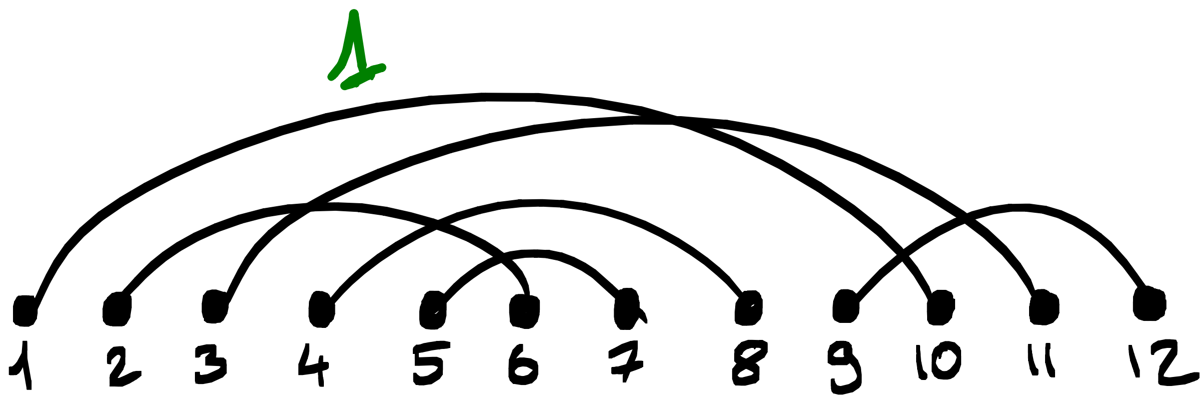

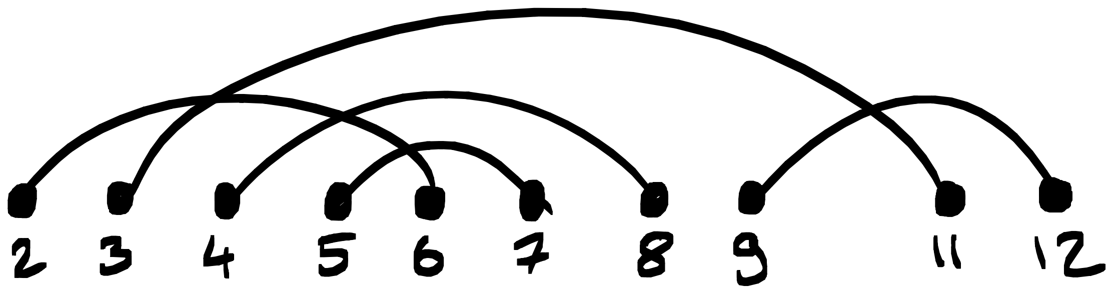

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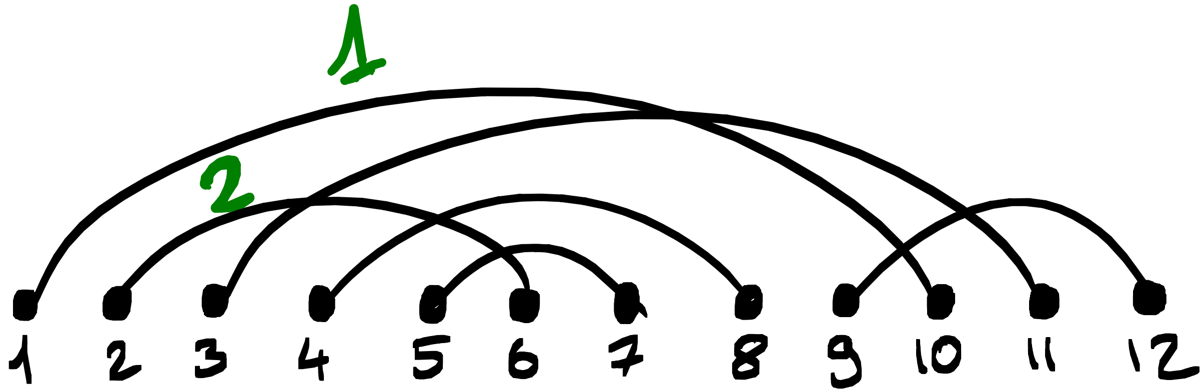

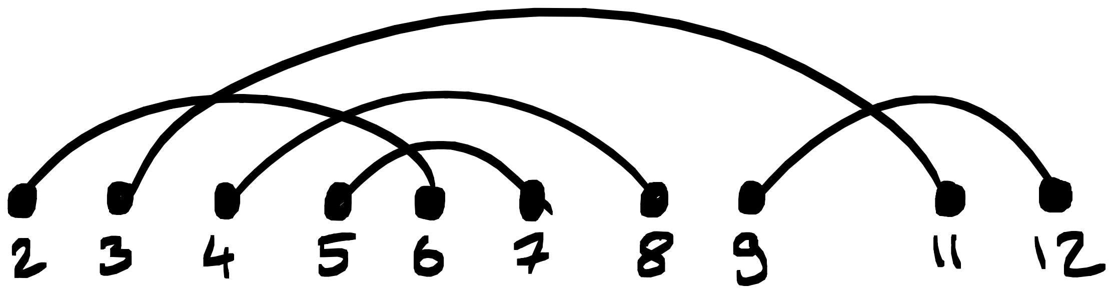

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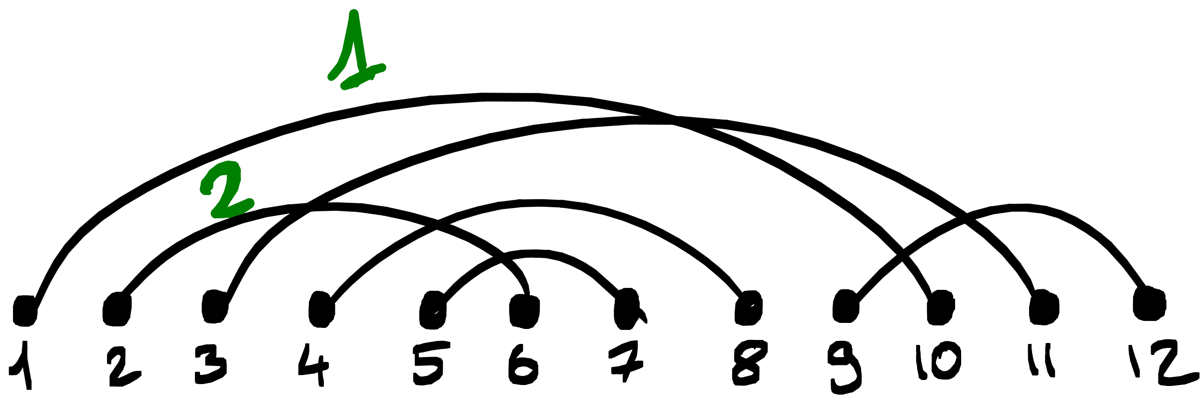

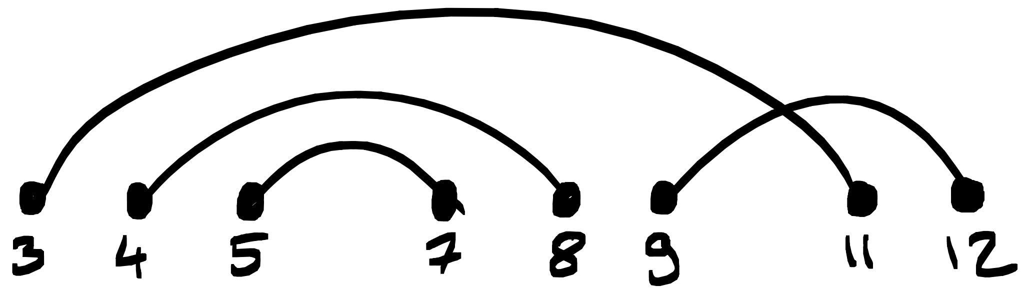

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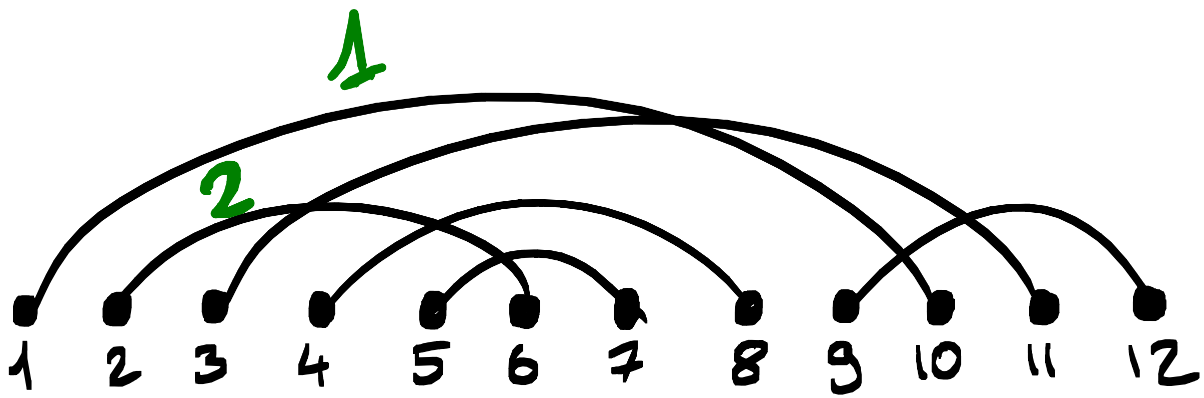

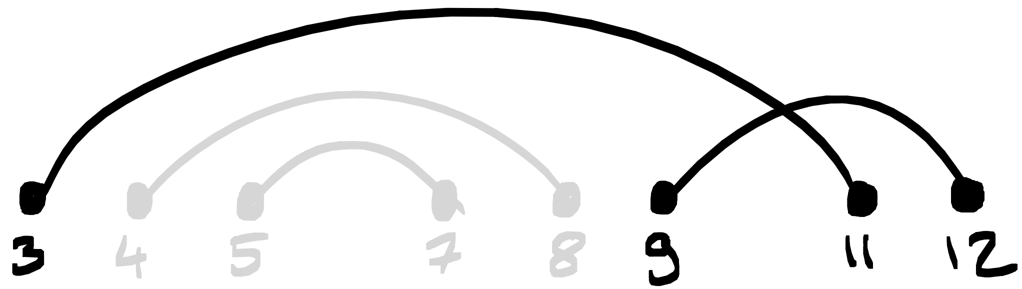

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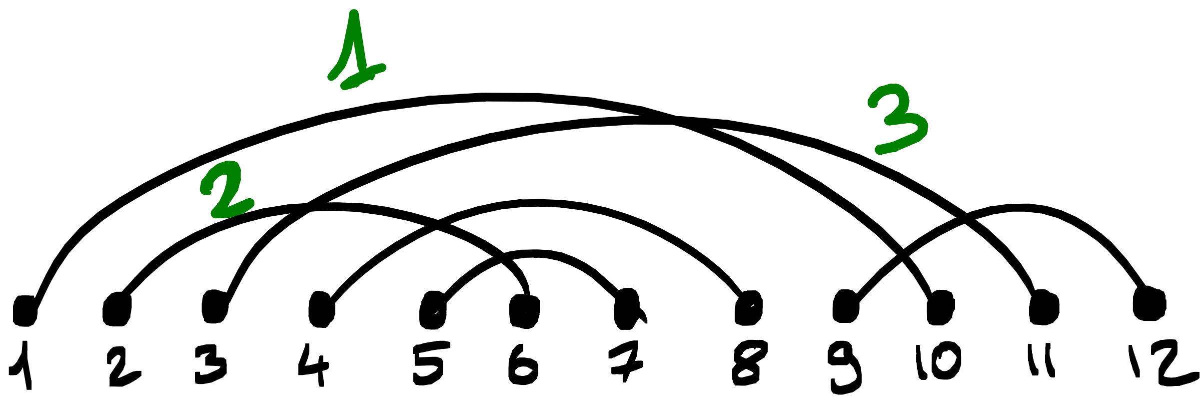

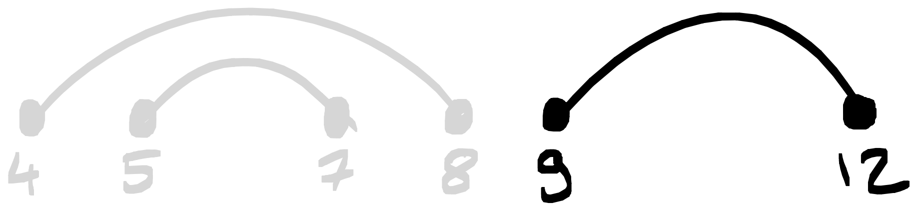

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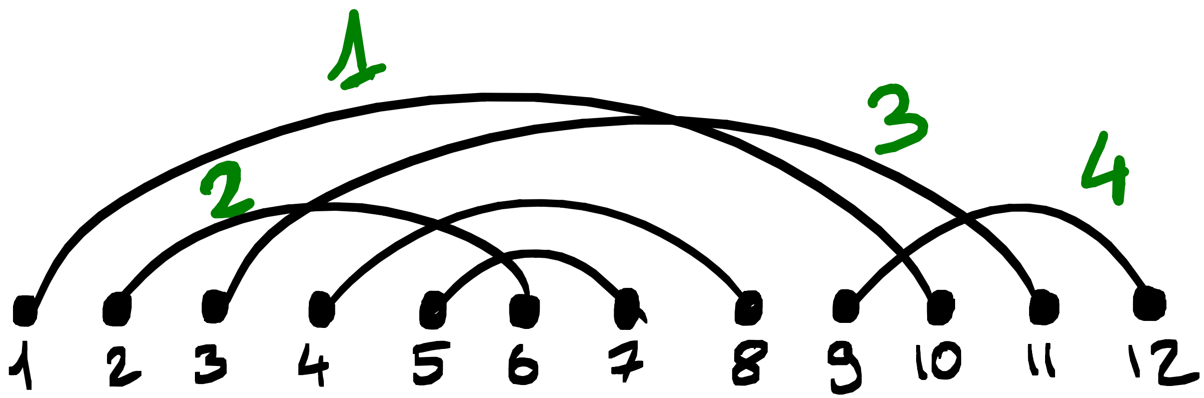

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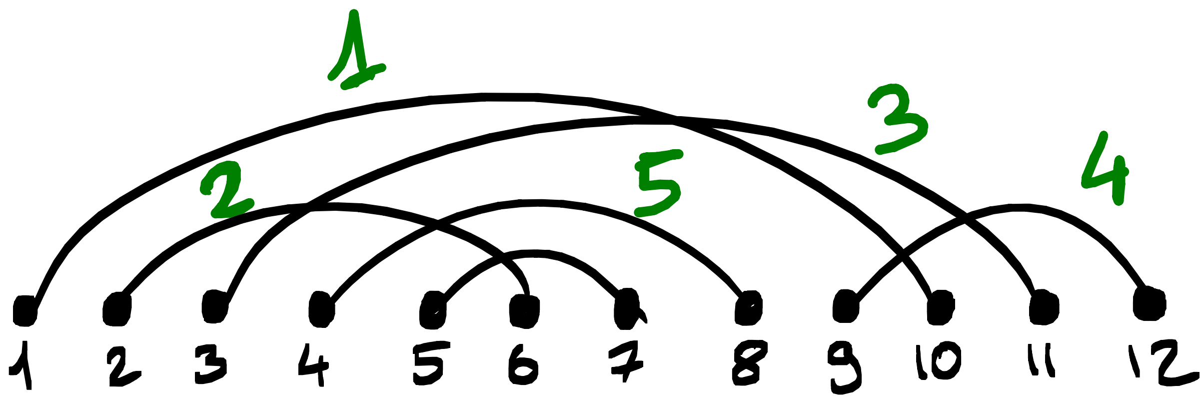

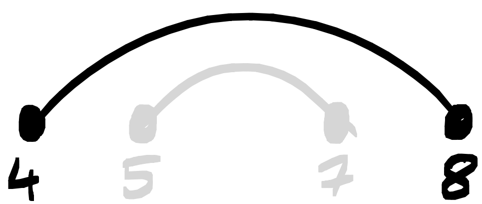

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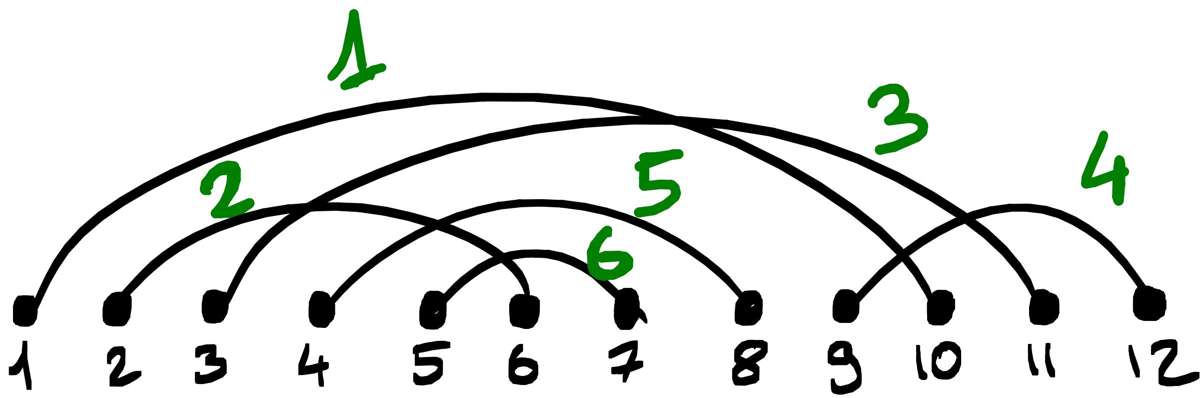

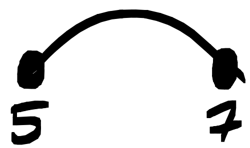

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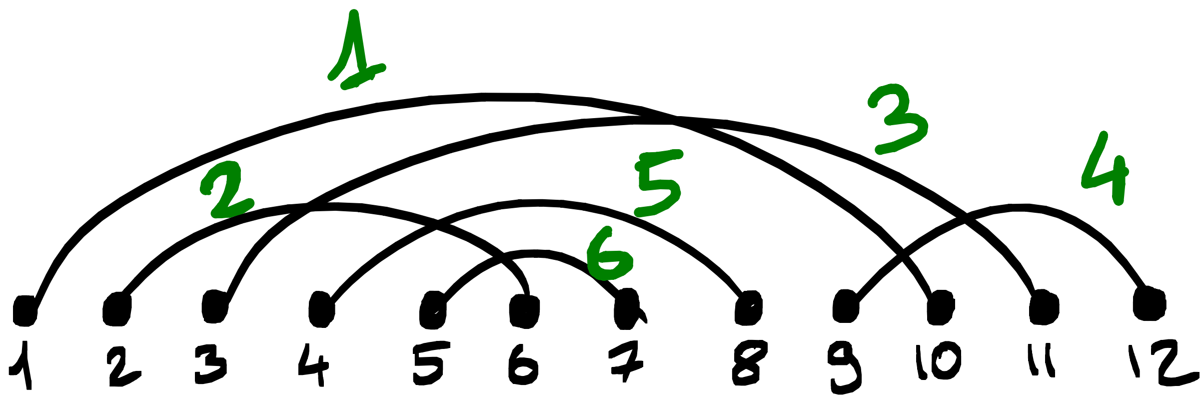

⚠ intersection order ≠ left-right order

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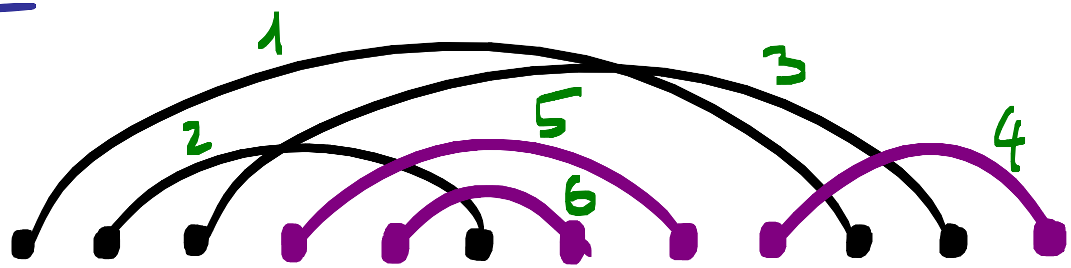

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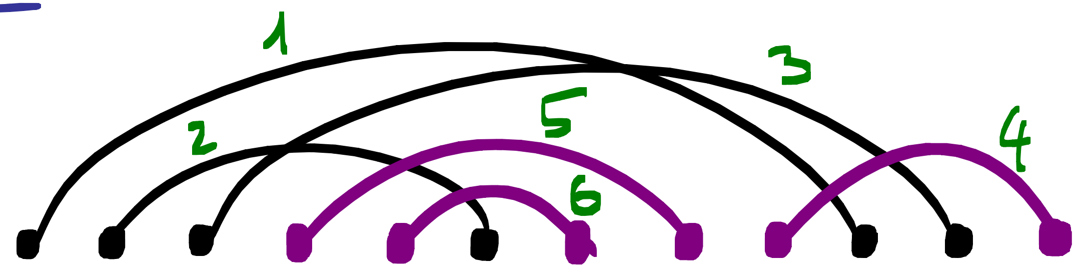

$k=3 \quad t_1=4 \quad t_2=5 \quad t_3=6$

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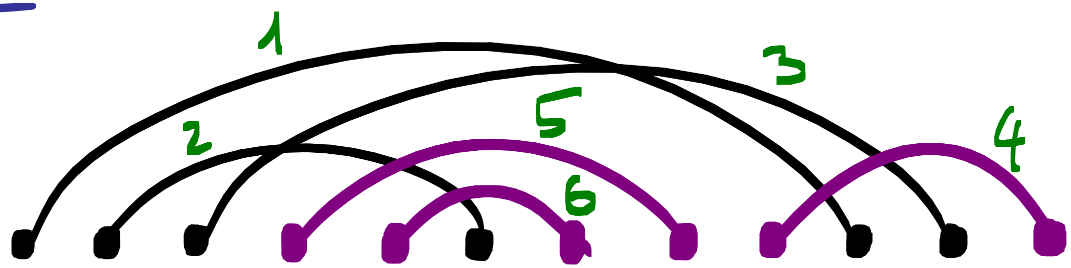

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(for $i \leq 4$)

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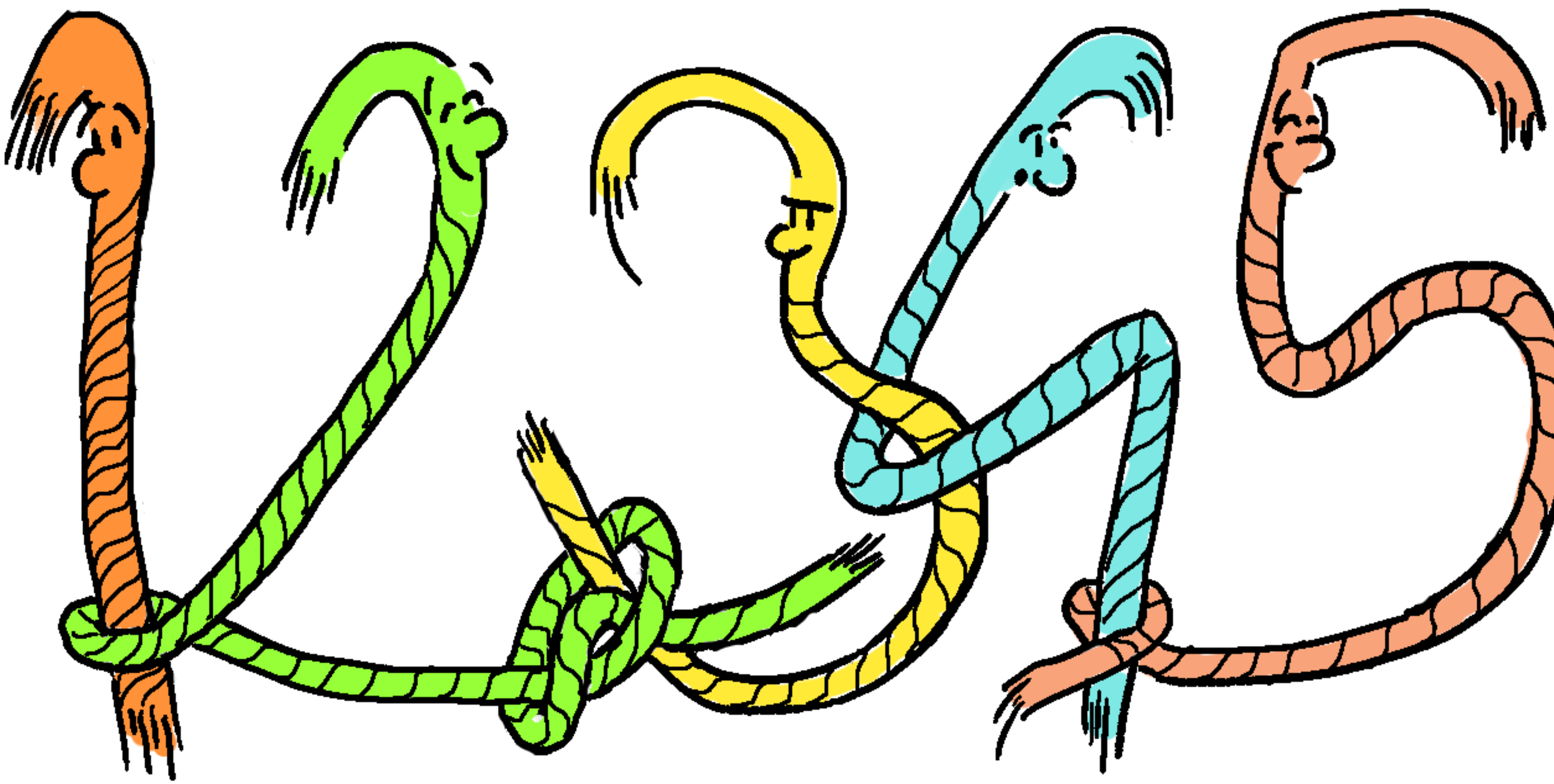
QUESTIONS

- → leading-log coefficients behaviour?
- → number of terminal chords?
- → position of the first terminal chord?
- → number of consecutive terminal chords?

ENUMERATION OF CONNECTED CHORD DIAGRAMS



HISTORICAL BACKGROUND

About the enumeration of chord diagrams:

- → Knot theory (Vassiliev invariants)
- → random graph generation
- → bio-informatics (RNA secondary structures)
- → cumulants
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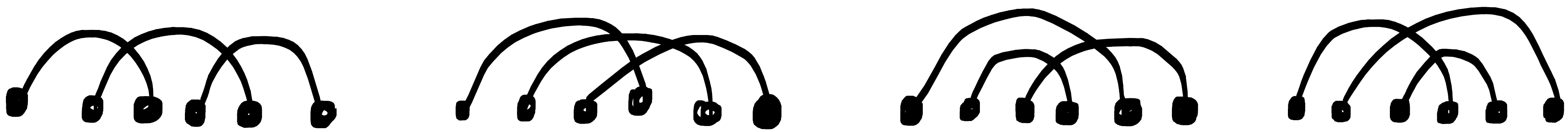
3. [Flajolet-Noy, 2000] = analytic combinatorics!!

STEIN FORMULA

c_n = number of connected diagrams with n chords

$c_1 = 1 \quad c_2 = 1 \quad c_3 = 4 \quad c_4 = 27$

For $n = 3$,



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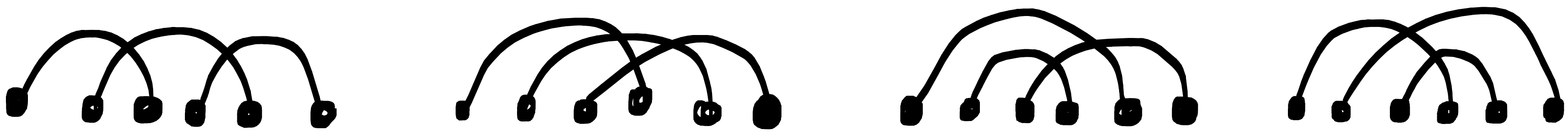

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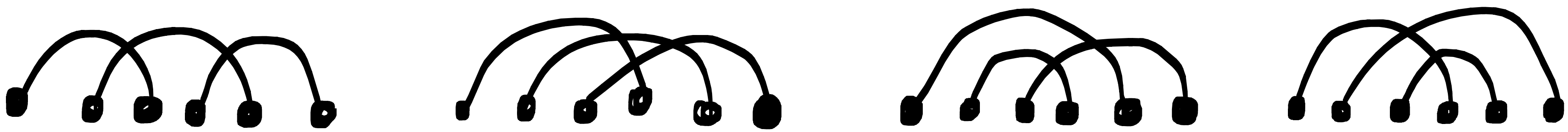

Theorem [Stein]

$$c_n = (n-1) \sum_{k=1}^{n-1} c_k \times c_{n-k}$$

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$\Big\downarrow /2$

Corollary: $$c_n = (n-1) \sum_{k=1}^{n-1} c_k \times c_{n-k}$$

STEIN FORMULA

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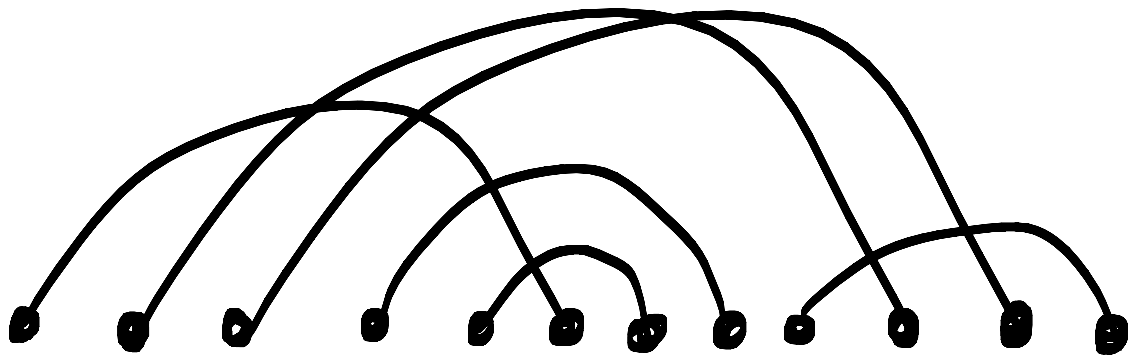

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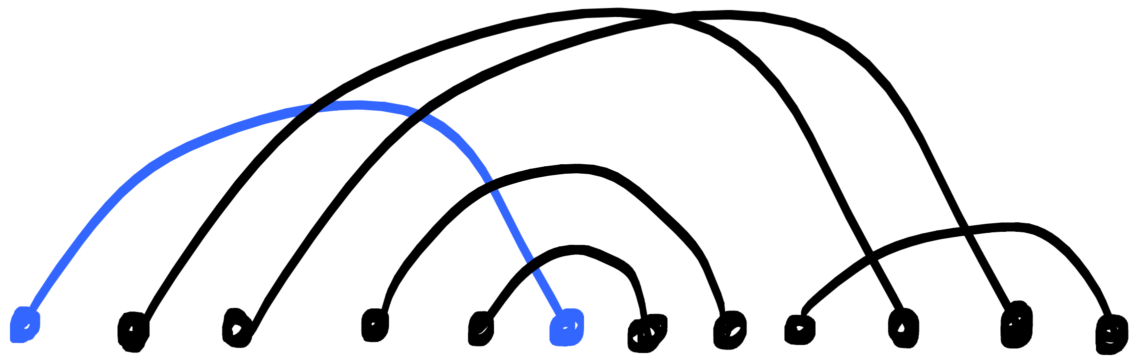

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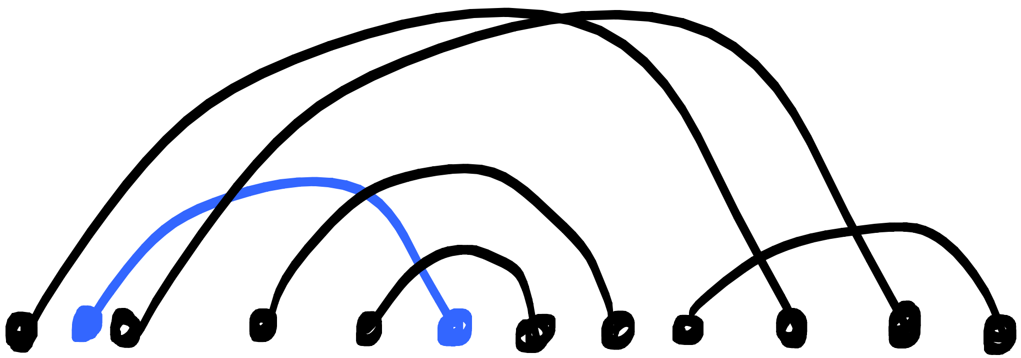

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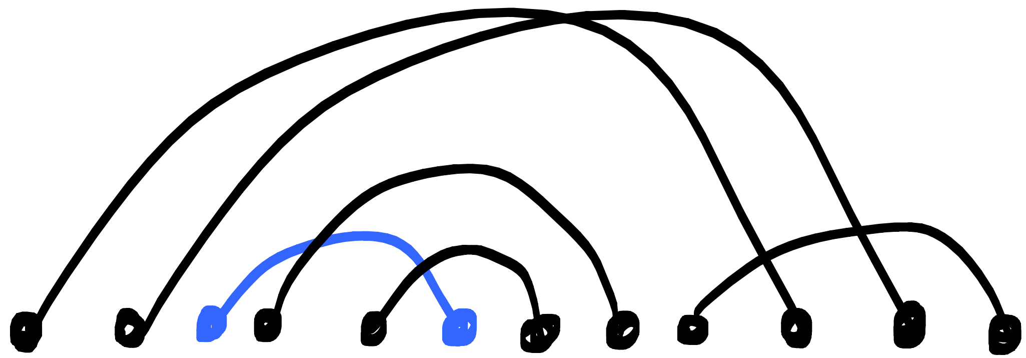

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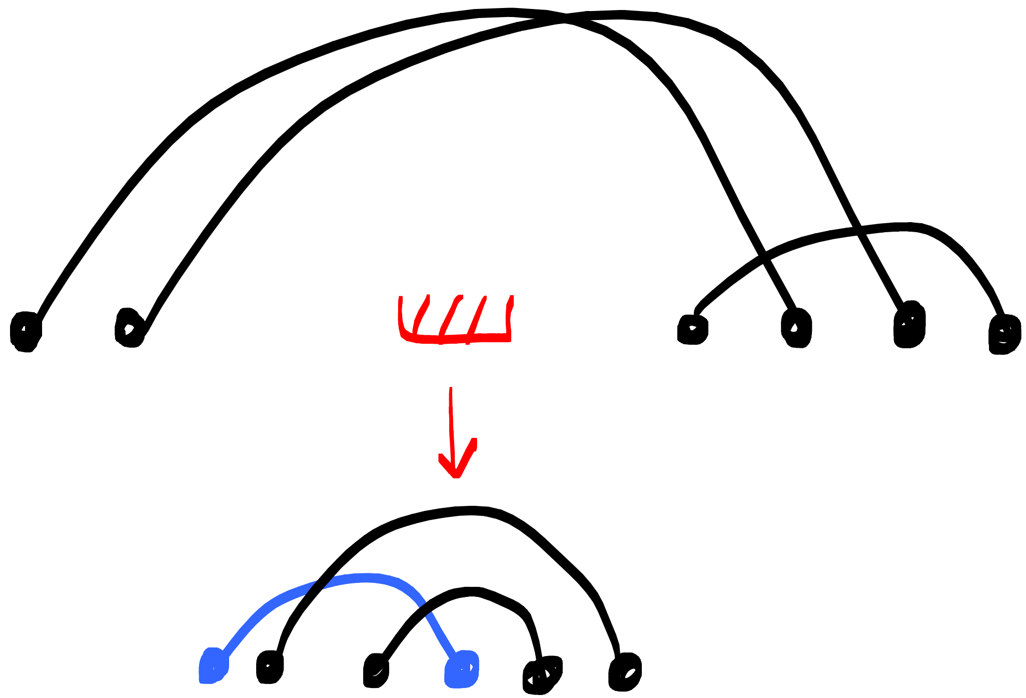

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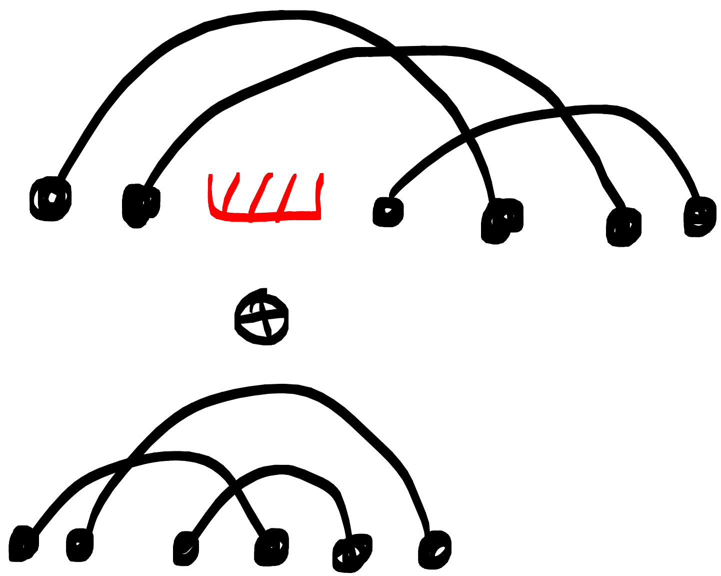

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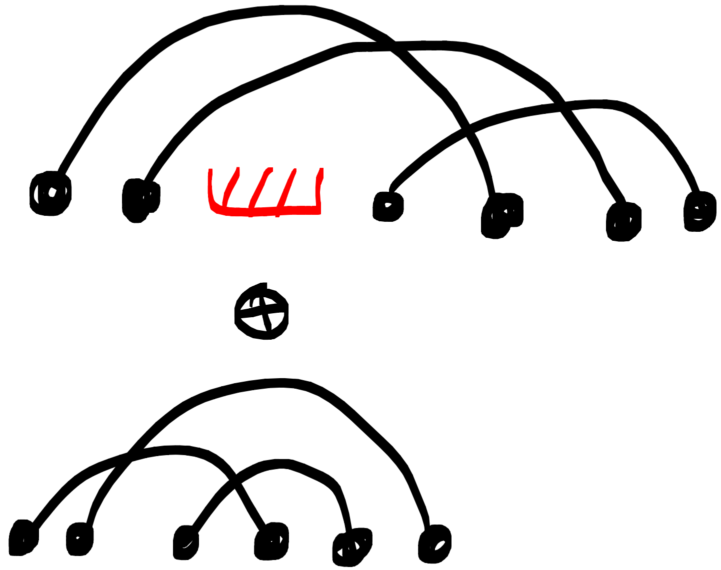
Proof:

If k chords,

then $(2k-1)$ possible insertions



c_n VS CATALAN

CONNECTED DIAGRAMS

$$c_n = (n-1) \sum_{k=1}^{n-1} c_k c_{n-k}$$

CATALAN

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c_n VS CATALAN

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$$c_n = (n-1)\sum_{k=1}^{n-1} c_k c_{n-k}$$

$$c_n \geqslant (n-1) \times c_1 \times c_{n-1}$$

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Consequence: - Ordinary Generating Functions :-(
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c_n vs CATALAN

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Consequence: - Ordinary Generating Functions are not adapted. :(

- No simple equation for the Exponential Generating Functions :(

ASYMPTOTIC BEHAVIOUR

[Stein-Everett]

$$c_n \sim \frac{1}{e} \times (2n-1)!!$$

Consequence : $\mathbb{P}(\text{diagram is connected}) \to \frac{1}{e}$

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- number of connected components $\sim$ Poisson(1)
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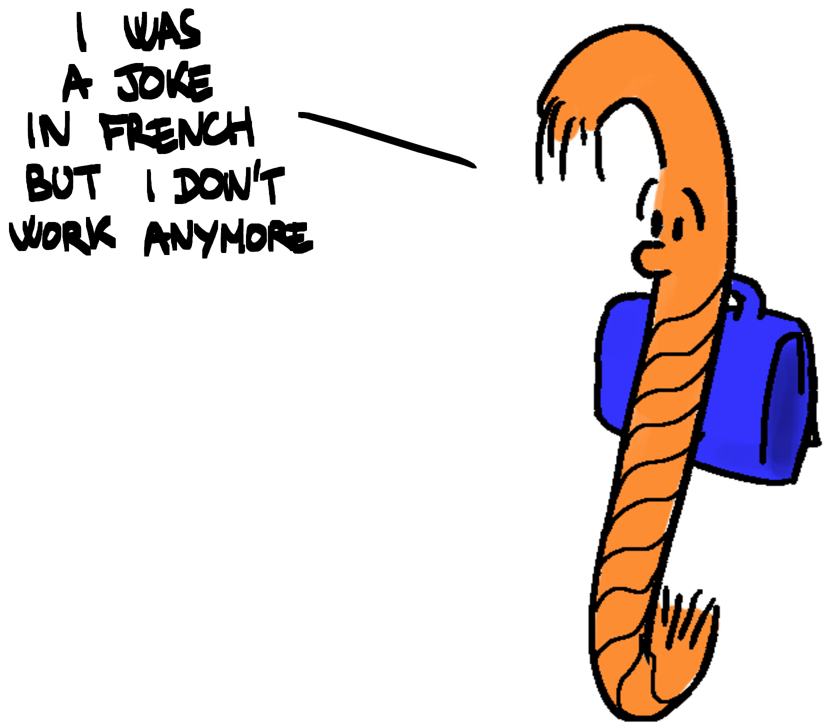
- number of connected components $\sim$ Poisson(1)
- n – size of the largest component $\sim$ Poisson(1)

Our humble contribution: $\frac{c_{n-1}}{c_n} = \frac{1}{2n} + \frac{1}{4n^2} - \frac{1}{4n^3} + o\left(\frac{1}{n^3}\right)$

STATISTICS ON TERMINAL CHORDS



LEADING-LOG TERMS

$$G(x,L) = 1 - \sum_{i \geq 1} \sum_{\substack{C \text{ connected} \\ \text{chord diagram} \\ \text{with terminal chords} \\ \text{in position } t_1 < t_2 < \ldots < t_k \\ \text{such that } t_1 \geq i}} \frac{L^i}{i!} x^{|C|} \beta_0^{|C|-k} \beta_{t_1-i} \beta_{t_2-t_1} \beta_{t_3-t_2} \cdots \beta_{t_k-t_{k-1}}$$

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Coefficient of $(Lx)^i x^{|C|-i}$ for i close to $|C|$?

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Coefficient of $(Lx)^i x^{|C|-i}$ for i close to $|C|$?

- $|C| = i$: leading-log expansion [Krüger-Kreimer]
- $|C| = i+1$: next-to leading-log expansion
- $|C| = i+\ell$: next-to$^{\ell}$ leading-log expansion

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Coefficient of $(Lx)^i x^{|C|-i}$ for i close to $|C|$?

- $|C| = i$: leading-log expansion [Krüger-Kreimer]

$\Leftrightarrow t_1 = |C|$

$\Leftrightarrow$ There is only one terminal chord.

ONLY ONE TERMINAL CHORD

number of connected diagrams with n chords and only one terminal chord

$= ?$

ONLY ONE TERMINAL CHORD

number of connected $\underbrace{\text{diagrams}}_{\text{and only one terminal chord}}$ with n chords

$$= (2n-3)!!$$

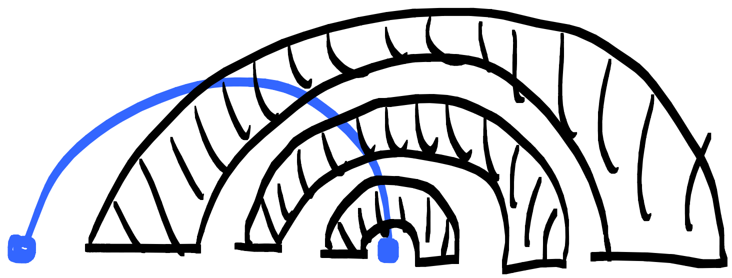
ONLY ONE TERMINAL CHORD

number of connected diagrams with n chords and only one terminal chord

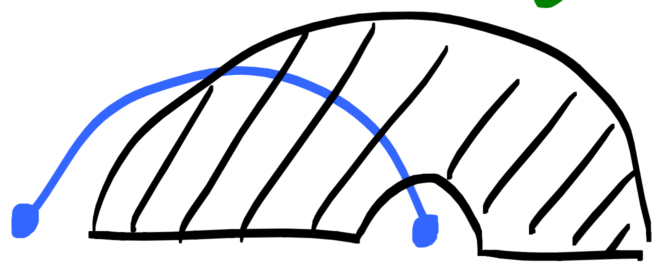
$$= (2n-3)!!$$

Proof:

One piece of size $n-1$



impossible



$2n-3$ possible locations

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number of connected diagrams with n chords and only one terminal chord

$$= (2n-3)!!$$

Proof:

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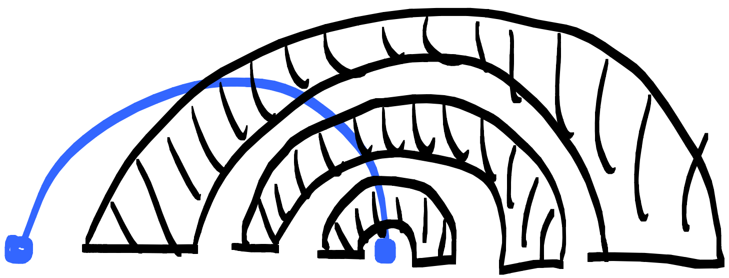

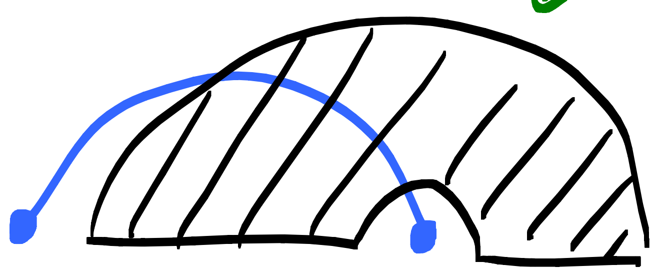

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$2n-3$ possible locations

Cor: n^{th} coeff of the leading-log expansion

$$= \frac{(2n-3)!!}{n!} f_0^{\,n}$$

NEXT-TO$^{\ell}$ LEADING-LOG TERMS

→ "Similar" recursions exist for the diagrams such that $t_1 \geqslant |C| - \ell$

→ Analytic combinatorics techniques works here.

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Theorem : For $\ell \geq 0$,

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$$\sim \frac{1}{\sqrt{\pi}\, 2^{\ell+1}\, \ell!} \times \frac{\ln(n)^{\ell}}{n^{\frac{3}{2}}}\, n!$$

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But how about $\beta_0^{|C|-k} \beta_{t_1-i} \beta_{t_2-t_1} \beta_{t_3-t_2} \cdots \beta_{t_k-t_{k-1}}$?

THE LAST ℓ CHORDS ARE TERMINAL

→ "Similar" recursions exist for the diagrams such that the last ℓ chords are terminal

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Here $f_0^{|C|-k} f_{t_1-i} f_{t_2-t_1} f_{t_3-t_2} \cdots f_{t_k-t_{k-1}} = f_0^{n-\ell+1} f_1^{\ell-1}$

NEXT-TO$^{\ell}$ LEADING-LOG TERMS

Diagrams such that the last ℓ chords are terminal are dominant among the diagrams such that $t_1 \geq |C| - \ell$.

Corollary: For $\ell \geq 0$,

n^{th} coeff of the next-to$^{\ell}$ leading-log expansion

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Only f_0 and f_1 matter!

NUMBER OF TERMINAL CHORDS

Average number of terminal chords ?

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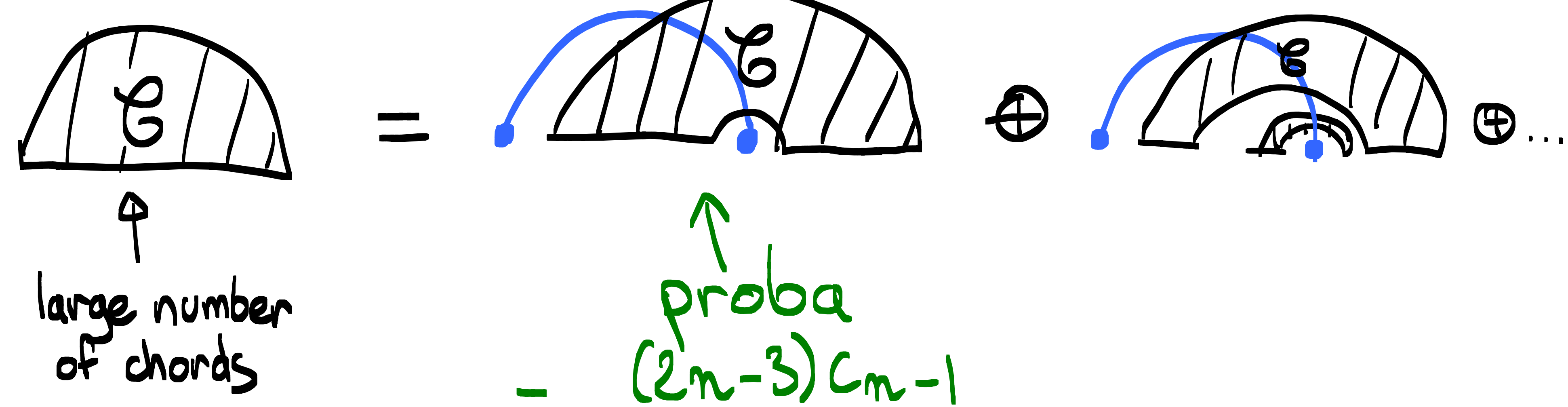
Idea:



$$= \frac{(2n-3)\,c_{n-1}}{c_n}$$

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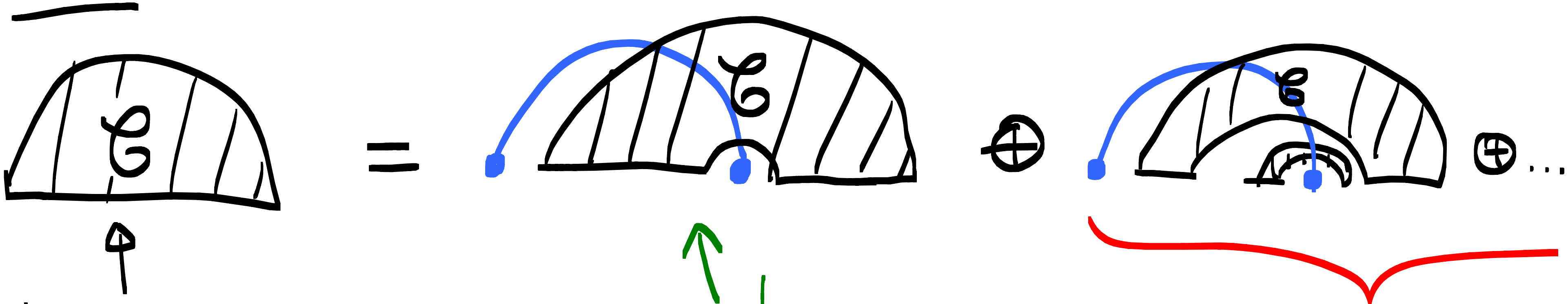

large number of chords

proba

$$= \underbrace{\frac{(2n-3)c_{n-1}}{c_n}}_{\to 1}$$

proba $\to 0$

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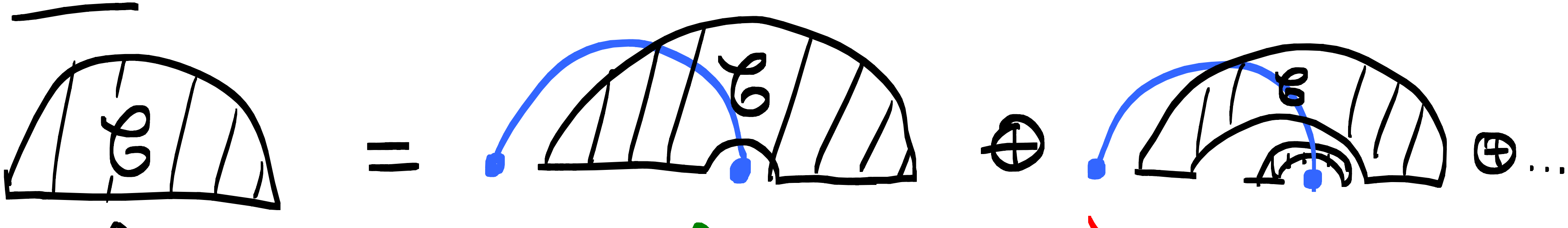

large number of chords

proba

$$= \underbrace{\frac{(2n-3)C_{n-1}}{C_n}}_{\to 1}$$

proba $\to 0$

Interesting but not sufficient...

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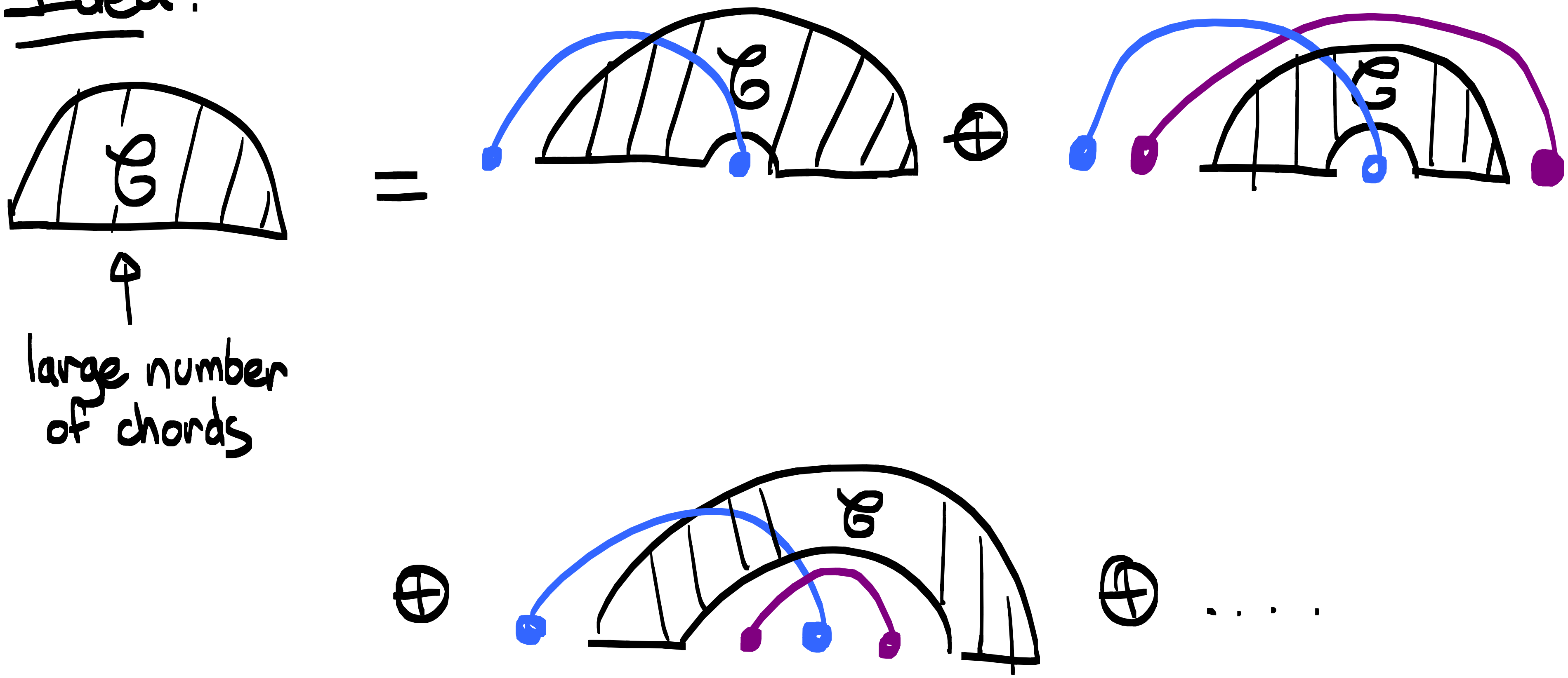

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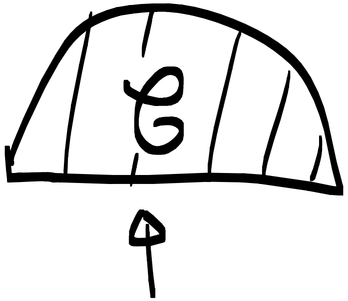
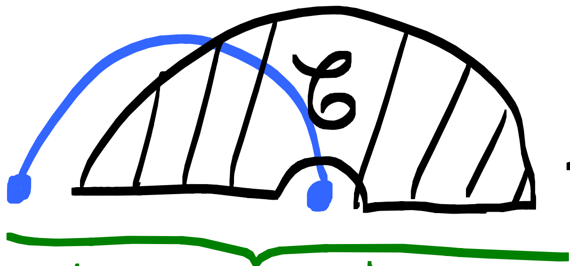


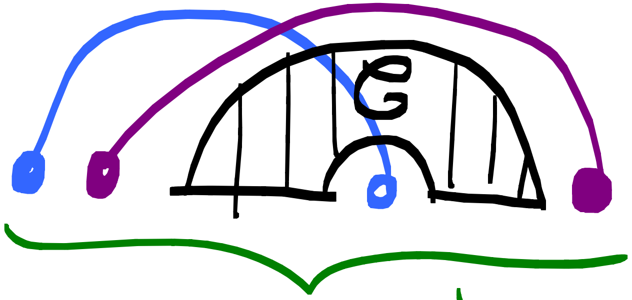
large number of chords

$=$



$$\text{proba} = \frac{(2n-3)\,c_{n-1}}{c_n}$$

$\oplus$



$$\text{proba} = \frac{(2n-5)\,c_{n-2}}{c_n}$$

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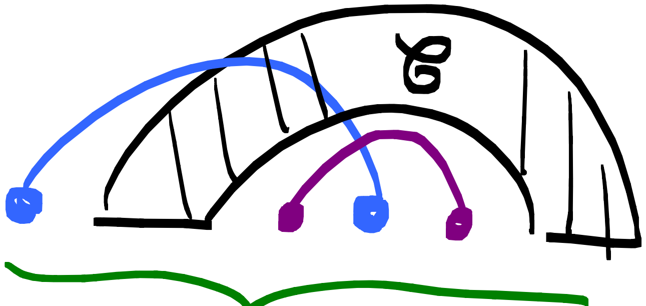

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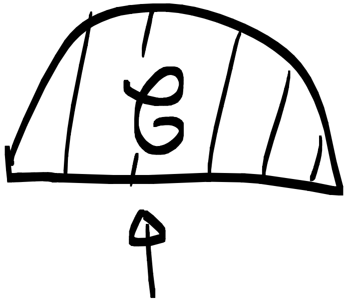

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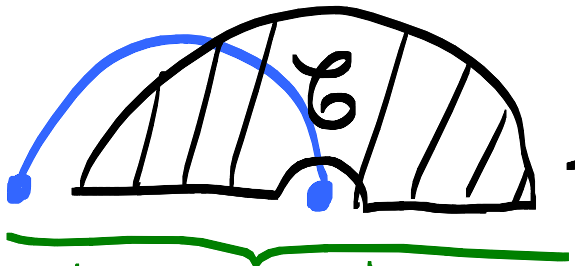
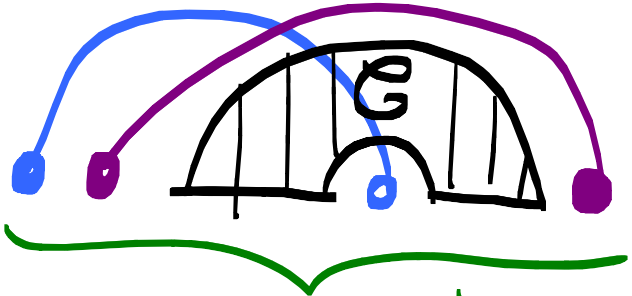


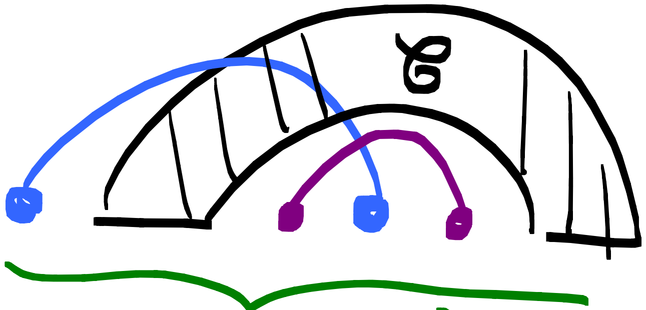
$$\text{proba} = \frac{(2n-3)\,c_{n-1}}{c_n} = 1 - \frac{1}{n} + o\left(\frac{1}{n}\right)$$

$\oplus$



$$\text{proba} = \frac{(2n-5)\,c_{n-2}}{c_n} \sim \frac{1}{2n}$$

$\oplus$



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$$\underbrace{\;\dots\;}_{= o\left(\frac{1}{n}\right)}$$

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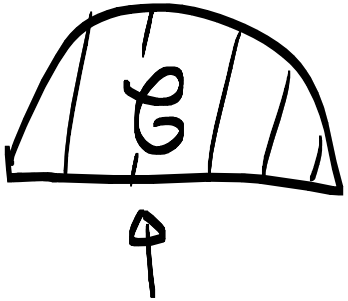
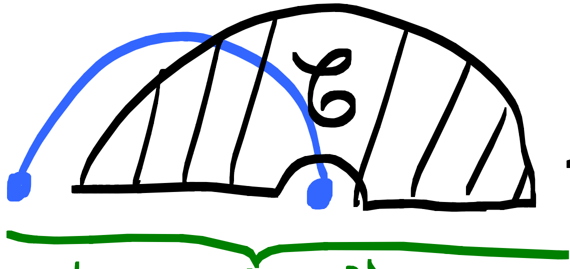


large number of chords

=



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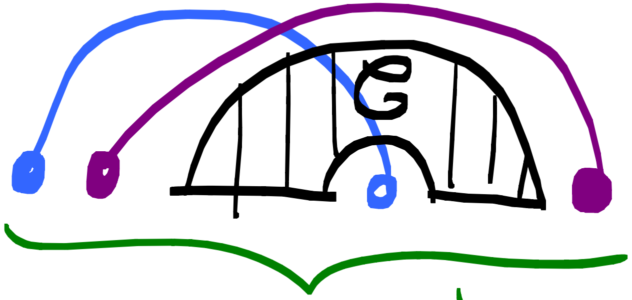

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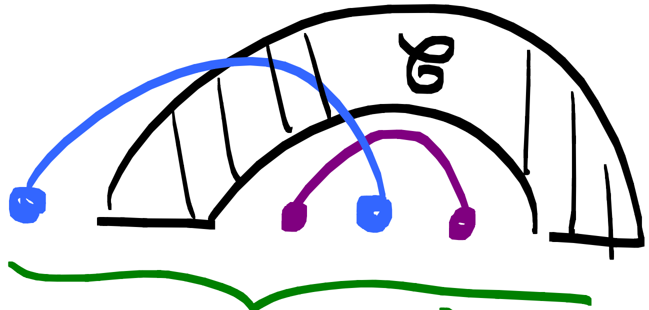


$$\text{proba} = \frac{(2n-5)\, c_{n-2}}{c_n} \sim \frac{1}{2n}$$

$\oplus$ ...

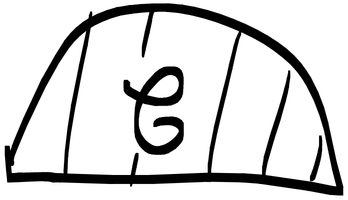
$$\cancel{= o\left(\frac{1}{n}\right)}$$

Let's forget that

NUMBER OF TERMINAL CHORDS

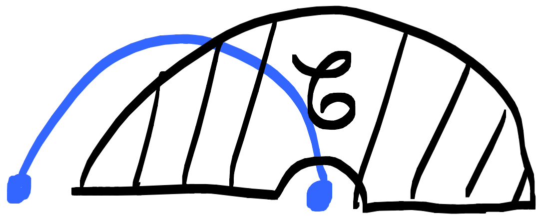
Set $p_{n,k} = \left(1-\frac{1}{n}\right) p_{n-1,k} + \frac{1}{n}\, p_{n-2,k-1}$

Idea:



large number of chords

$=$



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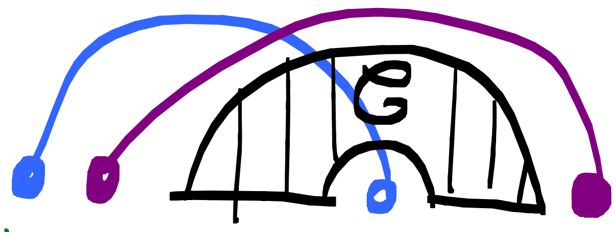

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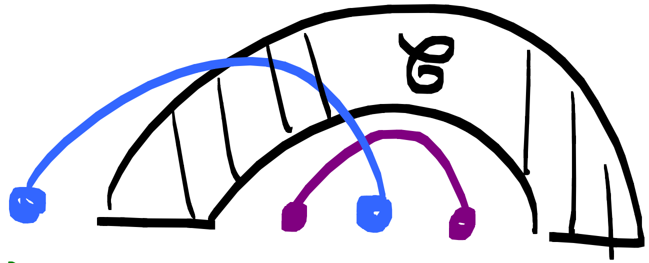

$\text{proba} = \frac{(2n-5)\,c_{n-2}}{c_n} \sim \frac{1}{2n}$

$\oplus$ $\cancel{\underbrace{\dots}_{= o\left(\frac{1}{n}\right)}}$

Let's forget that

NUMBER OF TERMINAL CHORDS

Set $p_{n,k} = \left(1 - \frac{1}{n}\right) p_{n-1,k} + \frac{1}{n} p_{n-2,k-1}$

Fact 1 : Let X_n be the random variable such that $\mathbb{P}(X_n = k) = p_{n,k}$

$$X_n \longrightarrow \text{Gaussian law.}$$

Fact 2 : The number of terminal chords "$\sim$" X_n

NUMBER OF TERMINAL CHORDS

Theorem : The number of terminal chords in a random connected diagram of size n asymptotically obeys to a Gaussian limit law of mean and variance $\sim \ln(n)$.

NUMBER OF CONSECUTIVE TERMINAL CHORDS

If the terminal chords are in position $t_1 < \ldots < t_k$, how many j's satisfy $t_j - t_{j-1} = 1$?

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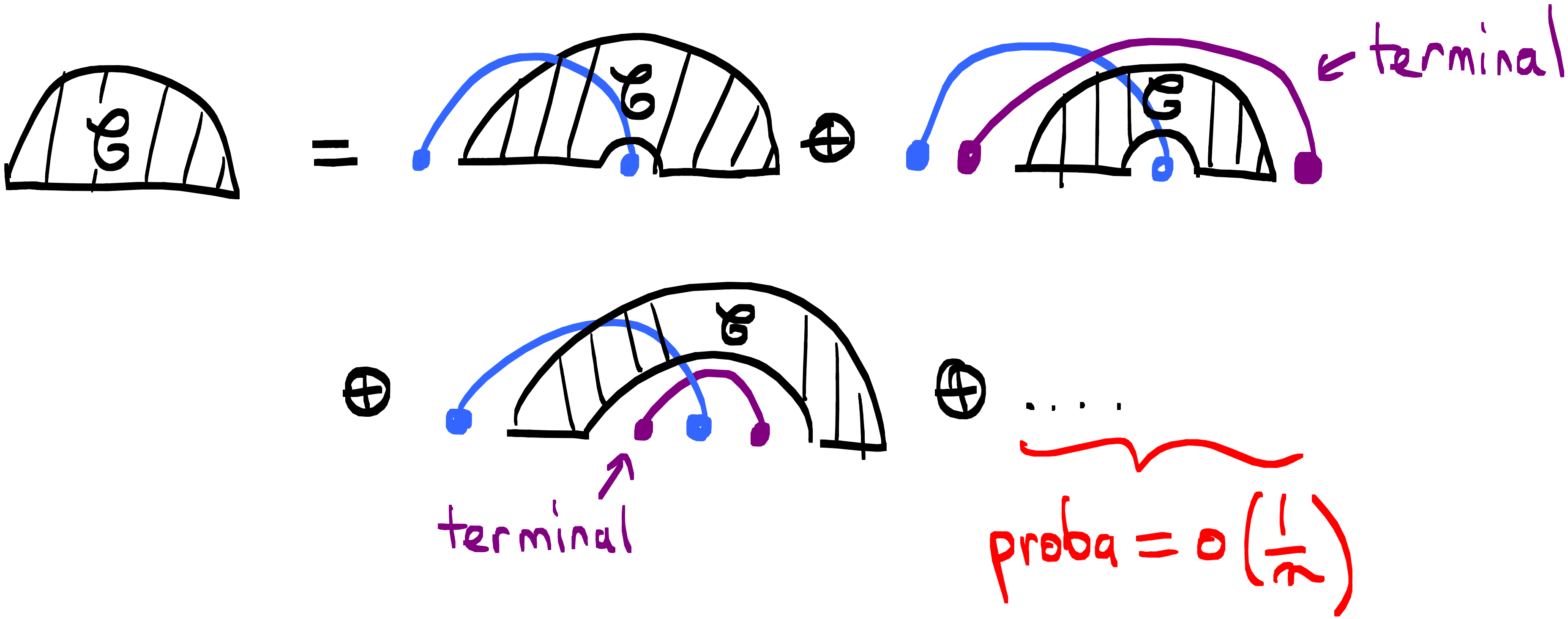

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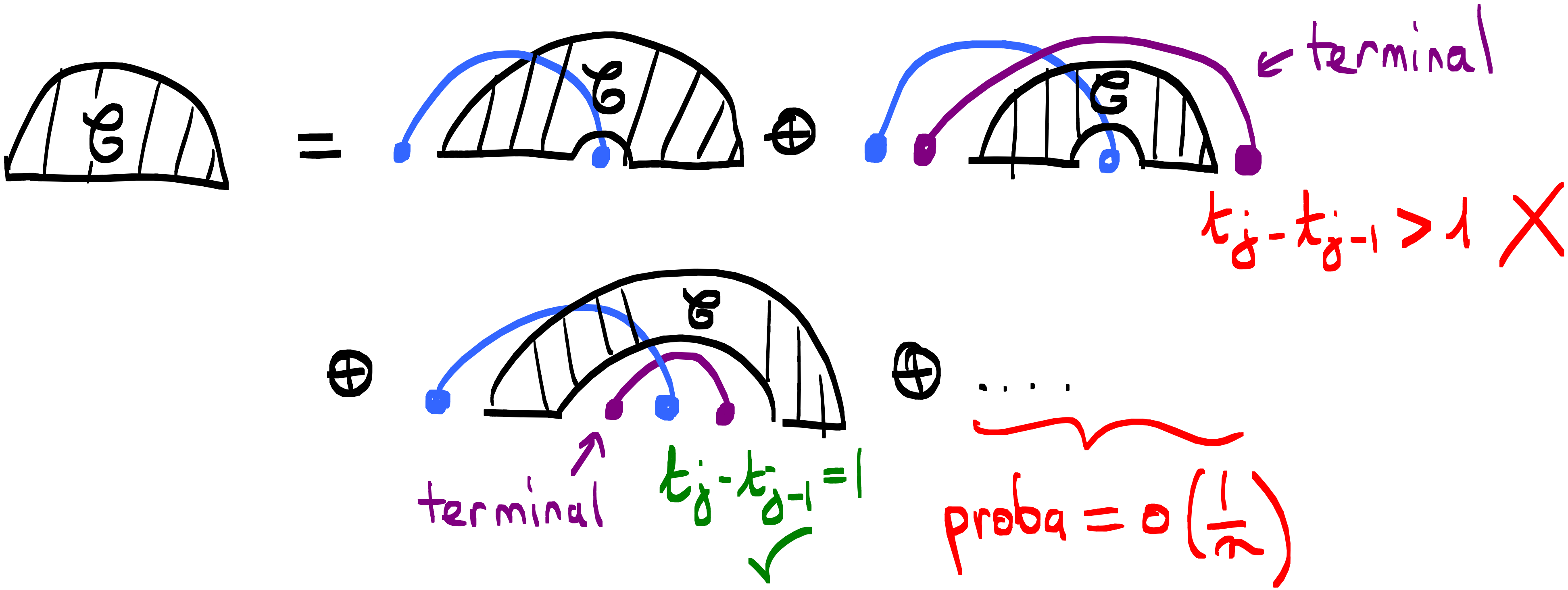

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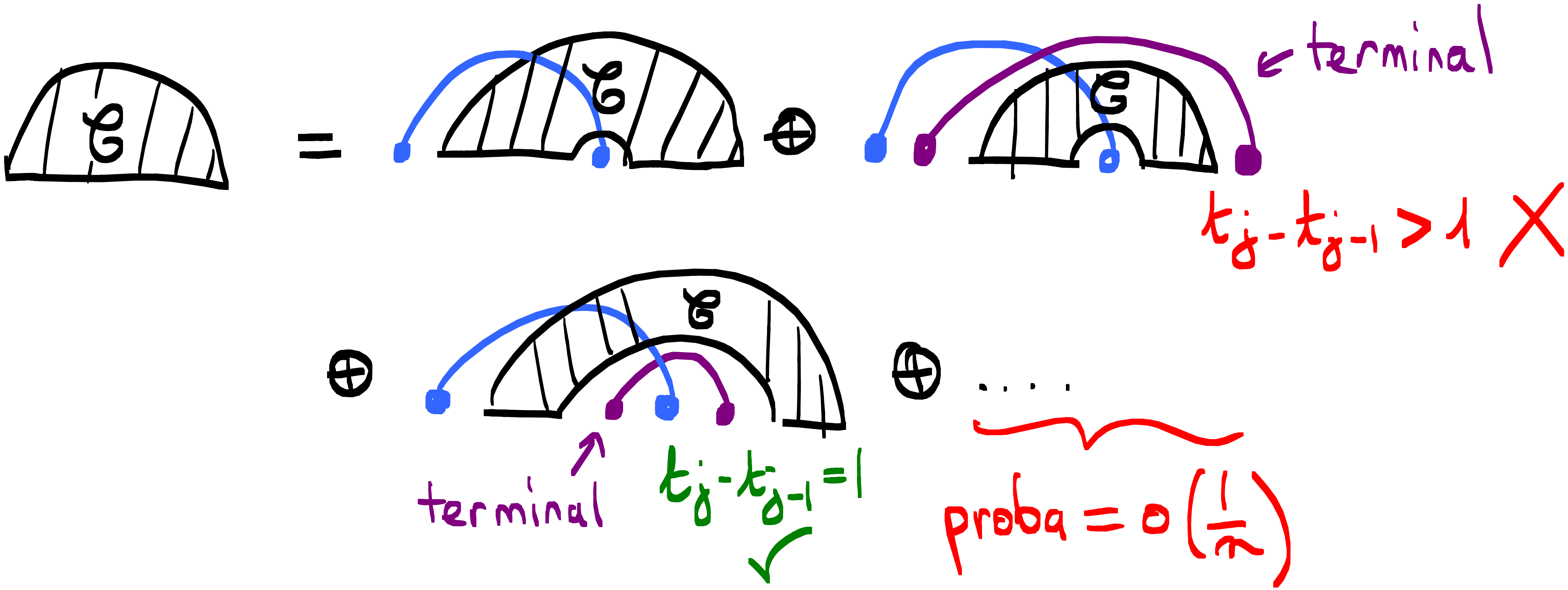

Theorem: Number of consecutive terminal chords $\longrightarrow$ Gaussian law of mean and variance $\sim \frac{\ln n}{2}$

NUMBER OF CONSECUTIVE TERMINAL CHORDS

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On average,

$$f_0^{|C|-k} f_{t_1 - i} f_{t_2 - t_1} f_{t_3 - t_2} \cdots f_{t_k - t_{k-1}} \sim f_0^{n - \ln n} f_{t_1 - i} f_1^{\frac{\ln n}{2}} \dots$$

→ confirms the importance of f_0 and f_1

Theorem: Number of consecutive terminal chords → Gaussian law of mean and variance $\sim \frac{\ln n}{2}$

POSITION OF THE FIRST TERMINAL CHORD.

t_1 = random variable returning the position of the 1^{st} terminal chord.

Theorem: $\mathbb{E}(t_1) \sim \frac{2}{3} n$

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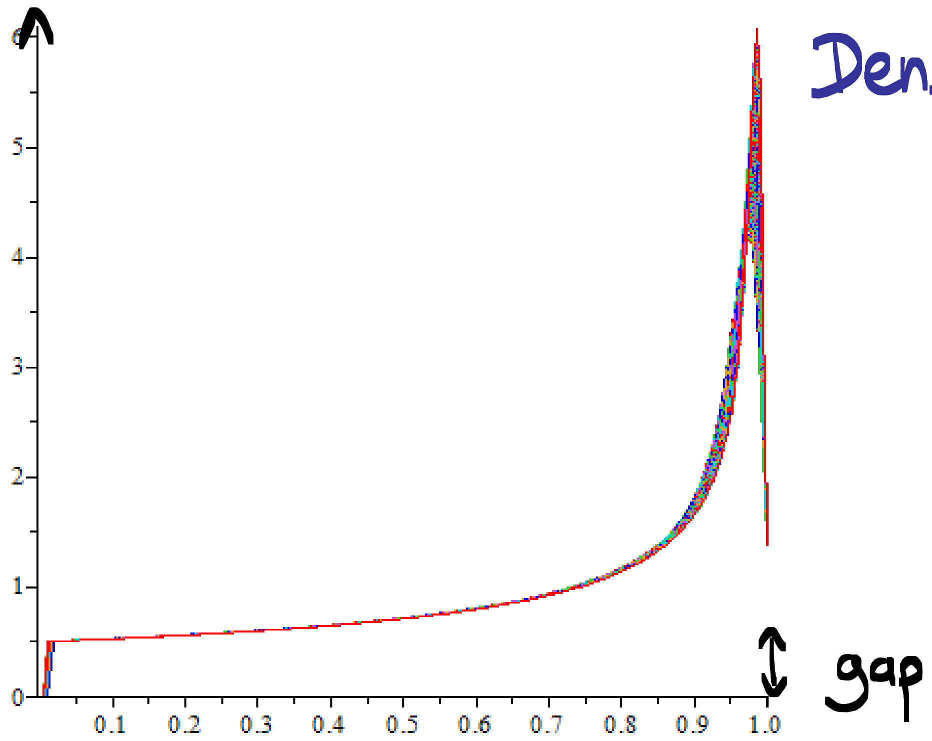
<u>Limit law</u>?



CONCLUSION

→ Recovers the results of Krüger and Kreimer

+ automaticity of the method
+ asymptotic behaviour

→ New combinatorial approach

→ Extension to Hihn-Yeats's results?

THANK YOU!