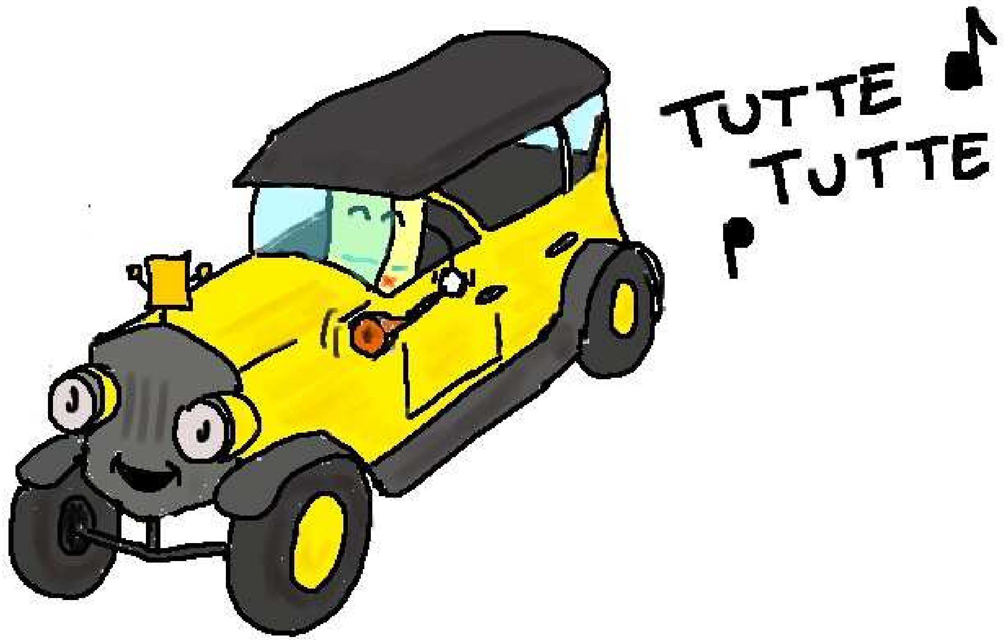
A GENERAL NOTION OF ACTIVITY FOR THE TUTTE POLYNOMIAL

COURTIEL Julien (PIMS/Simon Fraser University)

Workshop on the Tutte polynomial 2015



SEVERAL NOTIONS OF ACTIVITY

For a connected graph G,

Tutte polynomial $$T_G(x,y) = \sum_{\substack{T \text{ spanning} \\ \text{tree of } G}} x^{i(T)} \, y^{e(T)}$$

$i(T)$ = number of internal active edges
$e(T)$ = number of external active edges.

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Definition of active edge?

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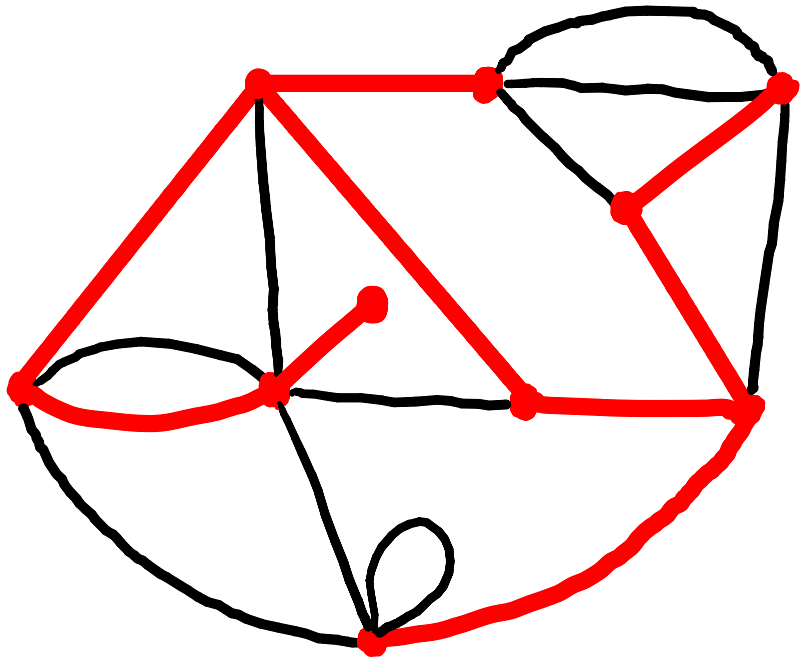
Definition of active edge?

1954	1996	2006	
Tutte	Gessel-Sagan	Bernardi	?
edge ordering	Depth-First Search	Graph embedding	

(just for external edges)

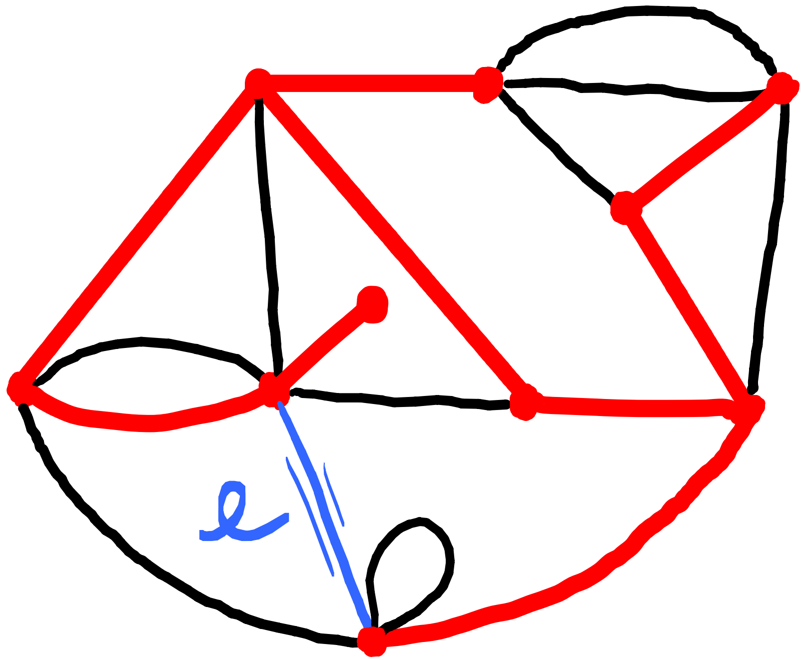
FUNDAMENTAL CYCLE / COCYCLE

Let T be a spanning tree and e an edge,



FUNDAMENTAL CYCLE / COCYCLE

Let T be a spanning tree and e an **external** edge,



fundamental cycle = unique cycle in $T \cup \{e\}$

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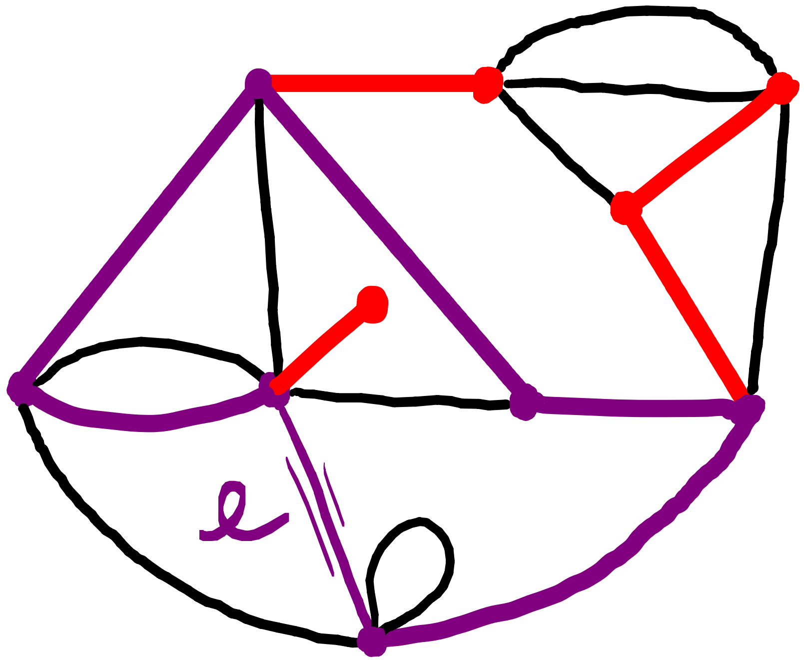

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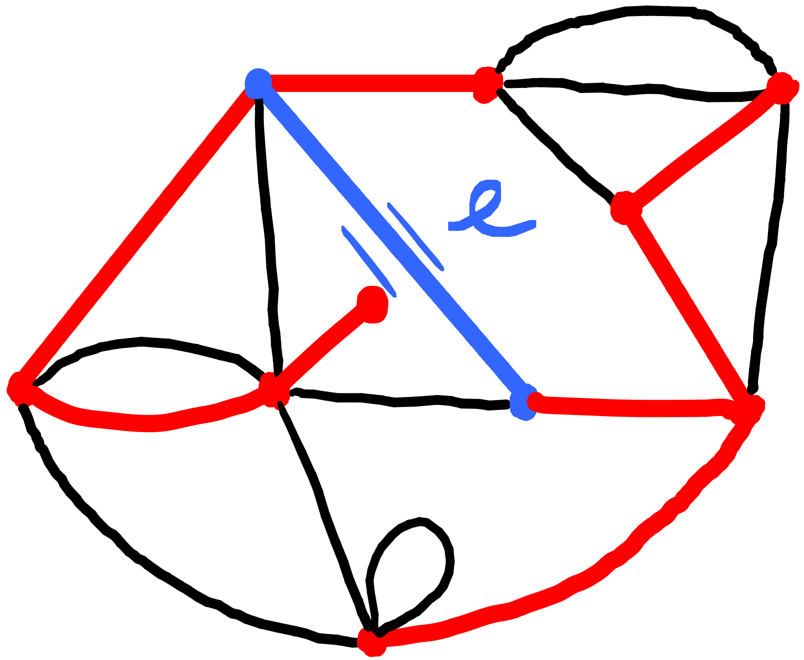

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FUNDAMENTAL CYCLE / COCYCLE

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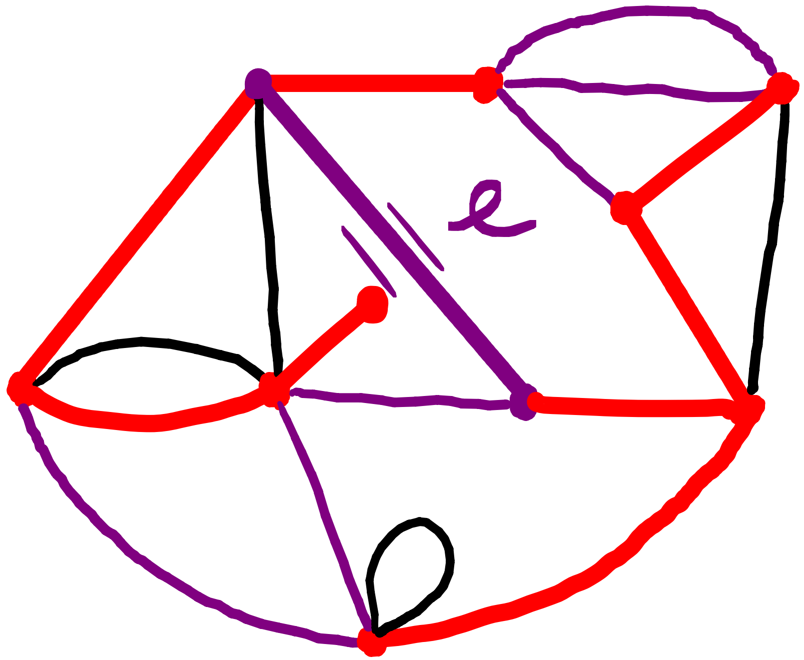

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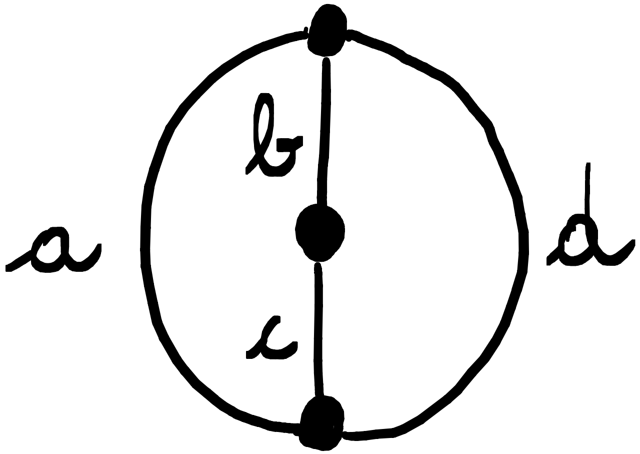
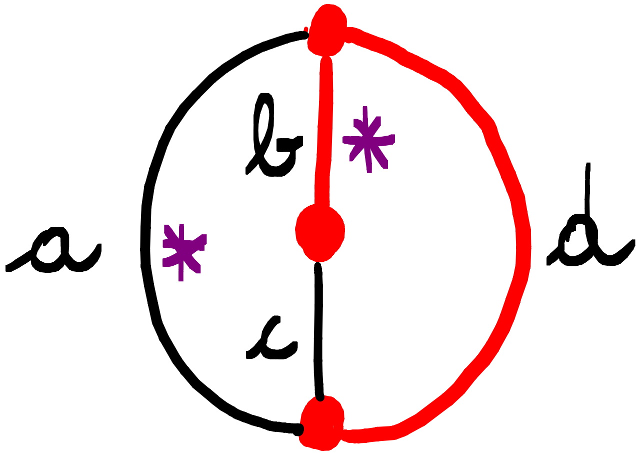
TUTTE'S ACTIVITY



We label and order the edges:

$$a < b < c < d$$

TUTTE'S ACTIVITY



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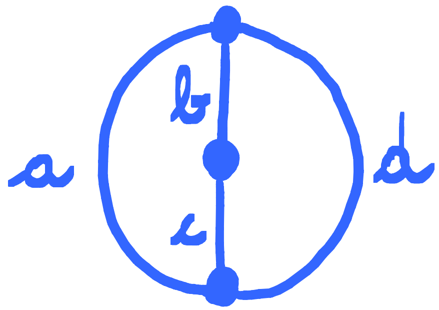
Active edge = minimal edge inside its fundamental cycle / cocycle

TUTTE'S ACTIVITY



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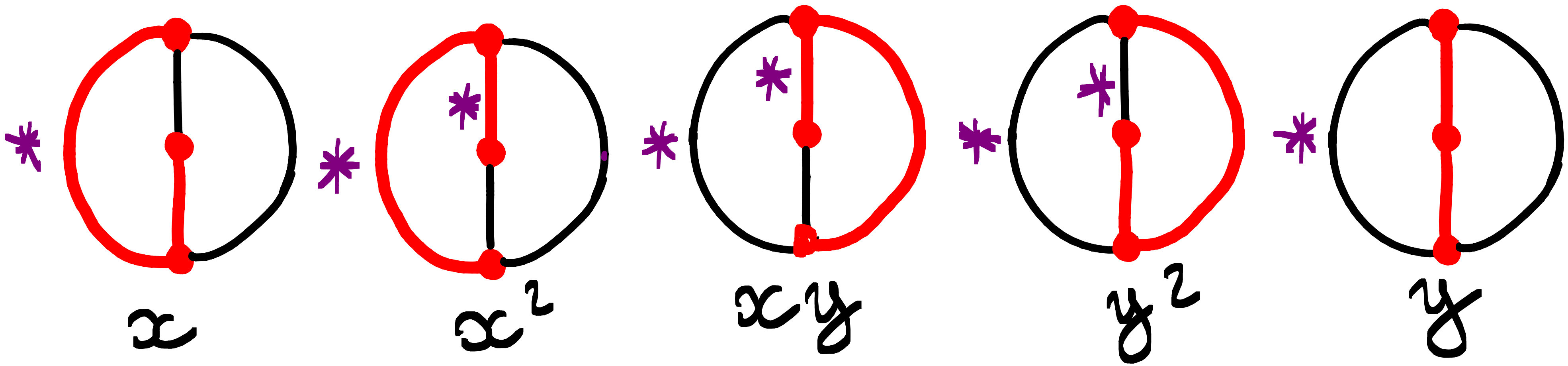
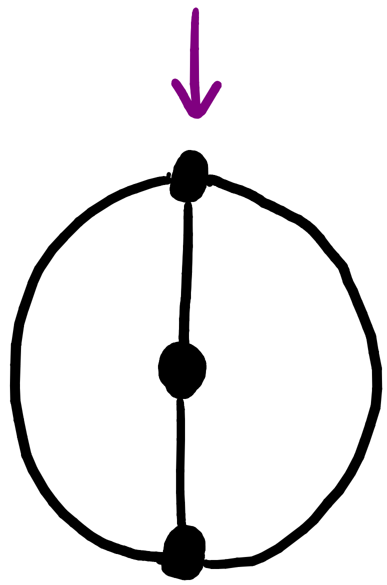


$$T_G(x,y) = x^2 + x + xy + y + y^2$$

BERNARDI'S ACTIVITY: TOUR OF THE TREE

We embed and root the graph:



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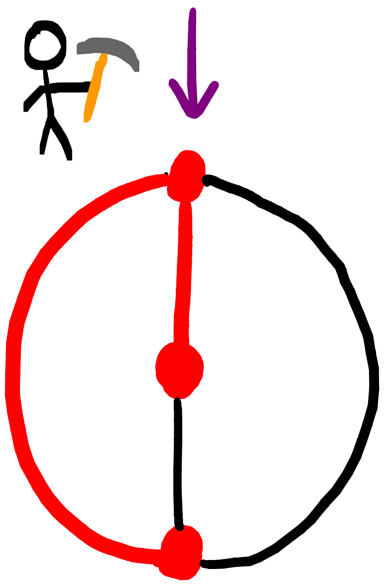
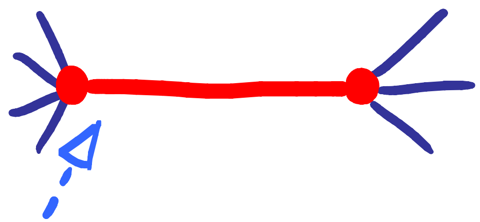
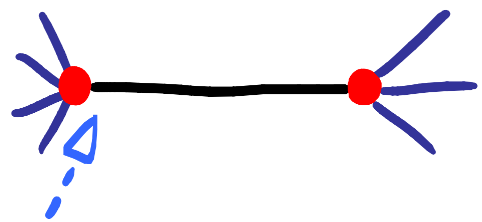


Rules:

inside the tree



outside the tree



BERNARDI'S ACTIVITY: TOUR OF THE TREE

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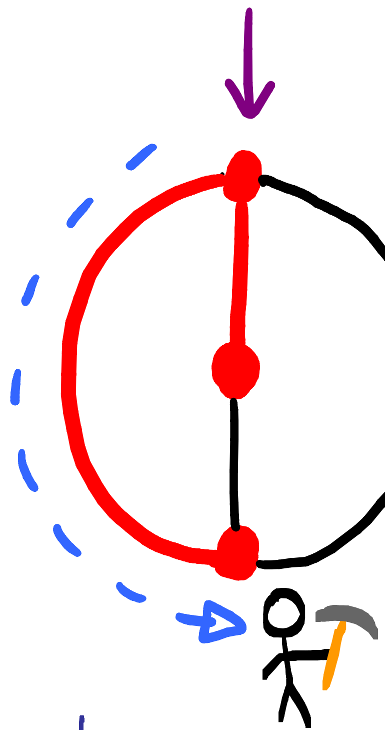
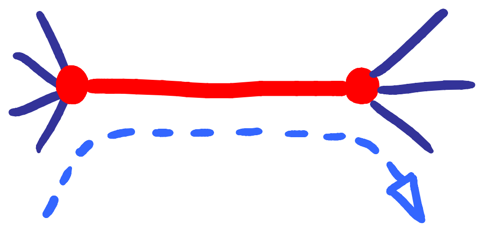


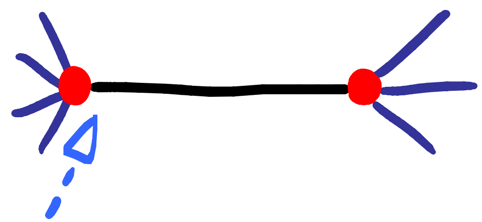
Rules:

inside the tree



We walk along.

outside the tree



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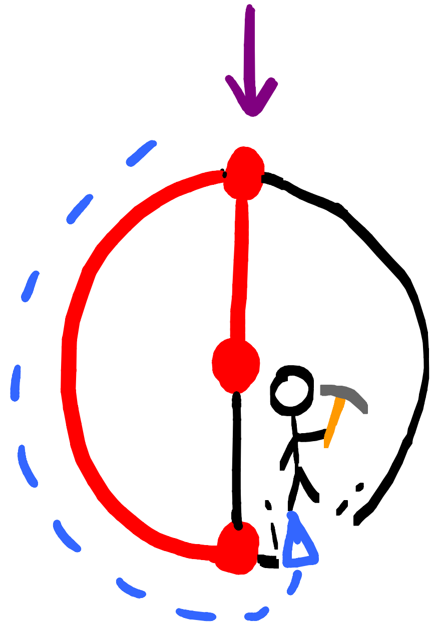

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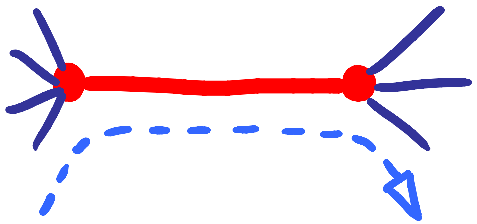
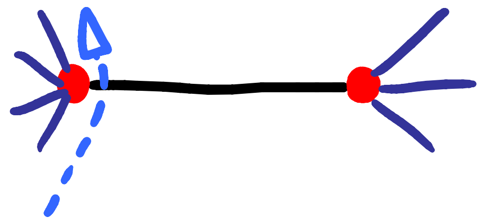


We walk along.

outside the tree



We cross.

BERNARDI'S ACTIVITY: TOUR OF THE TREE

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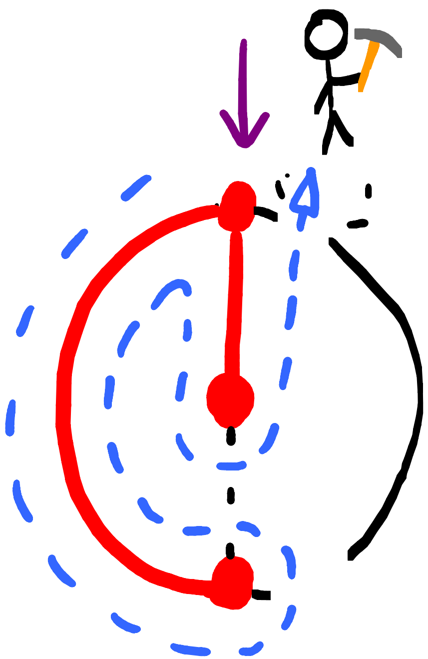

Rules:

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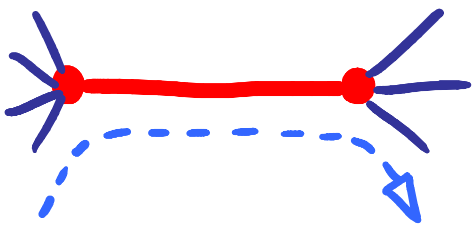
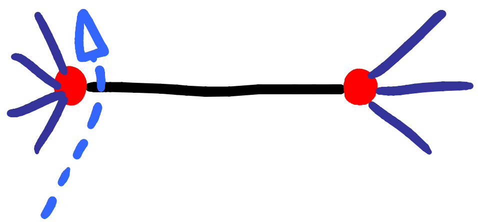


We walk along.

outside the tree



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BERNARDI'S ACTIVITY: TOUR OF THE TREE

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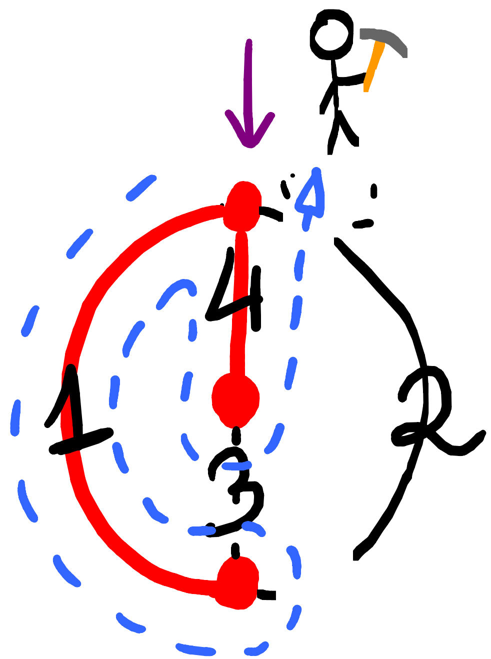

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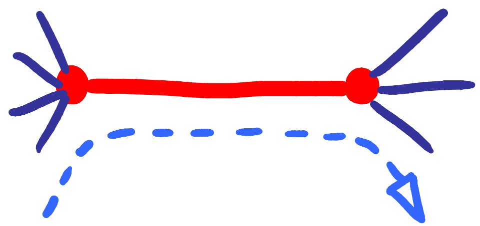
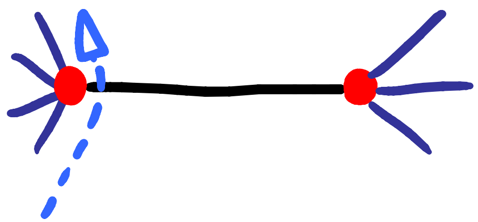


We walk along.

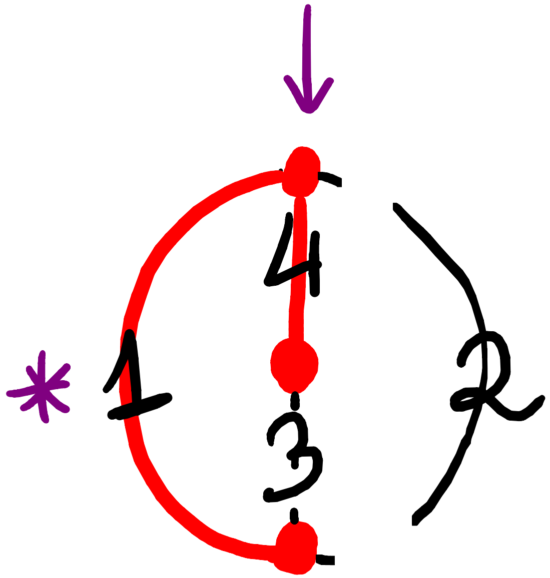
outside the tree



We cross.

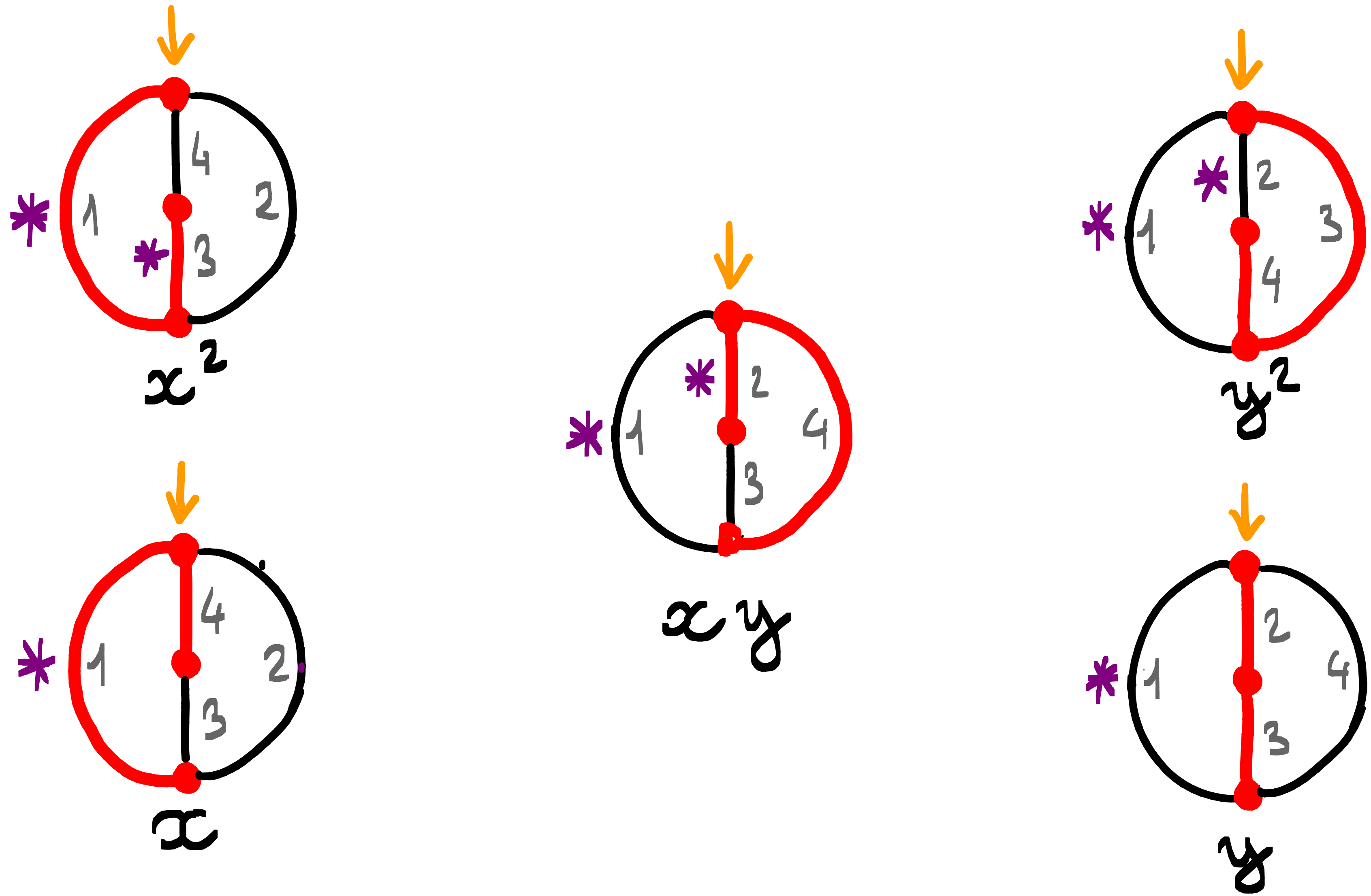
BERNARDI'S ACTIVITY: DEFINITION

We embed and root the graph:



Active edge = minimal edge inside its fundamental cycle/cocycle (for the first visit order)

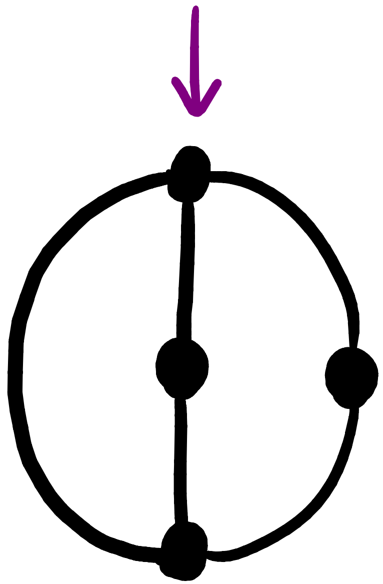
Bernardi's Activity: Definition



$$T_G(x,y) = x^2 + x + xy + y + y^2$$

NEW (?) ACTIVITY

We embed and root the graph, again.



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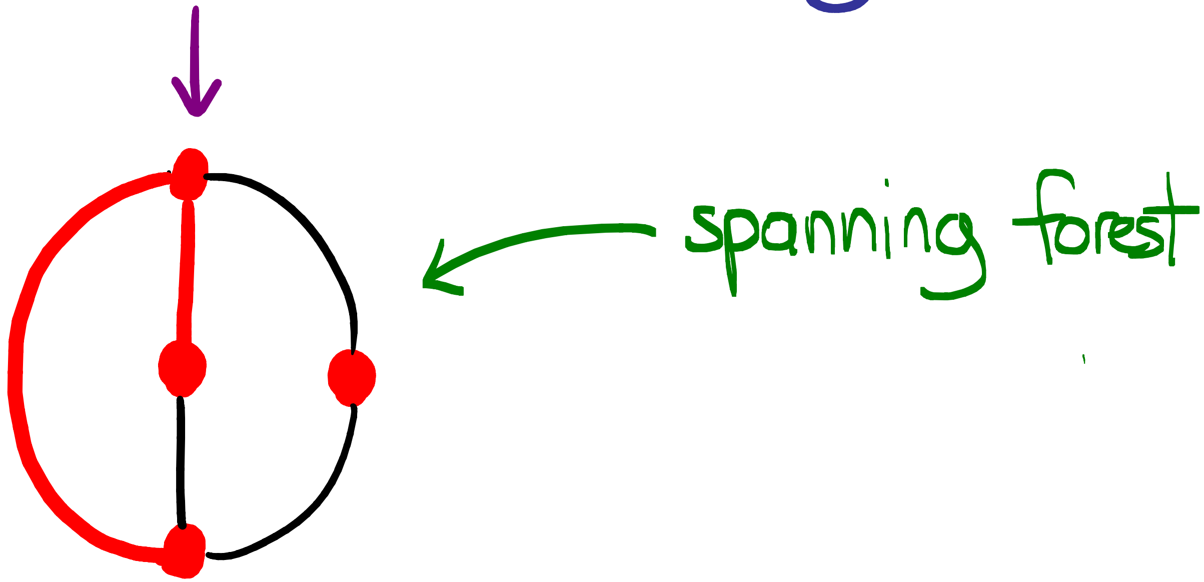
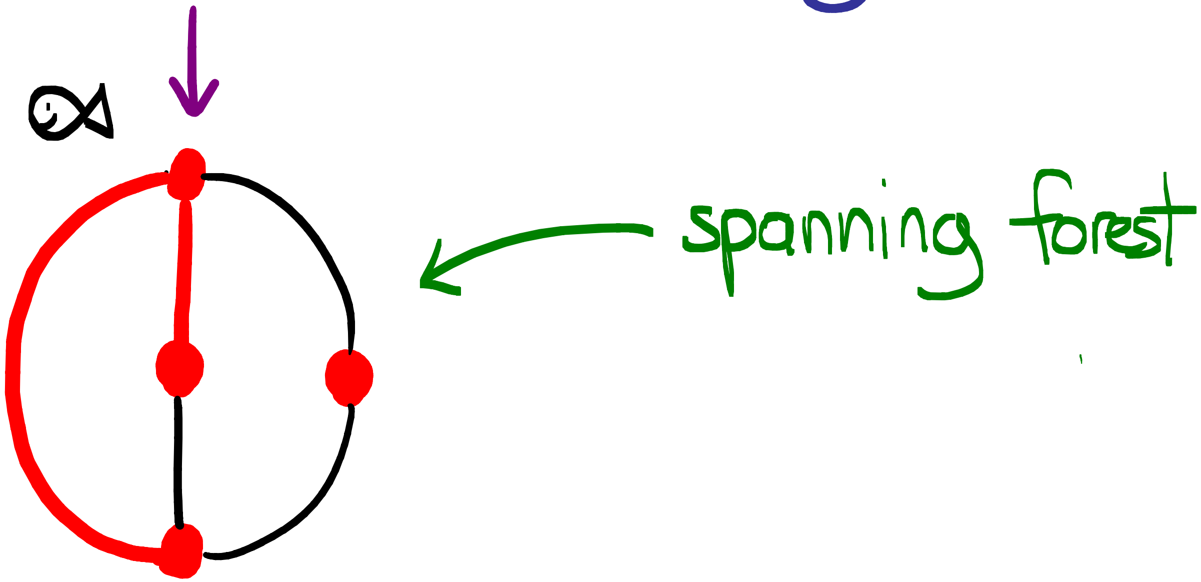


GOLDFISH ACTIVITY

We embed and root the graph, again.



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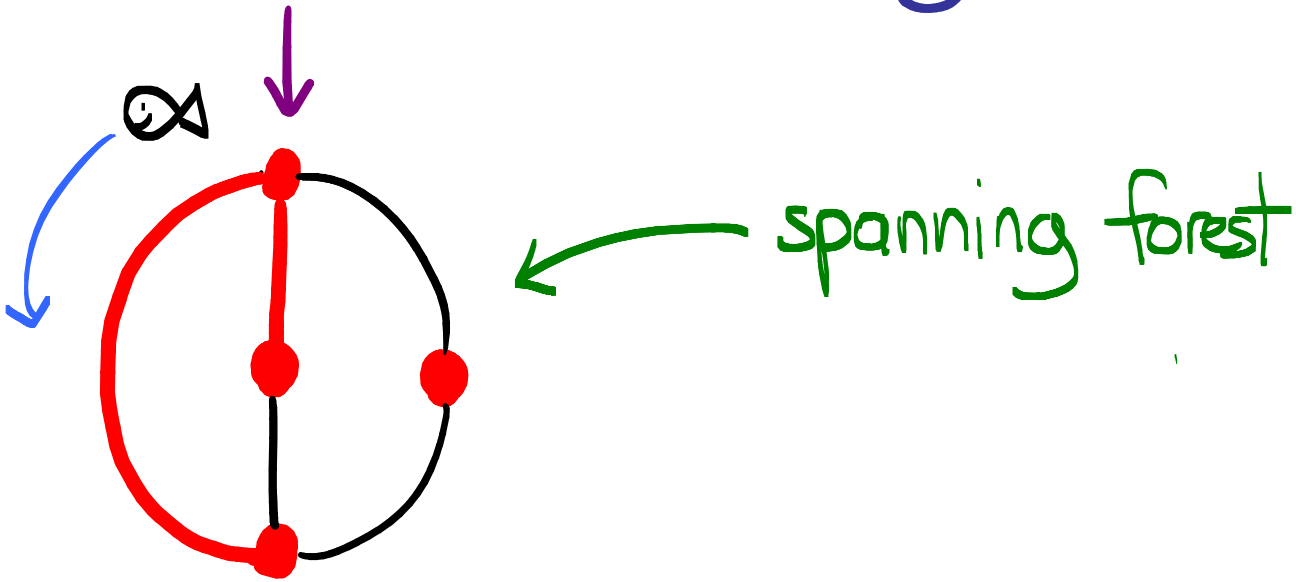
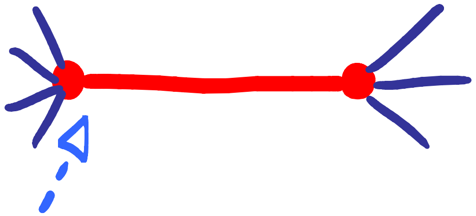
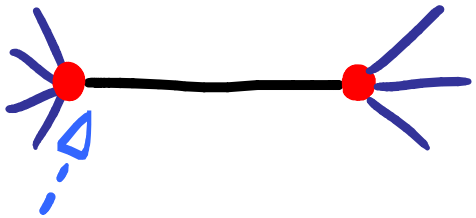


Rules:

inside the forest
or isthmus



outside the forest
and not isthmus



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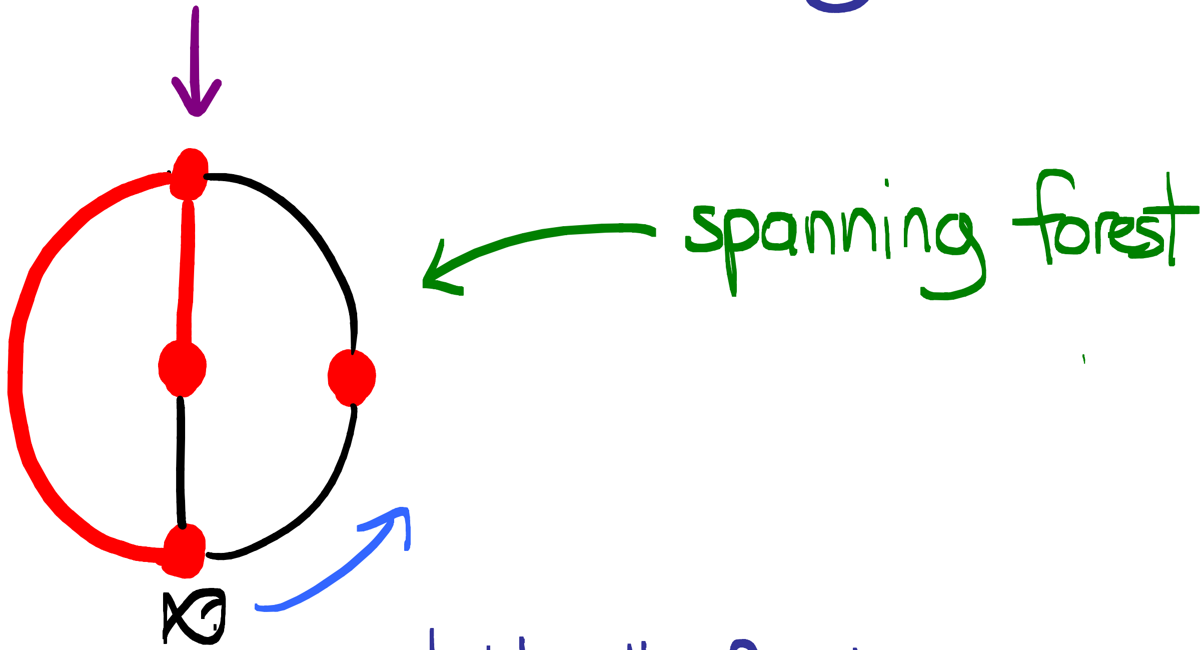

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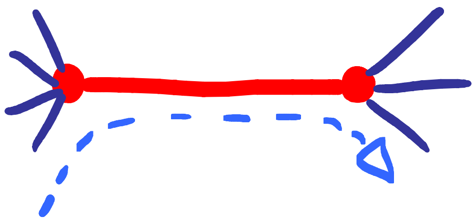
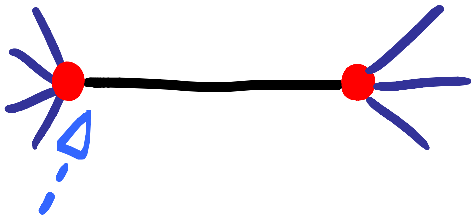


The fish swims along

outside the forest and not isthmus



GOLDFISH ACTIVITY

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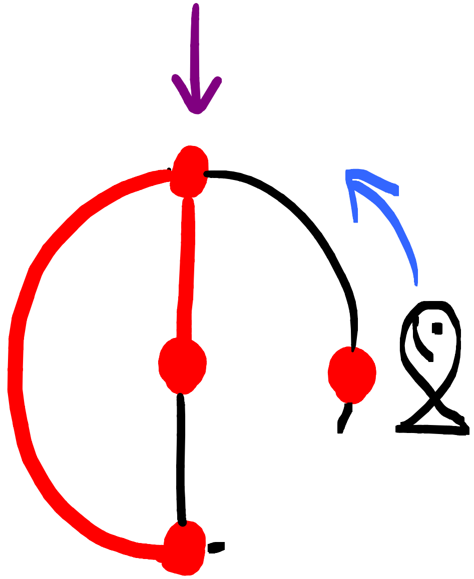
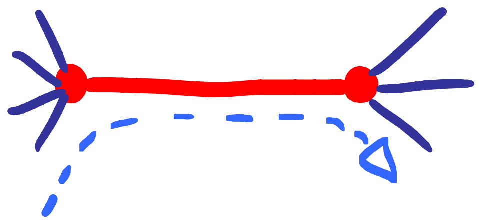


spanning forest

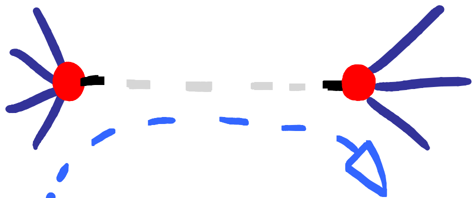
Rules:

inside the forest or isthmus



The fish swims along

outside the forest and not isthmus



The fish eats the edge while swimming along

GOLDFISH ACTIVITY

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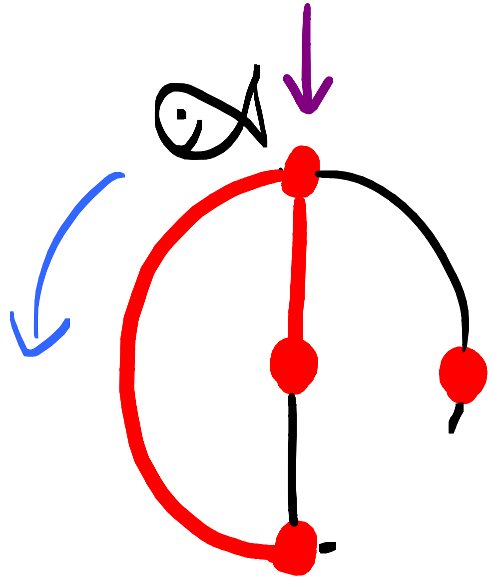

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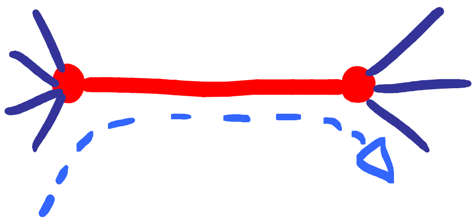

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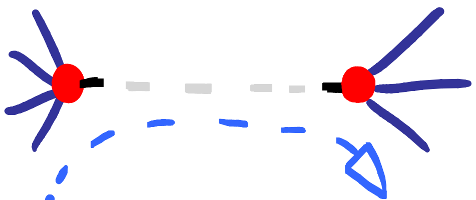

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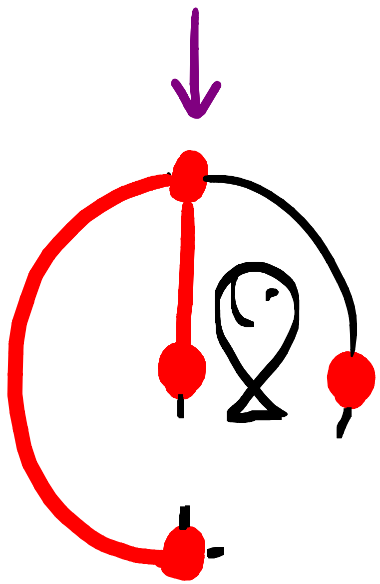

spanning forest

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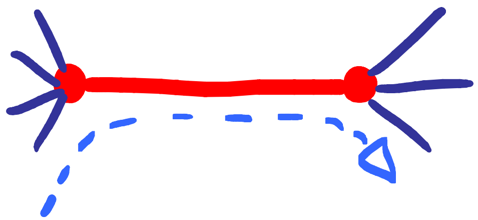
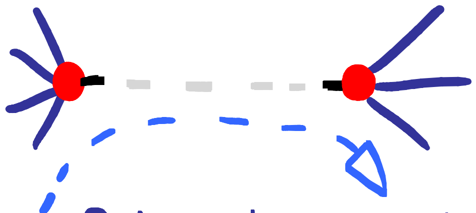


The fish swims along

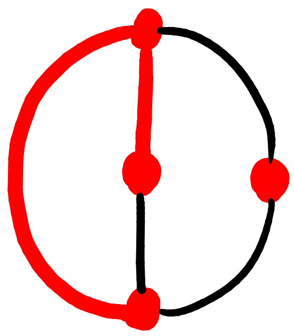
outside the forest and not isthmus

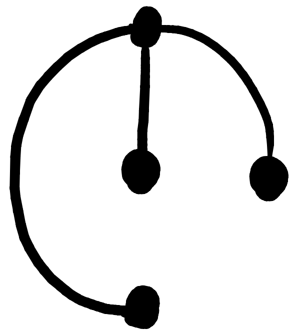


The fish eats the edge while swimming along

GOLDFISH ACTIVITY

spanning forest F $\longrightarrow$ tree $\tau(F)$





Rules:

inside the forest
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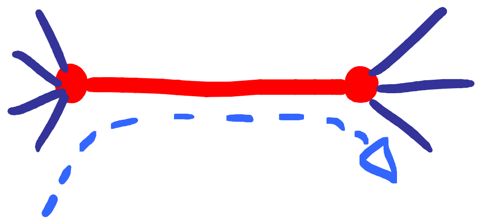

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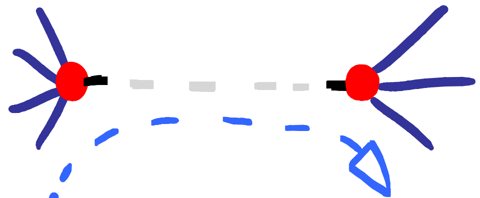

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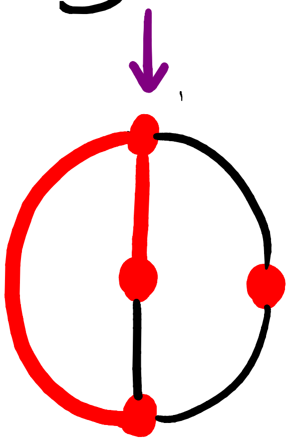
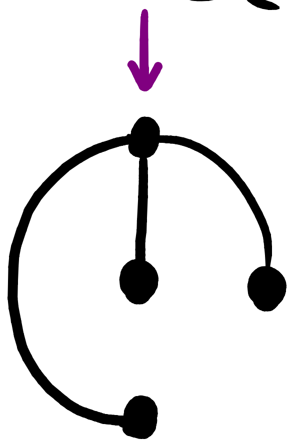
GOLDFISH ACTIVITY

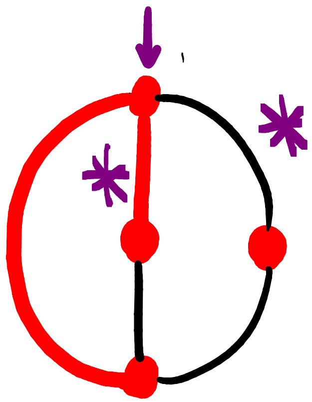
spanning forest $F \longrightarrow$ tree $\tau(F)$





Given a spanning tree T, an internal edge e is <u>active</u> if $\tau(T) = \tau(T \setminus e)$.

Ex:



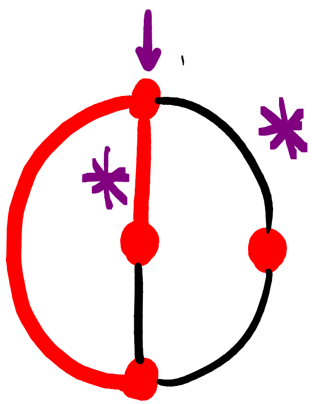
GOLDFISH ACTIVITY

Prop: $$T_G(x,1) = \sum_{T \text{ spanning tree}} x^{i(T)},$$

where $i(T)$ = number of internal active edges.

Given a spanning tree T,
an internal edge e is *active* if $\tau(T) = \tau(T \setminus e)$.

Ex:



QUESTION

Can we define a "meta-activity" that gathers the previous notions of activity?

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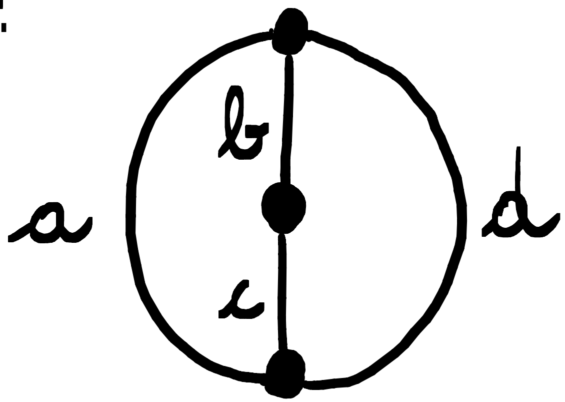
→ Yes, we can. Its name : Δ-activity.

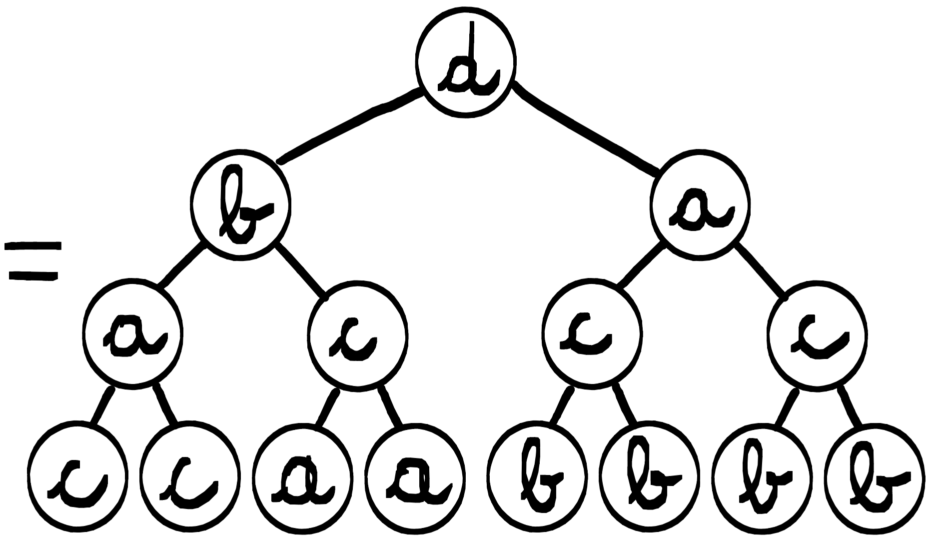
DECISION TREE

Let G be a graph.

Decision tree = plane binary tree Δ with a labelling $\mathrm{Vertices}(\Delta) \to \mathrm{Edges}(G)$ such that along every path starting from the root and ending at a leaf, the sequence of the labels forms a permutation of $\mathrm{Edges}(G)$.

Ex:

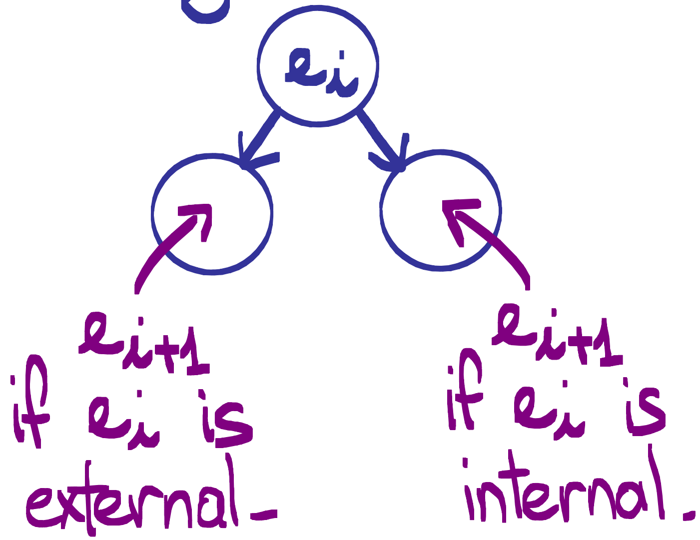


$\Delta =$ 

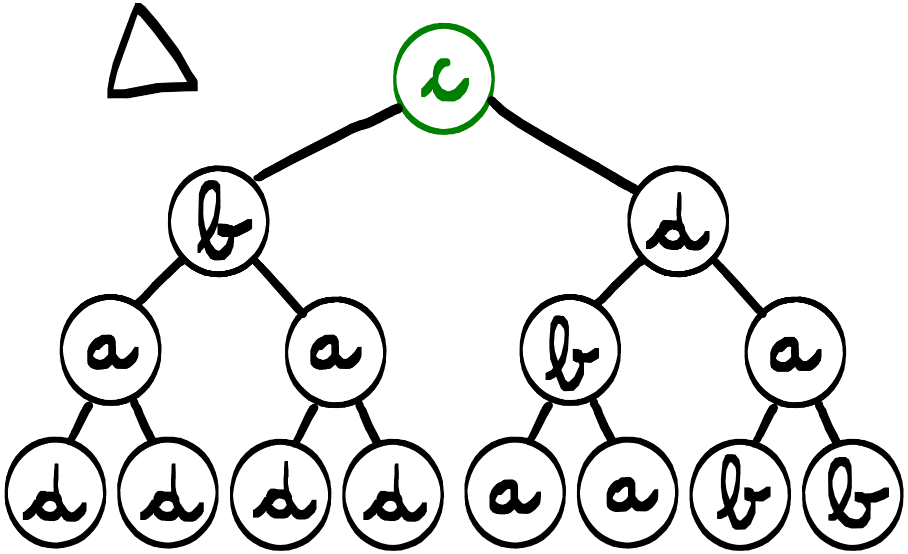
Δ-ACTIVITY

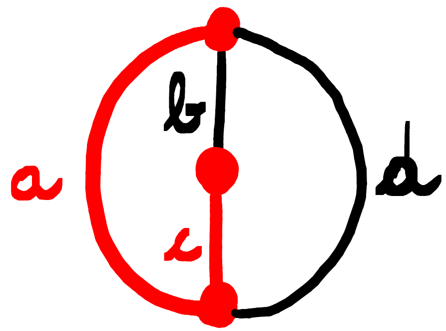
Given a spanning tree,
we define an order on the edges under the rule:

$e_1 =$ label of the root of Δ



Δ





c

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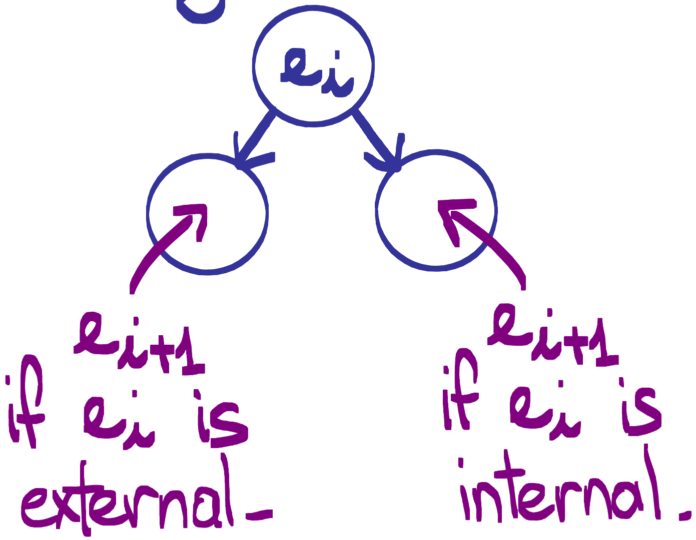
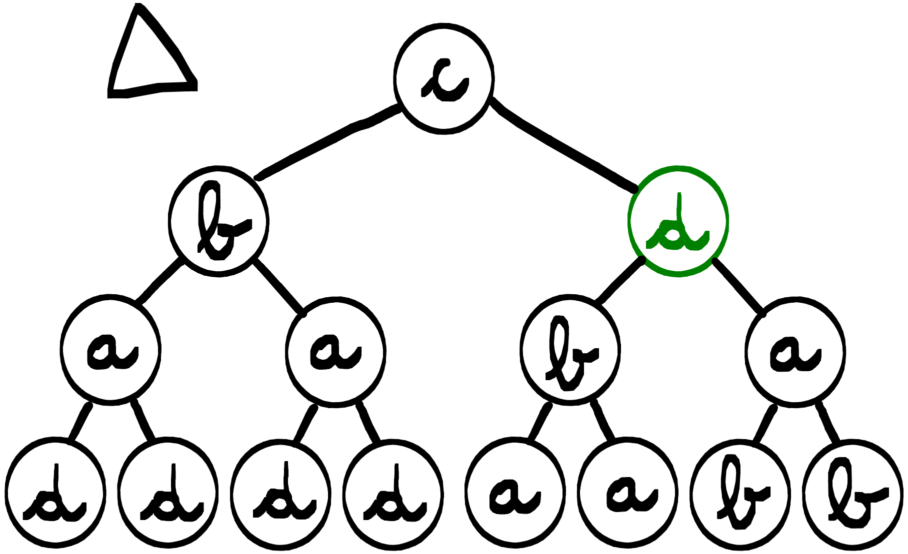
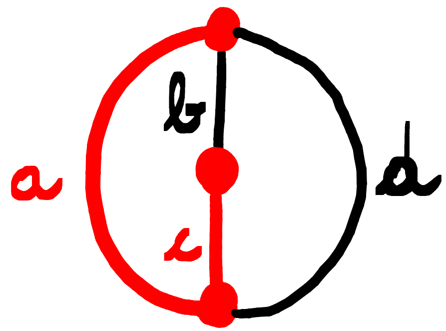


Δ



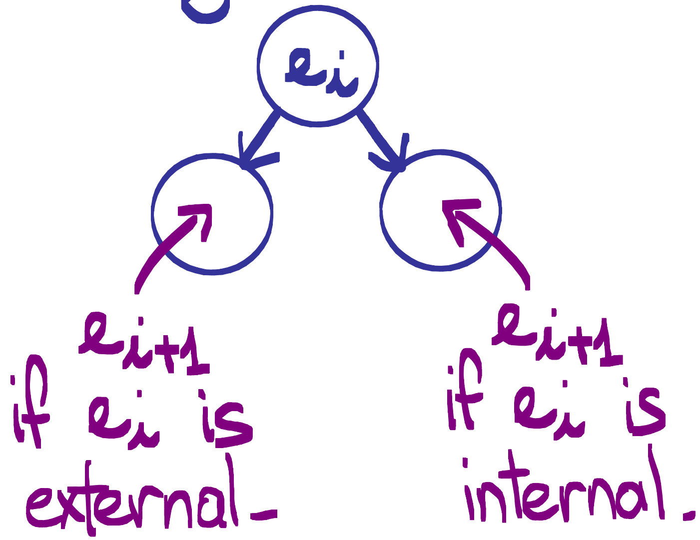


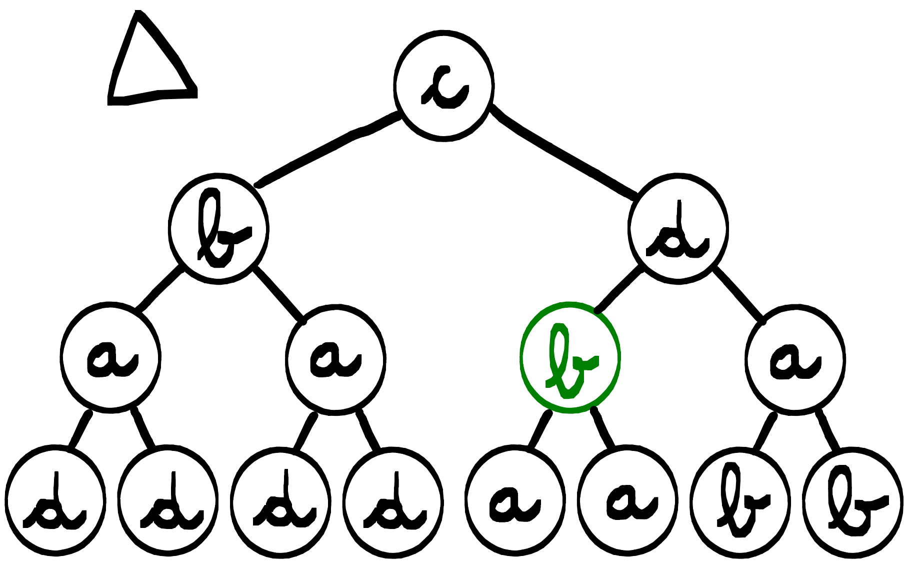
$c < d$

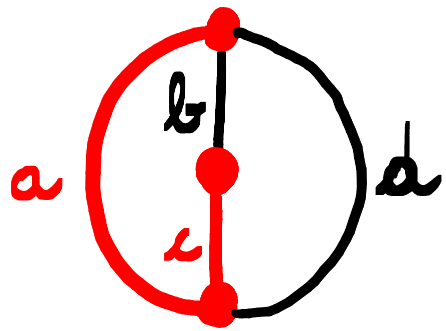
Δ-ACTIVITY

Given a spanning tree,
we define an order on the edges under the rule:

$e_1 =$ label of the root of Δ





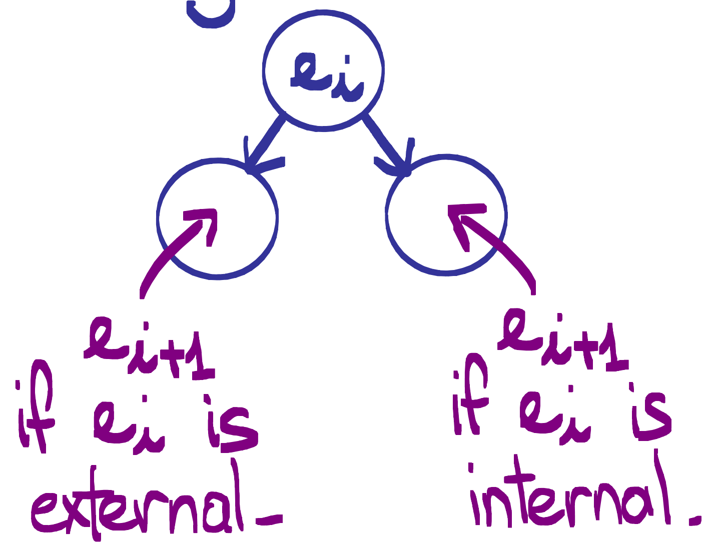


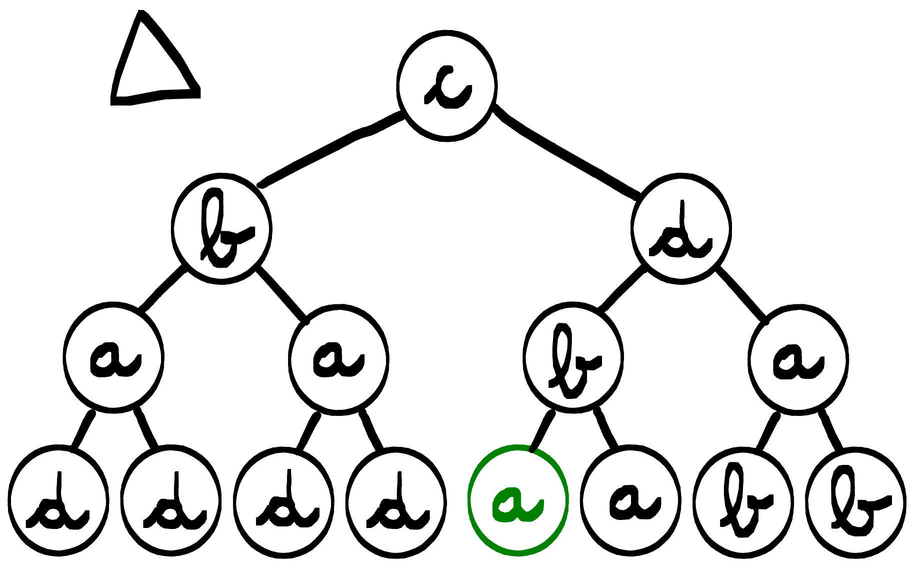
$$c < d < b$$

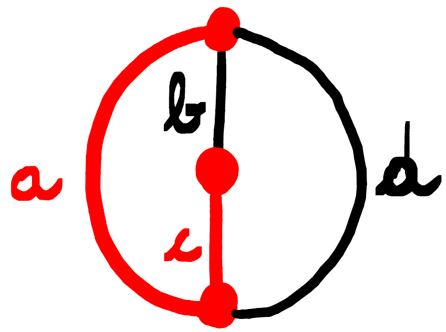
Δ-ACTIVITY

Given a spanning tree,
we define an order on the edges under the rule:

$e_1 =$ label of the root of Δ







$$c < d < b < a$$

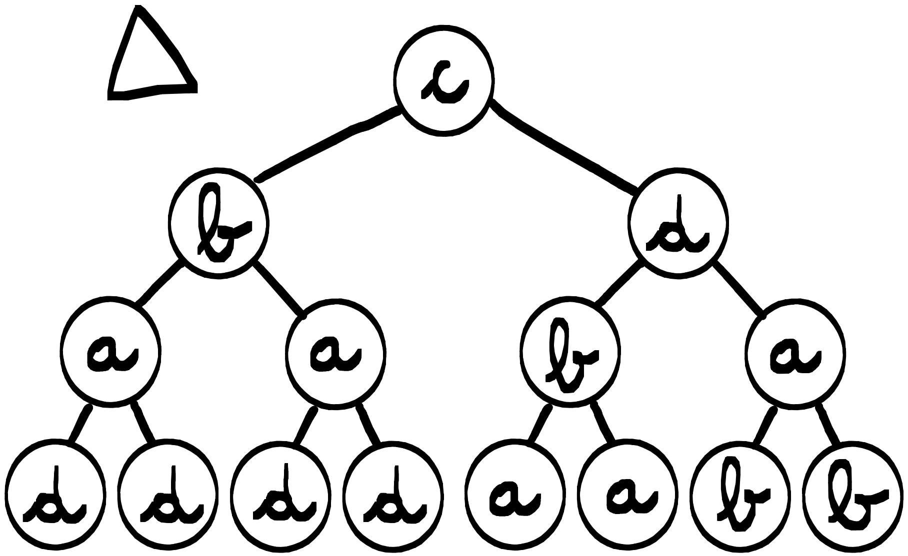
Δ-ACTIVITY

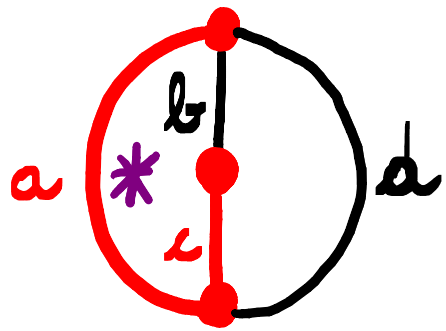
Δ-active edge = maximal edge inside its fundamental cycle / cocycle

Theorem For every graph G and decision tree Δ,

$$T_G(x,y) = \sum_{T \text{ spanning tree}} x^{i(T)} \, y^{e(T)}$$

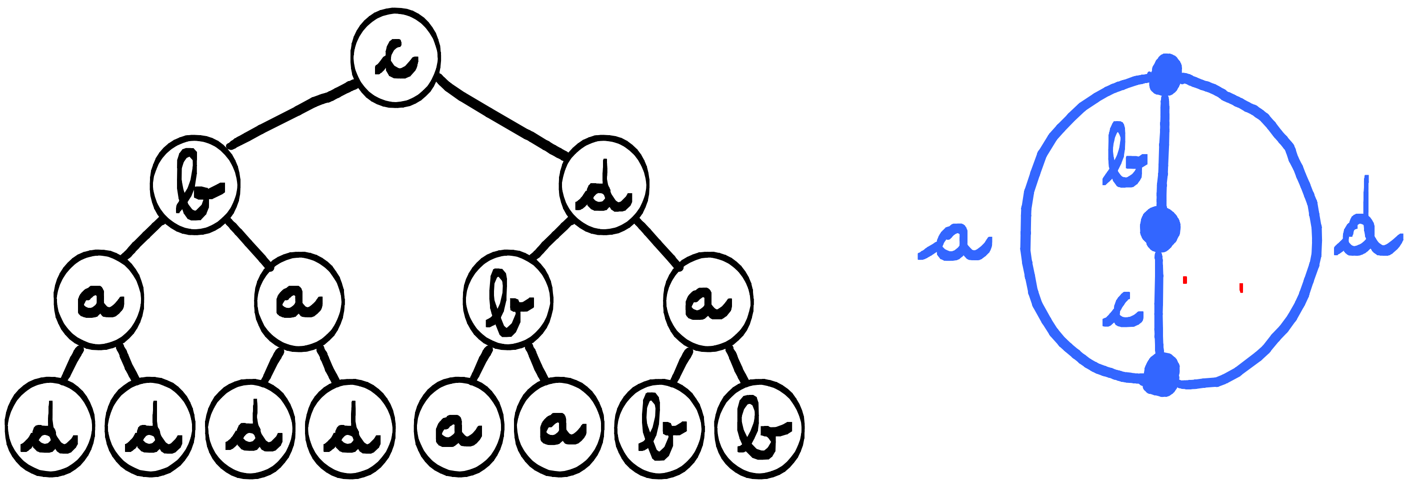
$i(T)$ = # internal Δ-active edges, $e(T)$ = # external Δ-active edges

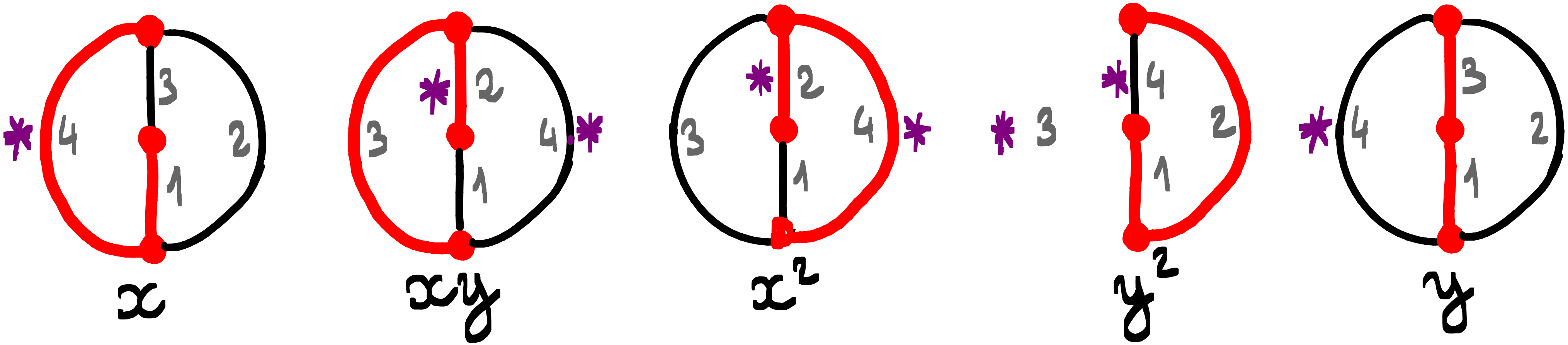




$c < d < b < a$

Δ-ACTIVITY

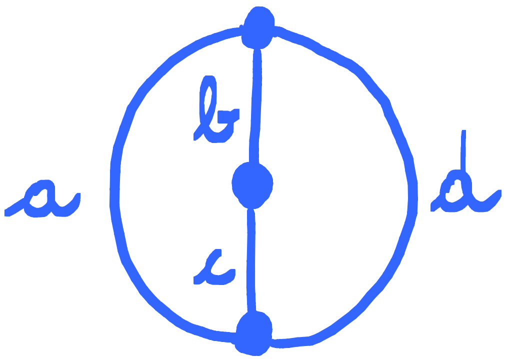




$$T_G(x,y) = x^2 + x + xy + y + y^2$$

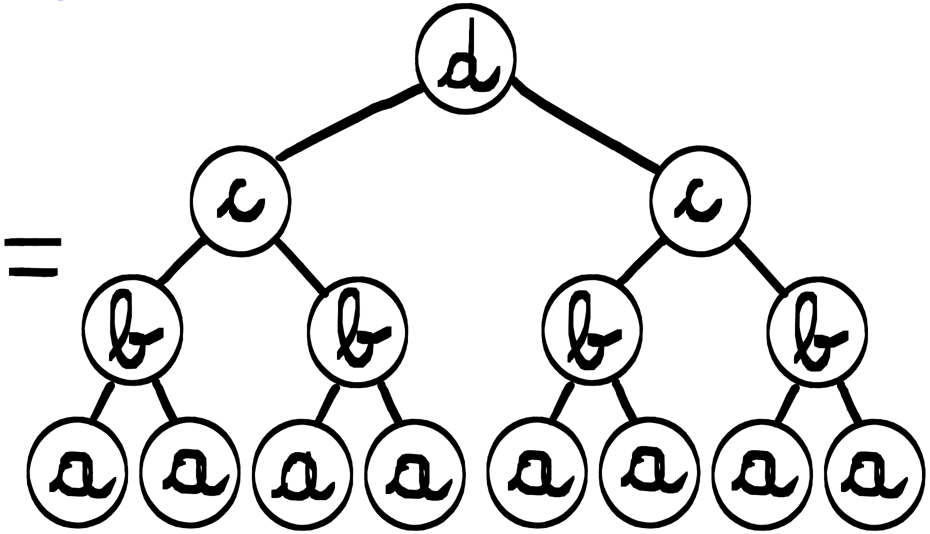
Δ-ACTIVITY

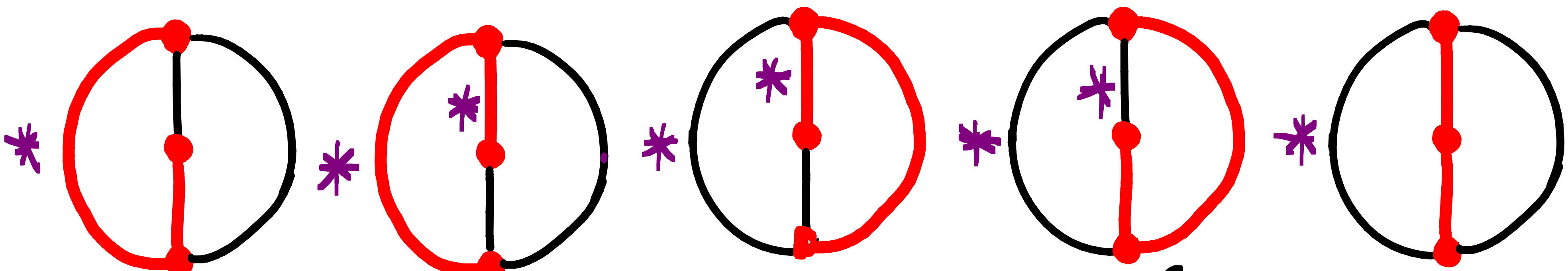
We recover the first activities : Tutte



$a < b < c < d$

$\Delta =$



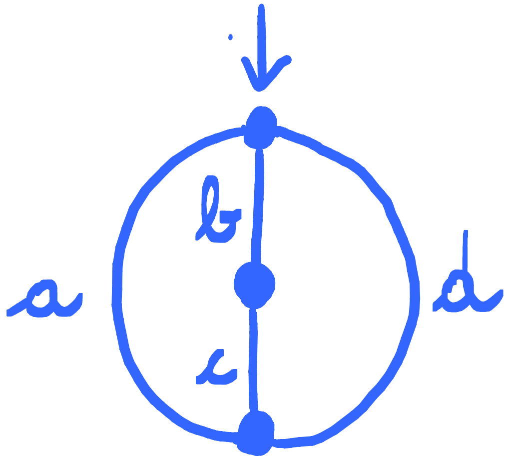


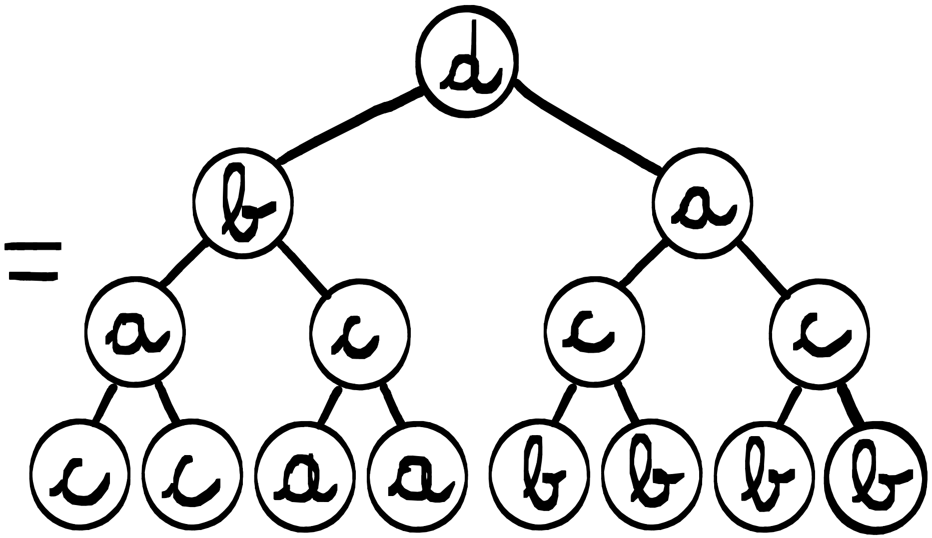
$x \qquad x^2 \qquad xy \qquad y^2 \qquad y$

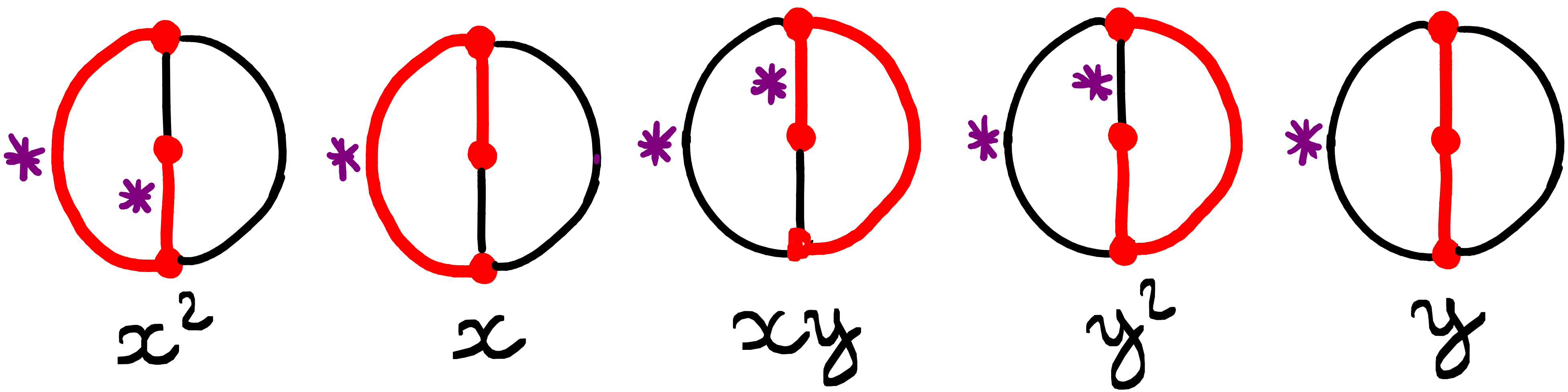
Δ-ACTIVITY

We recover the first activities :

Bernardi.



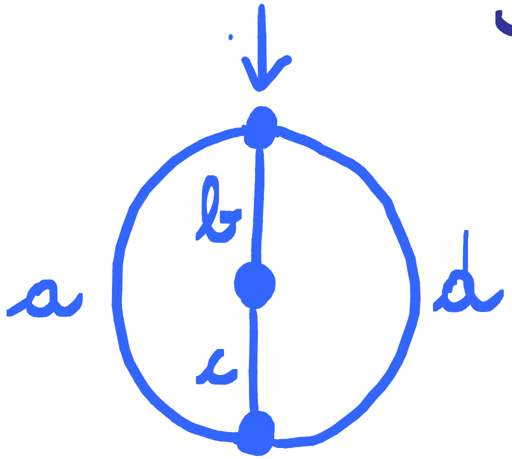
$\Delta =$ 

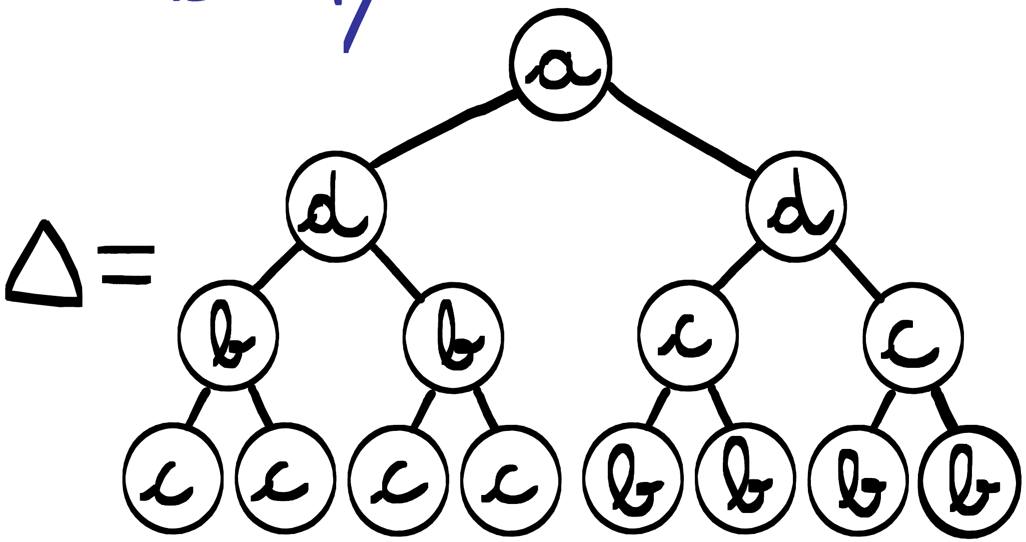


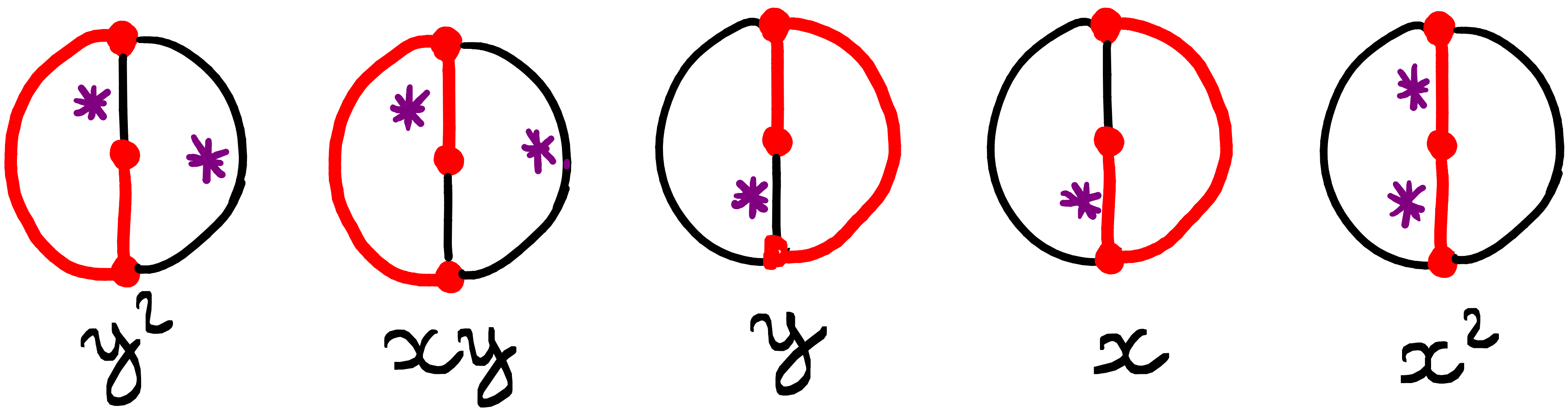
Δ-ACTIVITY

We recover the first activities :

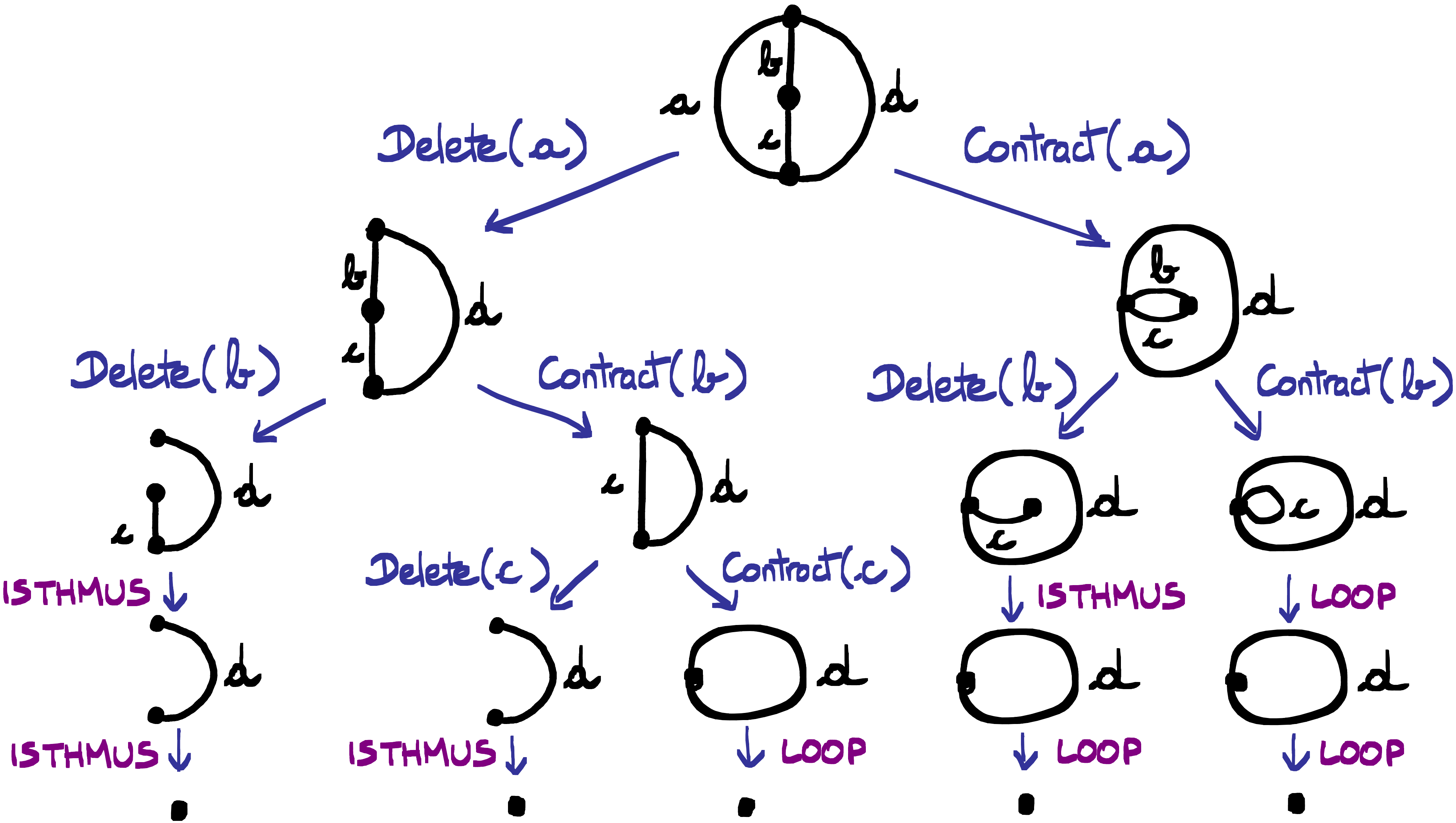
goldfish activity



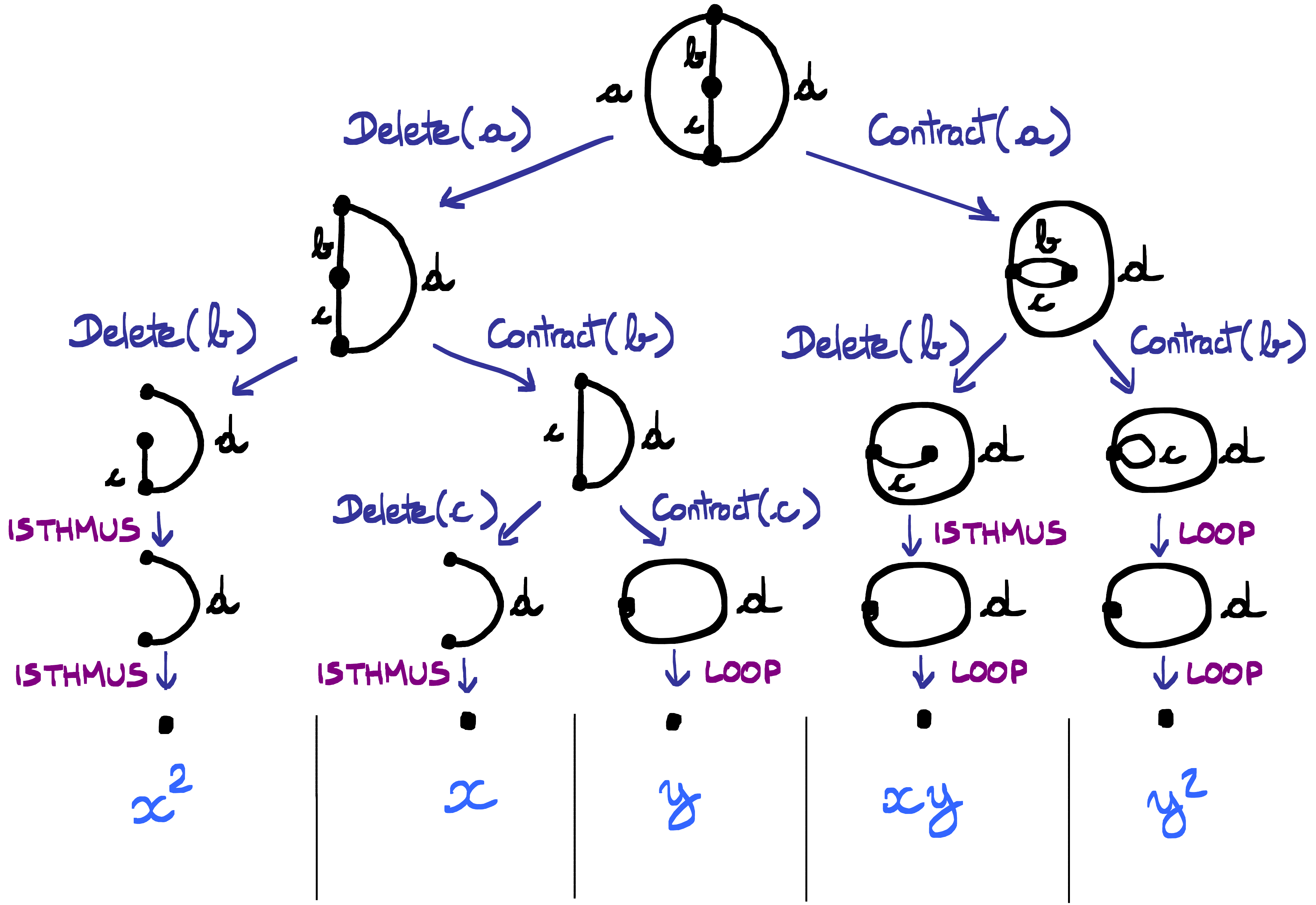




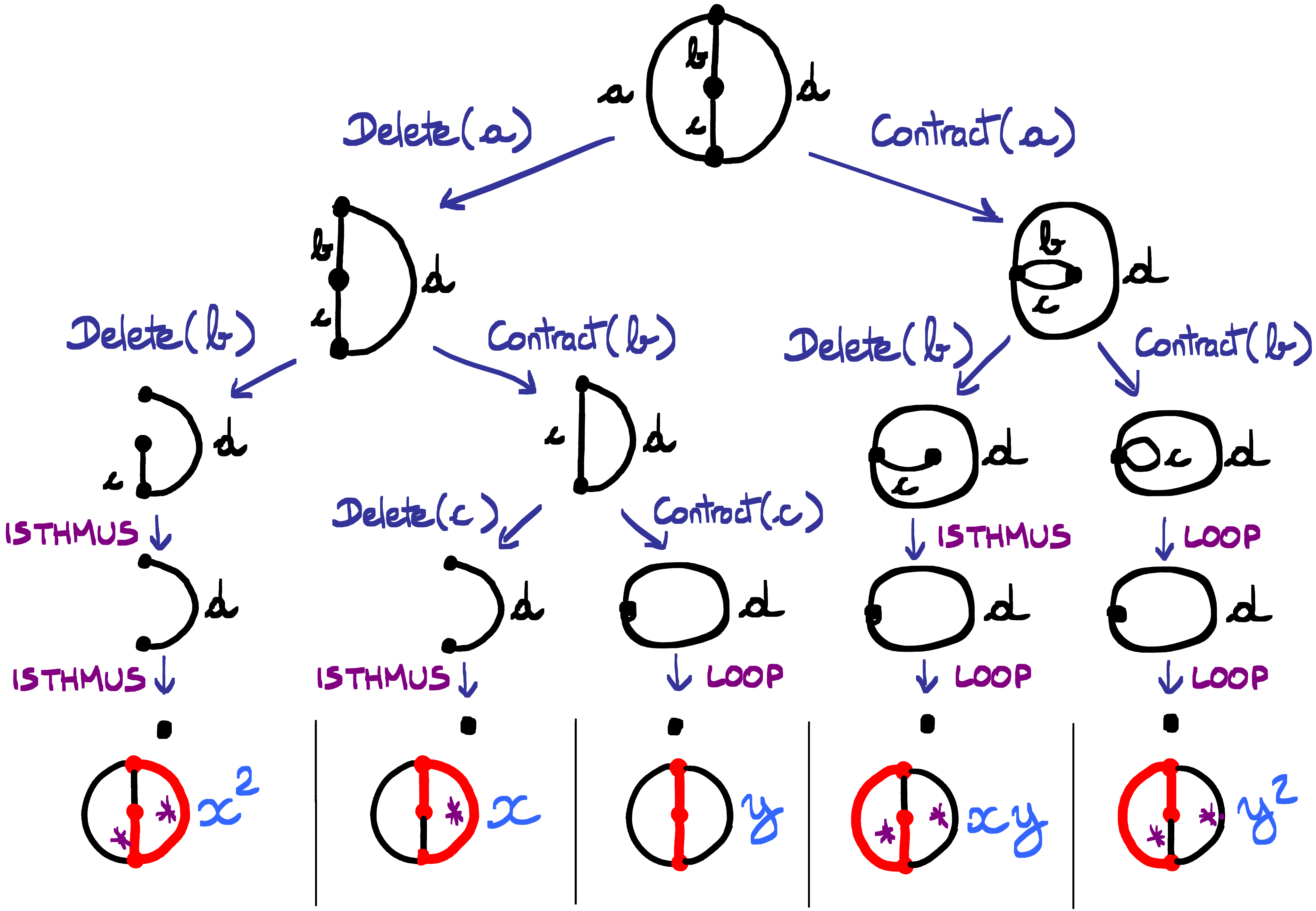
THE CLASSICAL DELETION-CONTRACTION TREE



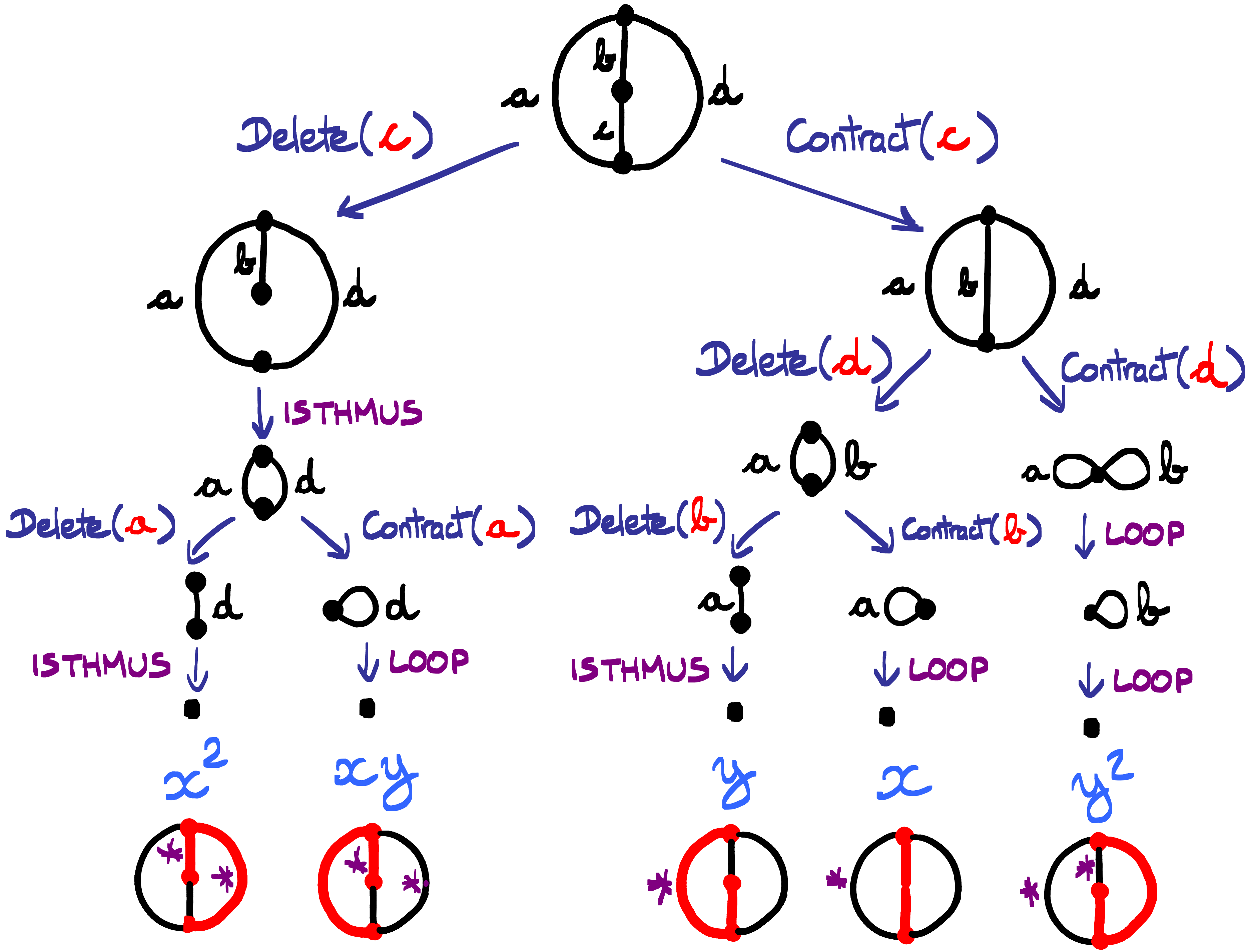
THE CLASSICAL DELETION-CONTRACTION TREE



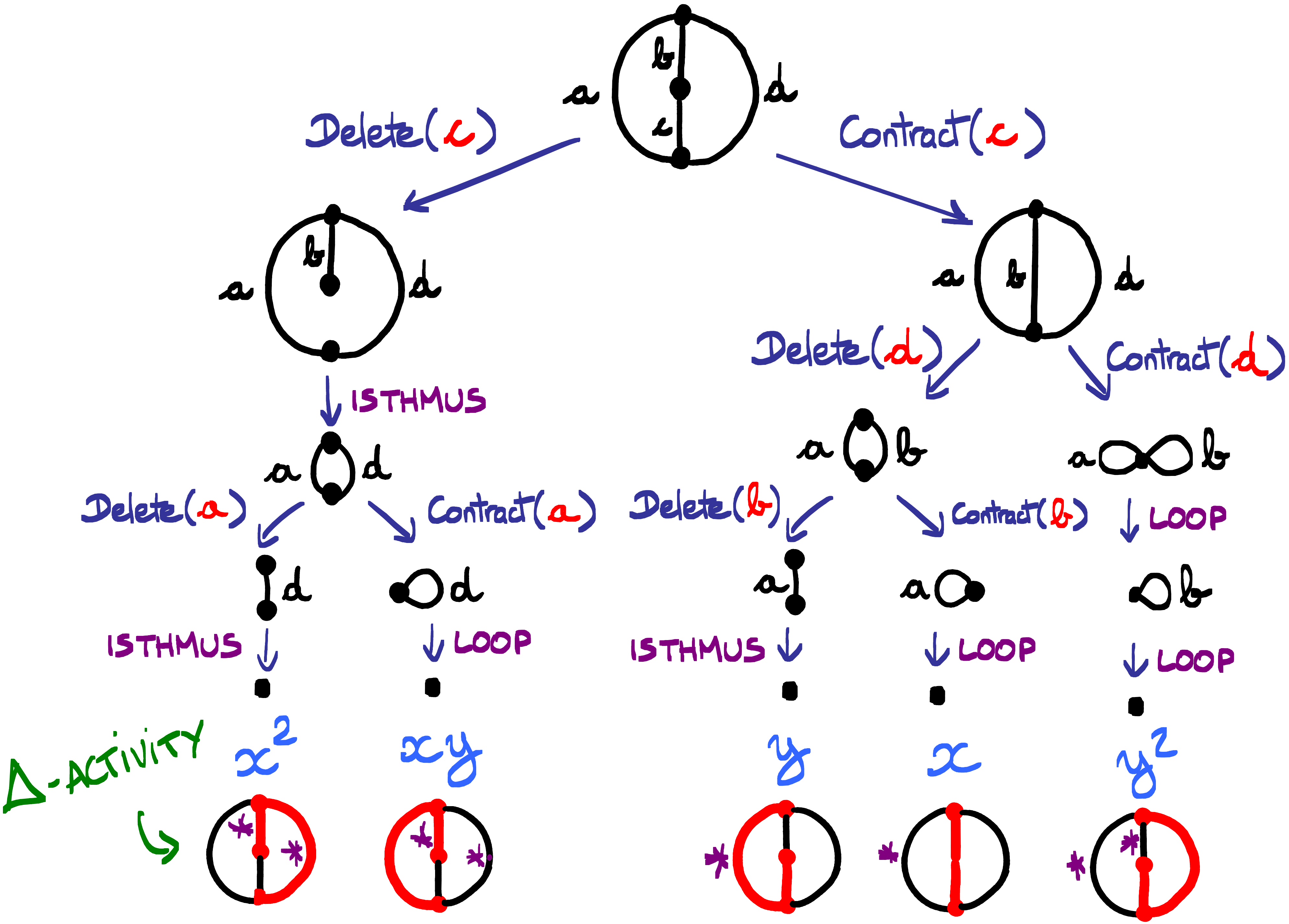
THE CLASSICAL DELETION-CONTRACTION TREE



A TWEAKED DELETION-CONTRACTION TREE



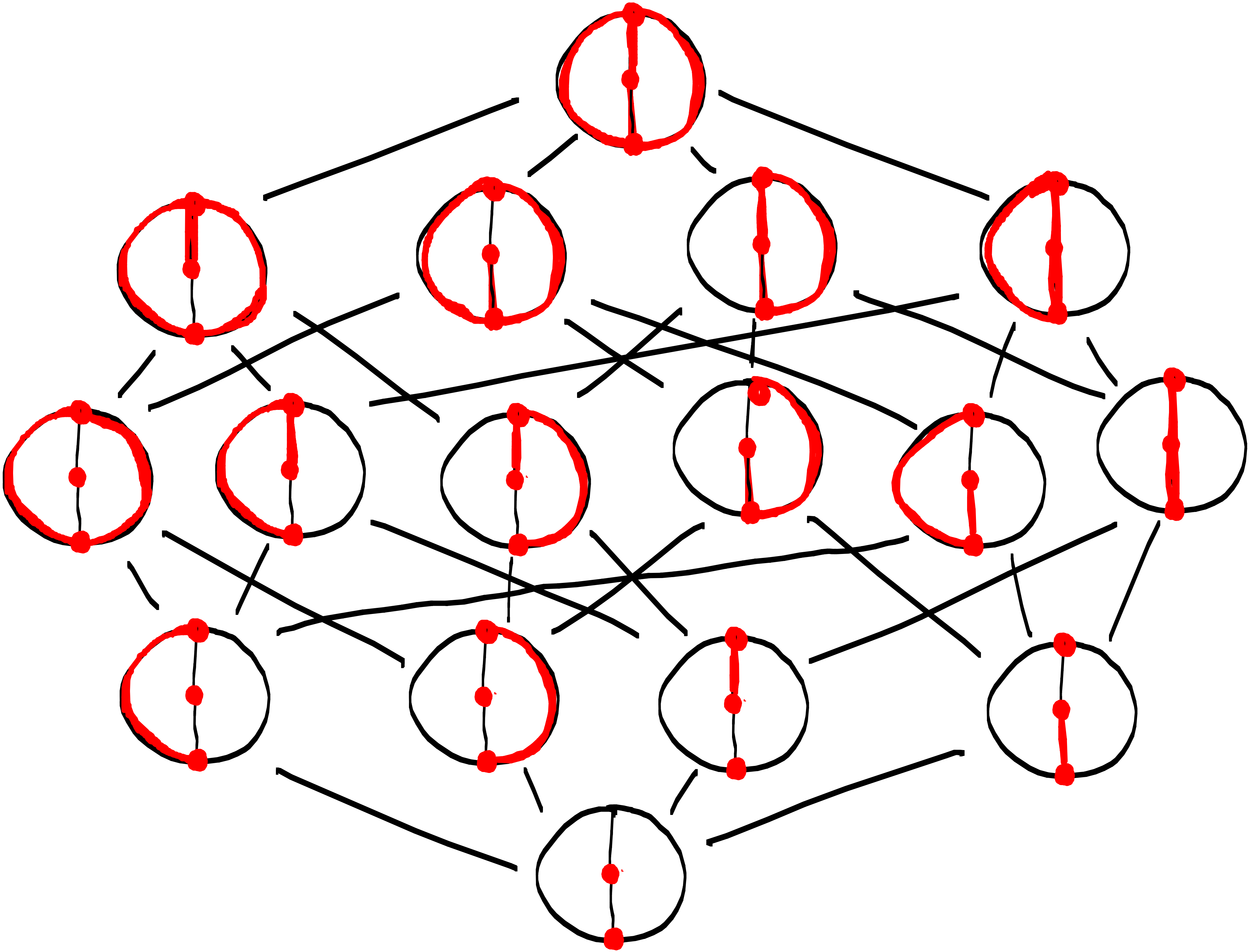
A TWEAKED DELETION-CONTRACTION TREE

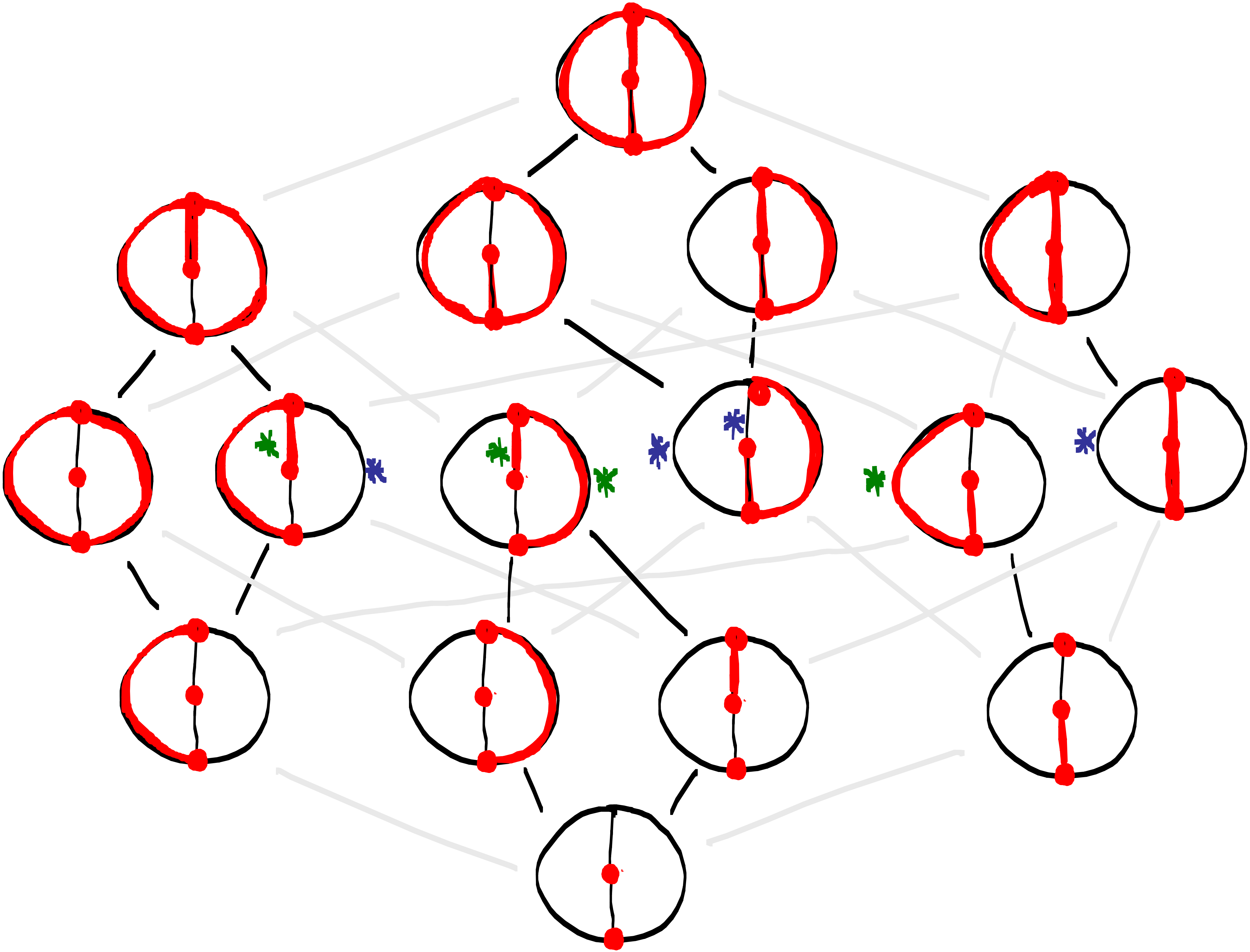


CRAPO'S PROPERTY

Crapo's property holds for Δ-activity:

$$\mathrm{Subgraphs}(G) = \biguplus_{T \text{ spanning tree of } G} [T \setminus \mathrm{Act}(T), T \cup \mathrm{Act}(T)]$$





A CONJECTURE FOR THE END

→ induces other "natural" activities.
(like Gessel-Sagan's activity)

Conjecture

Every activity that preserves Crapo's property is a Δ-activity.

THANK YOU!